Thermo-Elasticity - Solution

Note

This HW set, especially the last question, summarizes the first part of the course, and demands a working knowledge of linear elasticity. It also involves a small numeric calculation, allowing you to practice solving a physical problem with computational tools. We consider it a "semi-mid-term", and it will be given a larger weight in the final grade than the other sets. You are also given three weeks to complete it, so please start early and take it seriously.

1 Circular hole

Consider an infinite 2D material, from which a circular hole is taken out. The material is now heated by some amount. Will the hole shrink or expand?

Solution

If a material is heated by ΔT and all its boundaries are free, then the displacement field $\boldsymbol{r} \to (1 + \frac{\alpha_T \Delta T}{3})\boldsymbol{r}$, a simple homogeneous dilation, is a solution of the equations. Intuitively, this is so because α_T is nothing but the thermal expansion coefficient. I urge you to plug this solution into the equations and check that this is so: under the supposed deformation we have $\boldsymbol{\varepsilon} = \frac{\alpha_T \Delta T}{3}\delta_{ij}$. Simply plug that it in Hooke's law to get

$$\begin{aligned}\sigma_{ij} &= -K\alpha_T \Delta T \delta_{ij} + K \operatorname{tr}\boldsymbol{\varepsilon}\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\operatorname{tr}\boldsymbol{\varepsilon}\delta_{ij}\right) \\ &= -K\alpha_T \Delta T \delta_{ij} + K\alpha_T \Delta T \delta_{ij} + 2\mu\left(\frac{\alpha_T \Delta T}{3}\delta_{ij} - \frac{\alpha_T \Delta T}{3}\delta_{ij}\right) = 0\end{aligned} \tag{1}$$

Therefore, when heated and free of external forces, materials simply expand by homogeneous dilation and everything will be stress free. The same goes for the hole - it will expand.

Another way to think about it - consider the piece of material that was taken out. If you heat it by the same amount as the rest, it should fit perfectly.

2 Temperature and displacements

Consider a static infinite 3D material with a given arbitrary distribution of temperature $T(x, y, z)$, that decays at infinity: $T(\vec{r}) \to T_\infty$, as $|\vec{r}| \to \infty$. Before reading further it might by nice to try to estimate: if the temperature variation is localized, how does the displacement field decay at large $\boldsymbol{r}$? And the strain field?

Here's a nice way to gain intuition as to what temperature gradients do in thermo-elasticity: Prove that the displacement field is curl-free, i.e. is of the form $\vec{u} = \nabla\phi$, and that ϕ satisfies Poisson's equation $\nabla^2\phi = T$. Assuming you already have some intuition about electrostatics, this should help you gain intuition about thermo-elasticity.

Guidance: Begin with Navier-Lamé equation $(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = K\alpha\nabla T$. You can guess the correct form for u, and if it works then you're done because the solution is unique. Some vector-analysis identities might prove useful.

Solution

Since the "driving force" for the deformation is ∇T, i.e. a curl-free vector field, it is very reasonable to guess that $\vec{u}$ will also be curl-free. Otherwise this would mean that a chiral symmetry is broken. So we assume that $\vec{u}$ is curl-free, i.e. $u = \nabla\phi$. Before we plug that into the Navier-Lamé equation, we use the identity $\nabla^2\boldsymbol{A} = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla\times(\nabla\times\boldsymbol{A})$. We get

$$K\alpha\nabla T = (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = (\lambda+2\mu)\nabla(\nabla\cdot\boldsymbol{u}) - \mu\nabla\times\nabla\times\boldsymbol{u}$$

We now plug in $\boldsymbol{u} = \nabla\phi$, and the last term vanishes. We are left with

$$\begin{aligned} K\alpha\nabla T &= (\lambda+2\mu)\nabla\left(\nabla^2\phi\right) \\ &\Downarrow \\ 0 &= \nabla\Big[K\alpha T - (\lambda+2\mu)\nabla^2\phi\Big] \ . \end{aligned}$$

This means that the term in brackets is constant, i.e. ϕ satisfies Poisson's equation:

$$\nabla^2\phi = \frac{K\alpha}{\lambda+2\mu}T + C \ .$$

We see that C is a meaningless constant that can be swallowed into T, and since anyhow the only relevant feature of T is its gradient, we can safely set $C = 0$.

