1. Noise

1.1 Characterizing Noise

Noise is a random signal that gets added to all of our measurements. In 1D it looks like this:

![Signal-to-Noise (SNR)](image)

while in 2D it looks like the “snow” on your TV screen:

Noise is unavoidable. It comes from resistive elements in our system: the electronics, and even the patient who has some finite resistance. Microscopically speaking, it is because of the thermal fluctuations of our system: in the electronics and of the spins in the patient.

A noise signal, \( n(t) \), cannot be represented by an analytical function. To characterize noise, we need to speak in statistical terms. The two most important characteristics are the mean (also known as the average) and the variance of the noise. They are denoted by \( <n> \) and \( <n^2> \), respectively, and are illustrated below:

![Signal-to-Noise Ratio (SNR)](image)

In “well behaved” systems, the mean of the noise is 0; it sometimes gets added and sometimes gets subtracted from our signal at random. When \( <n> \) is non-zero the signal is said to be “biased”. Bias is easy to detect and remove, so we won’t focus on it here, and will assume the mean of our noise is 0.

The variance of our noise is basically its “size”. When the variance of the noise becomes as large as the signal being measured, it becomes extremely hard to discern the two. Ideally, we would like to make the variance as small as possible. In practice, we often settle for making it “small enough”; that is, small enough with respect to the signal we’re looking at, so as to make the features that interest us discernable. This chapter will mostly be devoted to ways of making the noise’s variance “small enough”.

1.2 Signal to Noise Ratio (SNR)

As noted before, noise gets added to every signal we measure.

![Signal-to-Noise Ratio (SNR)](image)

The signal to noise ratio of the noisy signal is defined as:

\[
\frac{\text{Signal}}{\text{Noise}} = \frac{\text{Signal}}{\text{Noise}}
\]
If the signal changes considerably from point to point, so will the SNR. In other words, the SNR is a function of position (if we’re dealing with images) or time (if we’re dealing with time signals). For example, take a look at this function:

\[
\text{SNR} = \frac{\text{magnitude of signal}}{\text{variance of noise}}
\]

Now we add some random noise to it:

The SNR of the first few “steps” is much lower than the SNR of the last few “steps”.

A low SNR means the noise is “big” and will make it hard for us to determine the magnitude of the signal accurately, or even see the signal. A high SNR means the noise is “small”.

Measuring the SNR in practice can be tricky. The best thing to do is to find some region in your signal/image where you know there’s only noise, and to use it to estimate your noise’s variance.

1.3 Adding Noise

Suppose you have several noisy, independent random signals, with the same variance, and you add them together. What will be the variance of the sum? Let’s do a little experiment. Here are four random signals:

\[
\begin{align*}
\text{Signal 1} & : 50, 100, 150, 200 \\
\text{Signal 2} & : -2, -1, 1, 2 \\
\text{Signal 3} & : 50, 100, 150, 200 \\
\text{Signal 4} & : -2, -1, 1, 2
\end{align*}
\]

and here is their sum:
You can observe visually that the variance of each individual noisy signal is about 0.5, and the variance of the sum is about 1. So, while we’ve added 4 random signals, we didn’t increase the variance by a factor of 4. In fact, we’ve increased it only by a factor of $\sqrt[4]{4} = 2$. This is a general fact about random signals:

Adding N random signals, each having the same variance, $X$, will yield a random signal with variance $\sqrt{NX}$.

I will not prove this fact in these notes. Intuitively, though, expecting an increase by a factor of N is unreasonable, because the noise sometimes adds constructively and sometimes destructively. This leads to a corollary:

Adding N measurements improves the SNR by a factor of $\sqrt{N}$.

This is because the signal multiplies by N, the noise’s variance by $\sqrt{N}$, and their ratio by $\sqrt{N}$:

$$\text{SNR} = \frac{\text{signal}}{\text{noise}} \rightarrow \frac{N \times \text{signal}}{\sqrt{N} \times \text{noise}} = \sqrt{N} \times \frac{\text{signal}}{\text{noise}}$$

This is sometimes also called signal averaging.

