Kinematics

1. Consider the following 2D deformation:

$$x_1(t) = \cosh(t)X_1 + \sinh(t)X_2 \ , \qquad x_2(t) = \sinh(t)X_1 + \cosh(t)X_2 \ .$$

   (a) Find the material velocity and the acceleration $\boldsymbol{V}, \boldsymbol{A}$ and express their spatial forms $\boldsymbol{v}, \boldsymbol{a}$. Remember to represent each field in the proper coordinates (i.e. $\boldsymbol{V}, \boldsymbol{A}$ in terms of $\boldsymbol{X}$ and $\boldsymbol{v}, \boldsymbol{a}$ in terms of $\boldsymbol{x}$). Plot schematically $\boldsymbol{V}$ and $\boldsymbol{v}$ at $t = -10, 0, 10$. Note how vastly different $\boldsymbol{V}$ and $\boldsymbol{v}$ are!

   (b) The acceleration $\boldsymbol{a}$ can also be calculated as a material derivative of the velocity:

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} \ .$$

   Calculate $\boldsymbol{a}$ using this expression, and show that the results coincide.

   (c) Calculate $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$ and $J = \det \boldsymbol{F}$ (we will use it in Q4).

2. Solve these apparent contradictions:

   (a) One may claim that $\nabla_{\boldsymbol{x}} \boldsymbol{v} \equiv 0$ because

$$\nabla_{\boldsymbol{x}} \boldsymbol{v} = \nabla_{\boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial}{\partial x_j} \frac{\partial x_i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial x_i}{\partial x_j} = \frac{\partial \delta_{ij}}{\partial t} = 0$$

   Is this true (hint: no)? What is wrong with this reasoning?

   (b) In Eq. (25) of TA session #2 we used the fact that $D_t \boldsymbol{x} = \boldsymbol{v}$ (there we denoted $\boldsymbol{x}$ by $\boldsymbol{r}$). One (say, Avraham Moriel) may claim that there's a factor of 2 missing, since

$$D_t \boldsymbol{x} \equiv \partial_t \boldsymbol{x} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{x} = \boldsymbol{v} + \boldsymbol{v} \boldsymbol{I} = 2\boldsymbol{v}$$

   Is this true (hint: no)? What is wrong with this reasoning?

3. We use quite freely in class $\boldsymbol{F}^{-1}$ and $\boldsymbol{F}^{-T}$ and so on. What is the physical meaning of the assumption that $\boldsymbol{F}$ is always an invertible matrix?

4. The purpose of this exercise is to prove the relation $\partial_t J = J \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}$, and in the meanwhile to get a better intuition about how tensorial derivatives work. This relation was used in class in deriving the mass continuity equation (Eq.(4.4) in the lecture notes).

If $\Phi(\boldsymbol{A})$ is a scalar function of a tensor, then its linear variation with respect to $\boldsymbol{A}$ is

$$\Phi(\boldsymbol{A}+d\boldsymbol{A}) = \Phi(\boldsymbol{A}) + d\Phi, \quad d\Phi = \frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} : d\boldsymbol{A} + O(d\boldsymbol{A}^2) \ ,$$

and the tensor $\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}$ is called the tensorial derivative.

Note that after a basis is chosen, the entries of $\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}$ are given by

$$\left(\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}\right)_{ij} = \frac{\partial \Phi}{\partial A_{ij}} \ .$$

That is, if Φ is thought of as a function of $A_{11}, A_{12}, \ldots,$ then the tensor $\frac{\partial \Phi}{\partial \boldsymbol{A}}$ is given, entry-wise, by the partial derivatives of Φ with respect to its arguments. Remember the definition $\boldsymbol{B} : \boldsymbol{C} \equiv \mathrm{tr}(\boldsymbol{B}\boldsymbol{C}^T)$. You may convince yourself that $\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}$ is indeed a tensor (i.e. that under a different choice of coordinates, the entries of $\frac{\partial \Phi}{\partial \boldsymbol{A}}$ transform as they should).