The solution of this equation you should be known from your undergrad. It is

$$\phi(\boldsymbol{r}) \sim \int \frac{T(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}' \ .$$

If $T(\boldsymbol{r})$ is localized, say a δ-function, then ϕ decays as r^{-1}, u as r^{-2} and $\boldsymbol{\varepsilon}$ as r^{-3}.

3 Thermally induced fracture

In 1993, Yuse & Sano published a remarkable paper regarding instabilities of thermally induced fracture (Yuse & Sano, Nature (362) 1993). They consider a strip of material which is pulled out of an oven **at a constant velocity** and cools down as it moves. The gradients of the temperature field induce fracture, as is seen in Figure 1.

To model the phenomenon, consider an infinite (in the x direction) 2D strip of width $2b$. The strip is subject to a y-independent temperature distribution $T(x)$, and is free of tractions at its boundaries

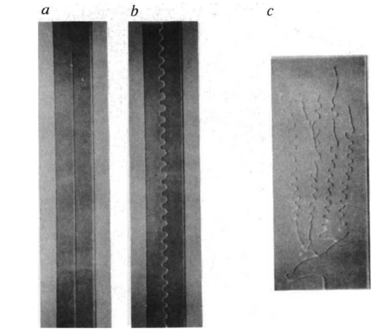

FIG. 2 Typical examples of three different types of cracks: a, a straight crack, b, oscillating crack, c, branched crack. Sample size (a and b) 10 × 100 × 0.11 mm³, (c) 24 × 60 × 0.13 mm³.

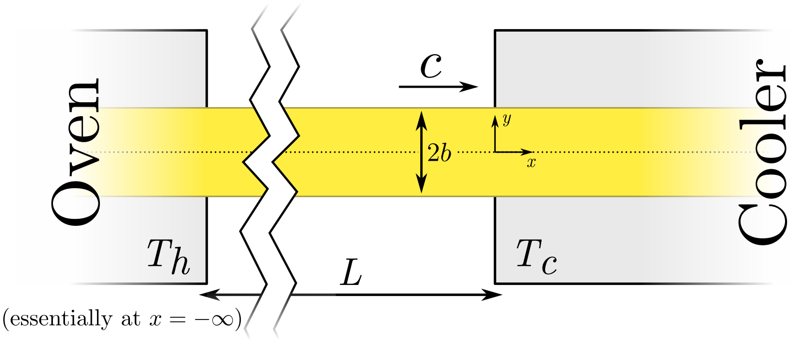

Figure 1: Left: Thermally induced fracture (from the paper). Right: the simplified model. Note the position of the origin of axes, and that the plot **is not to scale**: we assume $L \gg b$.

$y = \pm b$. Fracture will be considered later in this course. For now, we'll limit ourselves to finding an expression for the stretching component σ_{yy} along the strip's symmetry axis $y = 0$. This is the driving force that induces fracture.

(a) We begin by finding the temperature distribution. Write the heat diffusion equation in both the material ($\boldsymbol{X}$) and laboratory ($\boldsymbol{x}$) coordinates, and solve it in the laboratory coordinates for our problem. Assume that the cooler and the oven are strong enough such that $T(x > 0) = T_c$, and $T(x < -L) = T_h$. Assume also that L is much larger than any other length scale of the system. The heat diffusion constant D is of course given. Remember that you can always shift T by a global constant to get a simpler expression.

Solution

In the material coordinates $\{X, Y, \tau\}$, The diffusion equation is

$$\partial_\tau T = D\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) T$$

The change of variables from the material coordinates to the lab coordinates $\{x, y, t\}$, is defined by

$$x = X + c\tau, \quad t = \tau, \quad y = Y \ ,$$

and we find that the differentiation operators transform as

$$\frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tau}\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial X} = \frac{\partial t}{\partial X}\frac{\partial}{\partial t} + \frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial Y} = \frac{\partial t}{\partial Y}\frac{\partial}{\partial t} + \frac{\partial x}{\partial Y}\frac{\partial}{\partial x} + \frac{\partial y}{\partial Y}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$

