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Mathematical Preliminaries -

A general note

The purpose of the HW exercises is to give you hands-on experience with the course material. We try hard to ask questions that requite a conceptual process of understanding, rather than technical computation. Whenever some complicated calculations are required, please remember that it is only in order to convey the mathematical structure of the physical problems that we tackle, a structure that might elude the "passive listener" in the classroom. Accordingly, in the answers you hand in we do not require detailed calculations, unless they are crucial for the understanding.

Tensor integration - Archimedes law

1. Fluids exert forces on bodies that are submerged in them. At each point on the body's surface, denote the local normal by $\hat{\boldsymbol{n}}$. The force per unit area exerted by the fluid is given by $f_i = \sigma_{ij} n_j$. σ_{ij} is called the *stress tensor* of the fluid, and we'll deal with it extensively in the course.

   Consider a stationary (hydro-static), isotropic fluid that occupies the bottom half-space $z < 0$. The fluid is subject to a constant gravitational field $-g\hat{\boldsymbol{z}}$. At $z = 0$, we have $\sigma_{ij} = 0$.

   (a) The off-diagonal elements of σ_{ij} are called *shear stresses*. Almost by definition, in a stationary fluid the shear stresses must vanish. Therefore, for $i \neq j$ we must have $\sigma_{ij} = 0$ *for every choice of coordinate system*. Prove that this implies $\sigma_{ij} = -p(\boldsymbol{r})\,\delta_{ij}$, where $p(\boldsymbol{r})$ is a scalar field. Note: $p = -\frac{1}{3}\,\mathrm{tr}(\sigma)$ is called *pressure*.

   (b) By considering the force balance on a small cube of fluid and the translational symmetries of the system, show that the stress field satisfies the equation

   $$\partial_z \sigma_{zz}(x, y, z) = -\rho g$$

   where ρ is the fluid's density. Together with the results of (1a), conclude that the stress tensor is given by $\sigma_{ij} = -\rho g z \delta_{ij}$.

   (c) Consider a body of arbitrary shape and of volume V, which is submerged in the fluid. Calculate the magnitude and direction of the total force exerted by the fluid on the body by integrating $\sigma_{ij} n_j$ over the surface of the body. This force is called the *Buoyancy force*.

   (d) Imagine that we remove the body, so the volume that was previously occupied by it is now filled with fluid. Give a one-line derivation of the result you obtained in (1c) by analyzing this new setting (hint: the situation is *static*).

   (e) *Bonus*: Stand up and shout out loud "Eurika!!" (*Note: only filmed evidence will be considered for bonus purposes*).

Tensor differentiation - Stress tensor of a fluid

2. In the TA session we derived the Navier-Stokes equation

$$\rho\left(\partial_t + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = \vec{\nabla}P + \eta\nabla^2\vec{v} + \mu\vec{\nabla}\left(\nabla\cdot\vec{v}\right)$$

In class, we derived (or will derive next week) the formula for the material derivative

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \vec{v}\cdot\nabla A\ ,$$

where A is a tensor. So we see that the left-hand-side of the Navier-Stokes equation is nothing but the material derivative of the velocity field, that is, *inertia*. Find a tensor $\boldsymbol{\sigma}$ such that its divergence is the (minus of) right-hand-side of the NS equation. Thus the equation will take the form of a continuity equation

$$\rho\frac{D\vec{v}}{Dt} + \operatorname{div}\boldsymbol{\sigma} = 0\ .$$

Like in Question 1, the tensor $\boldsymbol{\sigma}$ is called the stress tensor, and its divergence is the total force exerted on a fluid element. Thus, the NS equation is nothing but a fancy way to write $F = ma$.

Bonus: There is more than one tensor whose divergence is the RHS of the NS equation, that is, the solution to this question is not unique. However, there's an extra physical requirement, which is that $\boldsymbol{\sigma}$ will be symmetric $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. Can you find a symmetric $\boldsymbol{\sigma}$?

Symmetries

3. In the TA session we've shown that in 3 dimensions a 2^{nd} rank isotropic tensor must be proportional to δ_{ij}, (in fact, this is true for all dimensions ≥ 3). However, in 2D this does not hold. Find the general form of an isotropic two-dimensional 2^{nd} rank tensor. What kind of symmetry do these tensors violate (those not proportional to the identity)?

Bonus: Can you think of an example of an isotropic 2D tensor that is not diagonal, from a real physical system?

4. **Invariants**: A scalar function of a tensor $f(\boldsymbol{A}) = f(A_{ij})$ or of a vector $g(\vec{v}) = g(v_i)$ is called invariant if its value is independent of the choice of basis. That is, if it has a proper geometric meaning which is independent of the particular basis that one happens to choose. Later in this course, we will be interested in scalar invariants of tensors. For example, the elastic energy is a scalar invariant of the strain tensor.

(a) Show that the trace is the only linear invariant scalar of a 2^{nd} rank tensor $\boldsymbol{A}$. That is, show that if $f(\boldsymbol{A})$ is an invariant function that is linear in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda \operatorname{tr} \boldsymbol{A}$ for some constant λ. Assume the dimension is ≥ 3.

(b) Show that the only quadratic invariants of a 2^{nd} rank tensor $\boldsymbol{A}$ are $\operatorname{tr}(\boldsymbol{A}^2)$, $(\operatorname{tr} \boldsymbol{A})^2$, and $\operatorname{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$. That is, show that if $f(\boldsymbol{A})$ is invariant and quadratic in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda_1 \operatorname{tr}(\boldsymbol{A}^2) + \lambda_2 (\operatorname{tr} \boldsymbol{A})^2 + \lambda_3 \operatorname{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$.

hint for the last two questions: Think about the discussion in the TA session regarding isotropic tensors, i.e. Section 2 in the TA lecture notes.

Angular velocity

5. In this short and nice exercise, we'll prove a theorem that you used without knowing it in your undergrad: we'll show that a rigid body that rotates has a well-defined angular velocity. That is, we'll show that a the velocity of a particle located at $\vec{r}_i$ can be written as

$$\vec{v}_i(t) = \frac{\partial \vec{r}_i(t)}{\partial t} = \vec{\omega} \times \vec{r}_i(t) \tag{1}$$

for some time-dependent vector $\vec{\omega}(t)$. (If you think this is trivial, try proving it without reading further!)

The meaning of the assumption that a body is rigid is that it can only move by global rotations (and translations, which we'll assume are not present). Therefore, the trajectory of the i-th particle must be given by

$$\vec{r}_i(t) = \boldsymbol{R}(t)\vec{r}_i(0) \ , \tag{2}$$

where $\vec{r}_i(0)$ is the particle's position at $t = 0$, and $\boldsymbol{R}(t)$ is a (time-dependent) rotation matrix.

(a) Show that every anti-symmetric matrix can be written as $\mathcal{E}_{ijk}\,\omega_k$ for some vector $\vec{\omega}$ ($\boldsymbol{\mathcal{E}}$ is the Levi-Civita tensor defined in the TA session).

(b) Show that $\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T$ is anti-symmetric at all t's (hint: orthogonality).

(c) Write $\vec{v} = \dot{\vec{r}}$ as a function of $\vec{r}(t)$ (and *not* as a function of $\vec{r}(0)$!!).

(d) Finish it off. Show that $\vec{v}(t) = \omega(t) \times \vec{r}(t)$ for some vector $\vec{\omega}$.