(a) Now choose $\Phi = \det$, and show that for invertible $\boldsymbol{A}$,

$$\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \det(\boldsymbol{A})\boldsymbol{A}^{-T}$$

where $\boldsymbol{A}^{-T}$ denotes the inverse of the transpose (or the transpose of the inverse - they're the same). Hints: (a) Start by writing $\boldsymbol{A}+d\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{I}+\boldsymbol{A}^{-1}d\boldsymbol{A})$. (b) Keep only the part of $\det(\boldsymbol{A}+d\boldsymbol{A})$ which is linear in $d\boldsymbol{A}$.

(b) Show that if $\boldsymbol{A}$ is a function of t, then

$$\frac{\partial}{\partial t}\Phi(\boldsymbol{A}(t)) = \frac{\partial \Phi}{\partial \boldsymbol{A}} : \partial_t \boldsymbol{A}$$

Up to now, these were general algebraic identities. Let's get down to business and look at a motion of a deformed body $\boldsymbol{x}(\boldsymbol{X})$, its deformation gradient $\boldsymbol{F}(\boldsymbol{X},t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$, the Jacobian $J(\boldsymbol{X},t) = \det \boldsymbol{F}(\boldsymbol{X},t)$ and the velocity field $\boldsymbol{v} = \frac{\partial \boldsymbol{x}}{\partial t}$.

(c) Show that

$$\frac{\partial \boldsymbol{F}}{\partial t} = \nabla_{\boldsymbol{X}} \boldsymbol{v} = \nabla_{\boldsymbol{x}} \boldsymbol{v}\, \boldsymbol{F} \ .$$

Note that this can be easily transformed to obtain Eq. (3.31) in the lecture notes.

(d) Use the results of (a)-(e) to prove the desired relation:

$$\partial_t J = J \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = J \,\mathrm{div}_{\boldsymbol{x}}\, \boldsymbol{v} \tag{1}$$

(You might want to remind yourself that tr(grad($\cdot$))=div($\cdot$)). Conclude that if a motion is volume preserving then $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0$.

(e) Verify this relation by calculation of $\partial_t J$ for the motion described in Question 1, first by calculating $\partial_t J$ from the formula (1) and then by differentiating the result of 1(c).

5. Consider a material that fills the whole space, except for a spherical cavity of initial radius Q, centered at the origin. At time $t = 0$ an explosive is detonated in the cavity and its radius varies as some specified function $q(t)$, resulting in a sphero-symmetric motion. That is, the motion is given by

$$\begin{aligned}
\boldsymbol{x}(t) &= \frac{r(t)}{R}\boldsymbol{X} = \frac{f(R,t)}{R}\boldsymbol{X} \\
r(t) &= f(R,t) = |\boldsymbol{x}(R,t)| \\
R(\boldsymbol{X}) &= |\boldsymbol{X}| \\
f(R = Q, t) &= q(t)
\end{aligned}$$

(a) Show that the deformation gradient is given by

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{x} = \frac{\partial f}{\partial R}\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \frac{f}{R}(\hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}) \ , \tag{2}$$

where $\hat{\boldsymbol{r}} = R^{-1}\boldsymbol{X} = r^{-1}\boldsymbol{x}$, and $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are the spherical unit vectors.
Hints:

- For a spherically symmetric function $g(r)$, $\nabla_{\boldsymbol{X}} g = \frac{\partial g}{\partial R}\hat{\boldsymbol{r}}$.
- $\boldsymbol{I} = \sum_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i$ for any set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of orthonormal vectors.

(b) If the motion is isochoric (volume-preserving), show that

$$f(R,t) = \sqrt[3]{R^3 + q(t)^3 - Q^3} \ .$$

You can show that either by using Eq.(2) to calculate the volume change, or by direct computation without going knowing the explicit form of $\boldsymbol{F}$ (doing both is better!).

(c) Calculate $\boldsymbol{v}$, expressed in terms of q and $\partial_t q(t)$.