Therefore, the diffusion equation takes the form

$$\partial_t T = D\nabla_{\boldsymbol{x}}^2 T - c\,\partial_x T \tag{2}$$

Assuming T is y-independent and the boundary conditions $T(x=0)=T_c,\ T(x=-L)=T_h$ the steady-state solution for $-L \le x \le 0$ is

$$T = \frac{\left(e^{\frac{c}{D}(L+x)} - 1\right) T_c - e^{\frac{cL}{D}}\left(e^{\frac{c}{D}x} - 1\right) T_h}{e^{\frac{cL}{D}} - 1} \tag{3}$$

We now define the diffusive length $d = D/c$, and use the fact that $L \gg d$ to simplify the above solution,

$$T \approx T_h + e^{\frac{x}{d}}\left(T_c - T_h\right) \ . \tag{4}$$

We set the zero of temperature at T_c, and denote $\Delta T \equiv T_h - T_c$. Incorporating the fact that $\Delta T = 0$ for $x > 0$ we obtain the temperature distribution

$$T(x) = -\Delta T\left(1 - e^{\frac{x}{d}}\right)\Theta(-x) \ . \tag{5}$$

Θ is Heaviside's theta function. Note that we don't care that T is not exactly constant for $x < -L$, because the correction is roughly $e^{-L/d}$, which we assume is a very small number.

(b) Show that the equations of plane-stress combined with static thermo-elasticity are

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] + \frac{1}{3}\alpha_T\Delta T, \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] + \frac{1}{3}\alpha_T\Delta T, \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}
\end{aligned} \tag{6}$$

Guidance: Start with the known Hooke's law derived in class, $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, T) = \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon} - \alpha K T \boldsymbol{I}$, invert it to the compliance form $\boldsymbol{\varepsilon}(\boldsymbol{\sigma}, T)$, and use the relations between λ, μ, K to E, ν (which are summarized in a nice table in Wikipedia).

Solution

The thermo-elastic Hooke's law was derived in class and it reads

$$\sigma_{ij} = \lambda\,\mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}_{ij} - \alpha K T \boldsymbol{I} \tag{7}$$

In what follows, we'd like to invert this relation, so we want to write everything in matrix form. Also, in the end we'd like to have relations of the form $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ (compliance form), so it is more natural to work with ν and E, rather than μ and λ. Using the conversions between $\mu, \lambda \to E, \nu$, Eq. (7) is written as:

$$\begin{pmatrix} \sigma_{xx} + \alpha K T \\ \sigma_{yy} + \alpha K T \\ \sigma_{zz} + \alpha K T \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & 1-2\nu & & \\ & & & & 1-2\nu & \\ & & & & & 1-2\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{pmatrix} \tag{8}$$

Inverting, we get

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{xz}\\ \varepsilon_{yz}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 1+\nu & & \\ & & & & 1+\nu & \\ & & & & & 1+\nu\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{pmatrix} + \frac{\alpha T}{3}\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix} \tag{9}$$

Where we used the fact $K = \frac{E}{3(1-2\nu)}$. The assumptions of plane stress are that $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. Thus, if we delete the 3rd, 5th and 6th rows of this equation, it reduces to

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 1+\nu\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} + \frac{\alpha T}{3}\begin{pmatrix}1\\1\\0\end{pmatrix}\ , \tag{10}$$

which is exactly the set of equations that we needed to derive. Exactly like the plane-stress/plane-strain discussion we had in class, note that this compliance matrix is obtained by deleting entries from the 3D compliance matrix (Eq. (9), but if we invert this relation in its 2D form (setting for a moment $T=0$), we get

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 1-\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} \tag{11}$$

which cannot be obtained by deleting entries from the 3D stiffness matrix (Eq. (8)).

(c) Prove the compatibility relation

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}\ , \tag{12}$$

and use it together with the definition of the Airy potential χ and Eqs. (6) to show that χ satisfies the equation

$$\nabla^2\nabla^2\chi = -\frac{1}{3}E\alpha_T\nabla^2 T \tag{13}$$

What is the symmetry of χ with respect to y? What are the boundary conditions that χ satisfies?