1.4 Noise in MRI

A well known theorem from statistical mechanics (the so-called Nyquist theorem) states that, the variance in the voltage in an electronic system is given by

$$\langle V^2 \rangle = 4kT R \Delta \nu$$

where $k = 1.38 \times 10^{-23}$ Joules/Kelvin is Boltzmann’s constant, $T$ is the temperature of the system (in Kelvin), $R$ is the its resistance (in Ohms) and $\Delta \nu$ is the range of frequencies we’re observing. For a typical 1D MRI experiment, where we acquire in the presence of a gradient $G_{\text{read}}$, $\Delta \nu = \gamma G_{\text{read}} \text{FOV}$:

$$\langle V^2 \rangle = 4kT C B_0^2 \gamma G_{\text{read}} \times \text{FOV}_{\text{read}}$$

What is $R$? There are two sources of resistance in an MRI experiment:

- The RF coils ($R_c$).
- The patient ($R_p$).

Both the coils and the patient are conductors, to a degree. When a magnetic field infringes upon a conductor it dissipates partially as heat. We are basically made out of water, which is a conductor. When a magnetic field tries to penetrate a conductor it creates “eddy currents” as it dissipates slowly. This is known as the skin effect. The currents induced in the patient then induce currents in the coils that are picked up as noise. This is called patient loading. It turns out that for high fields (~1 Tesla and above in practice), patient loading is more important than the intrinsic hardware noise.

Calculating a patient’s resistance is difficult and usually not very constructive, so we will simply treat it as a constant. The only important fact is that it is (approximately) proportional to the square of the main field: $B_0^2$. So

$$R = R_p + R_c = R_p \approx CB_0^2$$

where $C$ is some constant, whose value will be assumed unknown. It can be calculated, but it will not serve our purpose (which is inferring how changing the experimental parameters will affect the SNR). To sum up:

$$\langle V^2 \rangle = 4kTCB_0^2\gamma G_{\text{read}} \times \text{FOV}_{\text{read}}$$
Note I’ve added the “read” subscript, to emphasize that the range of frequencies we observe during acquisition is determined by the read gradient (and not, say, the slice selection or phase-encoding gradients).

1.5 Fourier Transforming Noise

The MRI signal is measured in k-space and consequently Fourier transformed to yield an image. The Fourier transform of noise is just ... more noise.

Don’t forget our FT is discrete: it’s carried over a finite number of points. Because every point in the original (k-space) function affects every point in the Fourier (image) space, this means the noise at some point \( r \) in our image is added up from all points in k-space. If we have a total of \( N \) points in k-space, then the variance of the noise in image space will increase as \( \sqrt{N} \). However, the discrete Fourier transform also has a factor of \( 1/N \) in its definition. Without going into the technical details, here is the bottom line that’s relevant for us:

Fourier transforming noise over a discrete set having \( N \) points decreases its variance by \( \sqrt{N} \).

This works in 2D and 3D as well. For a 2D grid having \( N_x \) points along the \( k_x \) axis and \( N_y \) points along the \( k_y \) axis, the noise’s variance will decrease by a factor \( \sqrt{N_xN_y} \).

2. SNR in MRI

2.1 Relative SNR

Being able to calculate the absolute SNR might be interesting theoretically but quite formidable, because the noise will ultimately depend on the hardware, patient, and image. Rather, one looks at how the SNR changes as we change the imaging parameters: the resolution, voxel size, FOV, acquisition time, gradients, \( B_0 \) (not really a parameter, but still interesting), etc. This is precisely what this section is all about.

2.2 Signal

We’ve spent the last section talking about noise, but haven’t said anything about signals. We’ve remarked how the finite sampling in k-space \( (k_{max}) \) causes blurring in the signal, so the signal at point \( r \) is actually an average of the signal in a voxel around it:

\[
I(r) = \omega_0 \int_{\text{voxel centered around } r} M_0(r')dr' \approx \omega_0 M_0(r) \Delta V
\]

where \( \Delta V \) is the size of the voxel. Remember the \( \omega_0 \) in front is there because of Faraday’s law (the signal is proportional to the time derivative of the magnetization, which is proportional to \( \omega_0 \). See chapter 2, section 3.2).