Solution

The compatibility relation, Eq. (12), is a trivial identity that follows from the definition $\epsilon_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. If we substitute ϵ_{ij} by Hookes law, Eq. (10), and then substitute σ_{ij} by derivatives of χ we get

$$\begin{aligned}
0 &= \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \\
&= \frac{\partial^2}{\partial y^2}\left(\frac{\sigma_{xx} - \nu\sigma_{yy}}{E} + \frac{\alpha_T T}{3}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\sigma_{yy} - \nu\sigma_{xx}}{E} + \frac{\alpha_T T}{3}\right) - 2\frac{1+\nu}{E}\frac{\partial^2}{\partial x \partial y}\sigma_{xy} \\
&\propto \frac{\partial^2}{\partial y^2}\left(\partial_{yy}\chi - \nu\partial_{xx}\chi\right) + \frac{\partial^2}{\partial x^2}\left(\partial_{xx}\chi - \nu\partial_{yy}\chi\right) + 2\left(1+\nu\right)\frac{\partial^2}{\partial y \partial x}\partial_{xy}\chi + \frac{\alpha_T E}{3}\nabla^2 T \\
&= (\partial_y^4 + 2\partial_{xx}\partial_{yy} + \partial_x^4)\chi + \frac{\alpha_T}{3}\nabla^2 T = \nabla^2\nabla^2\chi + \frac{\alpha_T E}{3}\nabla^2 T
\end{aligned} \tag{14}$$

(d) Solve Eq. (13) by Fourier transforming it in the x direction and imposing the boundary conditions. Express $\sigma_{yy}(x, y = 0)$ in an integral form. You should obtain an expression of the form

$$\sigma_{yy}(x, y = 0) = \int_{-\infty}^{\infty} T(x')\Psi(x - x')dx' \equiv T * \Psi \ , \tag{15}$$

where $*$ denotes convolution, and $\Psi(x)$ is the convolution kernel, for which you should have a closed expression (as an integral of something).

Solution

We need to solve Eq. (14) under the conditions

$$\sigma_{yy}\big|_{y=\pm b} = \sigma_{xy}\big|_{y=\pm b} = 0 \ . \tag{16}$$

Fourier-transforming the equation with respect to x gives

$$\left(k^4 - 2k^2\frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4}\right)\chi(k, y) = -E\alpha k^2 \hat{T}(k) \tag{17}$$

A particular solution of this nonhomogeneous problem is clearly $\chi = -\frac{E\alpha T(k)}{k^2}$. The homogeneous equation admits four independent solutions: $\chi = e^{\pm ky}, \chi = ye^{\pm ky}$. Note that the y-independence of T was used (and was crucial!). Because $\sigma_{yy} = \partial_{xx}\chi$ is an even function of y, χ must also be even in y. So we use even combinations of the solutions we found to get

$$\chi(k, y) = A(k)\cosh(ky) + yB(k)\sinh(ky) - \frac{E\alpha}{k^2}\hat{T}(k) \ , \tag{18}$$

The boundary conditions (16) translate to

$$\partial_{xx}\chi\big|_{y=b} = 0 \Rightarrow \quad A\cosh(kb) + Bb\sinh(kb) - E\alpha\frac{\hat{T}(k)}{k^2} = 0 \ , \tag{19}$$

$$\partial_{xy}\chi\big|_{y=b} = 0 \Rightarrow \quad Ak\sinh(kb) + B\sinh(kb) + Bbk\cosh(kb) = 0 \ . \tag{20}$$

This is a set of linear equations which is solved by

$$A(k) = E\alpha\frac{\hat{T}(k)}{k^2}\underbrace{2\frac{bk\cosh(bk)+\sinh(bk)}{2bk+\sinh(2bk)}}_{\equiv\hat{\Phi}(k)} = E\alpha\frac{\hat{T}(k)}{k^2}\hat{\Phi}(k) \tag{21}$$

$$B(k) = -2E\alpha\frac{\hat{T}(k)}{k}\frac{\sinh(bk)}{2bk+\sinh(2bk)} \tag{22}$$

Note the definition of Φ, the use of which will become clear immediately. σ_{yy} along the symmetry line $y=0$ is given by

$$\begin{aligned}\sigma_{yy}(x,y=0) = \partial_{xx}\chi(x,y=0) &= \mathcal{F}^{-1}\left\{-k^2\chi(k,y=0)\right\} \\ &= \mathcal{F}^{-1}\left\{-k^2\left(A(k)-\frac{E\alpha\hat{T}(k)}{k^2}\right)\right\} \\ &= \mathcal{F}^{-1}\left\{E\alpha\hat{T}(k)\left(1-\hat{\Phi}(k)\right)\right\}\end{aligned}$$

Where $\mathcal{F}^{-1}$ is the inverse Fourier transform. We now define $\hat{\Psi}(k)\equiv 1-\hat{\Phi}(k)$ and use the convolution theorem to finally obtain

$$\sigma_{yy}(x,y=0) = \mathcal{F}^{-1}\left\{E\alpha\hat{T}(k)\hat{\Psi}(k)\right\} = E\alpha T(x) * \Psi(x) \tag{23}$$

where $*$ denotes convolution and $\Psi(x) = \frac{1}{2\pi}\int e^{-ikx}\Psi(k)dk$.

In order to proceed, we non-dimensionalise our equations. We renormalize lengths by b, that is, we define $\tilde{k}\equiv bk$, $\tilde{x}\equiv x/b$, and also $\tilde{\sigma}\equiv\sigma/E$, $\tilde{T}\equiv\alpha T/\Delta T$. We thus obtain

$$\hat{\Psi}(\tilde{k}) = 1-2\frac{\tilde{k}\cosh(\tilde{k})+\sinh(\tilde{k})}{2\tilde{k}+\sinh(2\tilde{k})} \tag{24}$$

$$\Psi(\tilde{x}) = \int e^{-i\tilde{k}\tilde{x}}\hat{\Psi}(\tilde{k})\frac{d\tilde{k}}{2\pi} = \delta(\tilde{x}) - \underbrace{\int 2\cos\left(\tilde{k}\tilde{x}\right)\frac{\tilde{k}\cosh(\tilde{k})+\sinh(\tilde{k})}{2\tilde{k}+\sinh(2\tilde{k})}\frac{d\tilde{k}}{2\pi}}_{=\Phi(\tilde{x})} \tag{25}$$

$$\tilde{\sigma}_{yy}(\tilde{x},y=0) = \frac{1}{b}\tilde{T}(\tilde{x}) * \Psi(\tilde{x}) = \frac{1}{b}\tilde{T}(\tilde{x}) * [\delta(\tilde{x})-\Phi(\tilde{x})] = \frac{1}{b}\left[T(\tilde{x}) - T(\tilde{x}) * \Phi(\tilde{x})\right] \tag{26}$$

In Eq. (25) the $e^{i\tilde{k}\tilde{x}}$ was replaced with $\cos(\tilde{k}\tilde{x})$ since the integrand is even. Also, note that since $\Psi(x)$ has a δ-function singularity, the convolution is done by convolving with Φ and subtracting T, as is shown in Eq. (26).

We see that $\Psi(\tilde{x})$ is a "universal" function that can be computed once, and then the stress field resulting from an arbitrary temperature distribution can be obtained by convolving the temperature with the kernel $\Psi(x)$ (a different terminology would be that $\Psi(x)$ is the Green's function of the problem).

Note also how useful is the non-dimensionalization: The kernel as a function of $\tilde{x}$ will not change when b or D/c change. This will come in only through the temperature profile:

$$T(\tilde{x}) = T\left(\frac{x}{b}\right) = \Delta T\left(1-e^{\frac{b}{d}\tilde{x}}\right)\theta(-x)\ .$$

So the only numeric value we have to use is the dimensionless ratio b/d. Other than that, all the functions can be pre-calculated.

A sanity check: if $T(x) = const$ then σ_{yy} should be zero. According to Eq. (23), the stress will be

$$\sigma_{yy} \propto \int \Psi(x)dx \propto \hat{\Psi}(k=0)$$

Indeed, it is easily seen that $\hat{\Psi}(k=0) = 0$.

(e) Calculate numerically $\sigma_{yy}(x, y=0)$ for three cases: $b \ll D/c$ (very narrow strip), $b \approx D/c$ (intermediate) and $b \gg D/c$ (very wide strip). Is the scale of variation of σ determined by b or by D/c?

Solution

The function $\Psi(x)$ cannot be computed analytically. On the website there's a Mathematica notebook with very detailed explanations about the numerics. I also wrote a MATLAB script that does the numerics, but without explanations. The results are shown in Fig. S1.