2.2 SNR in 3D Imaging

Here is a 3D GRE sequence:

![3D GRE sequence diagram]

Suppose we’ve:
1. Collected \( N_x, N_y, \) and \( N_z \) points along the 
   \( k_x, k_y, k_z \) axes.
2. Have a voxel size \( \Delta V = \Delta x \Delta y \Delta z \) (note the voxel 
   doesn’t have to be a cube, i.e., \( \Delta x \) isn’t necessarily 
   equal to \( \Delta y \) or \( \Delta z \), etc).
3. Because we’re reading along the \( x \)-axis, 
   we have a bandwidth of \( \Delta v = \gamma G_x \text{FOV}_x \).
4. The readout time along \( x \) is \( T_x \) (see 
   sequence above).
5. We acquire the same image \( N_{\text{acq}} \) times 
   (for signal averaging).

Then

\[
\text{SNR}(r) = \frac{\text{signal}(r)}{\text{noise}} \propto \frac{M_0(r) \Delta V \sqrt{N_{\text{acq}}}}{\sqrt{N_x N_y N_z}}
\]

Using

\[
T_x = N_x \delta t = N_x \delta k \gamma G_x \frac{N_x \gamma k \text{FOV}_x}{N_x \Delta V} = \frac{N_x}{\Delta V}
\]

we get

\[
\text{SNR}(r) \propto M_0(r) \Delta V \sqrt{N_{y_{\text{acq}}}}
\]

The SNR of a 3D scan (assuming read 
   is along the \( x \)-axis)

This is also valid for 3D spin-echo imaging.

2.3 SNR IN 2D IMAGING

2D imaging is just like 3D imaging, with one 
   omission: you don’t sample along the \( 3^{\text{rd}} \) 
   dimension. Rather, you use a slice-selective 
   gradient. Let’s take the slice-selective direction 
   to be \( z \), with \( x \) & \( y \) being the read & phase axes, 
   respectively. This means that we need to omit \( N_z \) 
   from the above expression, because we’re not 
   Fourier-transforming over that direction.

Furthermore, note that \( \Delta V \), the voxel’s volume, 
   has a thickness equal to the slice’s thickness along 
   the \( z \)-axis (let’s call it naturally \( \Delta z \)):

\[
\Delta z
\]

Thus:

\[
\text{SNR}_{2D}(r) \propto M_0(r) \Delta V \sqrt{N_y N_{\text{acq}}} T_x = \frac{\text{SNR}_{3D}}{N_z}
\]

In terms of SNR, 3D imaging is superior to 2D 
   imaging. Should we always use 3D? That depends. 
3D often requires more scans to achieve the same 
   slice thickness, to avoid aliasing (you need smaller 
   \( \Delta k \)'s to cover the entire FOV along the \( 3^{\text{rd}} \) axis; in 
2D imaging, you don’t care about aliasing because 
you’re selectively exciting a slice and imaging all of 
   it). Another problem is “ringing” artifacts having 
   to do with the Fourier reconstruction. As a rule of 
thumb, for “thick” slices (a few mm and above), 
you should go for 2D imaging. For “thin” ones (~ 
   mm and thinner), go for 3D.

There are also other, phenomenon-specific reasons for going 2D; 
in MR “time-of-flight” angiography sequences, for example, 
slowly flowing blood yields better contrast in slice- 
   selective, rather than 3D, imaging.

2.4 SNR DEPENDENCE ON \( B_0 \)

One thing the above equations don’t show is the 
   dependence of the SNR on \( B_0 \). Recall that, for 
   high fields (which interest us),

\[
\text{noise} \sim \sqrt{\langle V^2 \rangle} \sim \sqrt{B_0^2} = B_0
\]

while

\[
\text{signal} \sim \omega_0^2
\]

(one \( \omega_0 \) comes from \( M_0 \), the other comes from the 
   Faraday’s law: the signal we acquire is proportional 
to the time derivative of \( M_{xy} = e^{\omega t} \)). Hence:

\[
\text{SNR}(B_0) \sim \frac{B_0^2}{B_0} = B_0
\]

The SNR should increase linearly with the field. In 
   practice, this is only approximate.