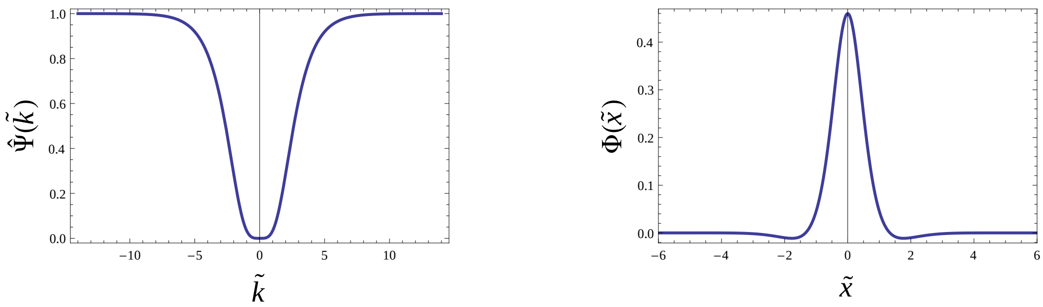

Figure S1: The functions $\hat{\Psi}(\tilde{k})$ and $\Phi(\tilde{x})$ (one can't really plot $\Psi(x)$ as it contains a δ function). Note that the width of $\Psi(\tilde{x})$ is roughly 1, or in other words, the width of $\Psi(x)$ is roughly b. The stress is numerically obtained by substracting $T(x)$ from the convolution $T(x) * \Psi(x)$

To obtain the solution for σ_{yy}, we need to convolve $\Psi(x)$ with $T(x)$. Note that the width of $T(x)$ is d, while the width of $\Psi(x)$ is b. Therefore, for $b \gg d$, $T(x)$ essentially looks like a step-function. So,

$$\sigma_{yy}(x, y=0) = \int_{-\infty}^{\infty} \Psi(y)T(x-y)dy \propto \int_{-\infty}^{x} \Psi(y)dy$$

The numerical calculation of this integral is shown in Fig S2. Note the discontinuity at $x = 0$ which is due to the $\delta(x)$ term in Ψ. For $b \approx d$ and $b \ll d$, one needs to numerically convolve. The result is also shown in Fig S2. Note that the length scale of the stress variation is always b – the width of the kernel – and not d.

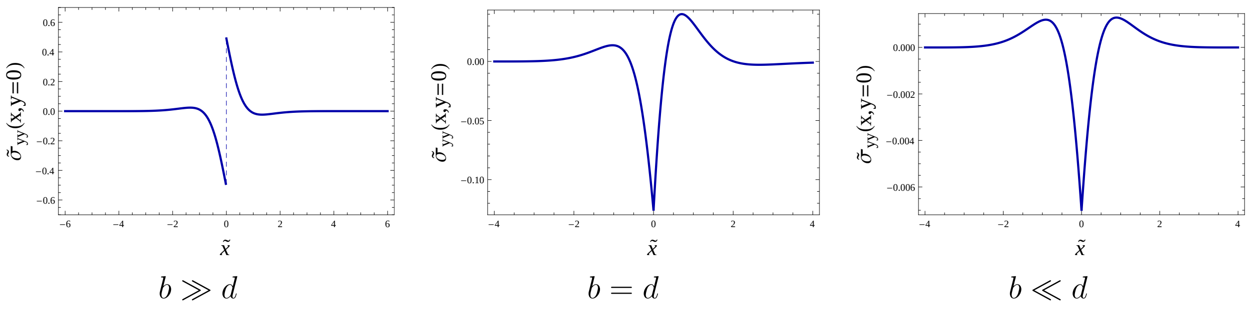

Figure S2: Numerical solution of $\sigma_{yy}(x, y = 0)$ for various conditions. Note that the stress increase as d/b decreases. This is because the stresses are roughly proportional to $\partial_{\tilde{x}\tilde{x}}T$ and when b is small the derivative with respect to $\tilde{x}$ is large.

(f) BONUS: Try to solve (e) by guessing an ansatz of the form $\chi(x, y) = f(y)g(x)$ where g has the same x-dependence as $T(x)$. Plugging it into Eq. (13) should give you a differential equation on $f(y)$ which is solvable. The solution is drastically different from the one you obtained in (e), but is an exact solution of Eq. (13) in the region $x > 0$. How do you resolve this apparent contradiction?