Non-Equilibrium Continuum Physics **HW set #3**

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Elasticity and Thermo-Elasticity

Note

This HW set, especially the last question, is purposely longer and more difficult than the previous ones. It summarizes the first part of the course, and demands a working knowledge of linear elasticity. It also involves a small numeric calculation. We consider it a "semi-mid-term", and it will be given a larger weight in the final grade than the other sets. You are also given a whole month to complete it, so please start early and take it seriously.

We believe that if you successfully solve these problems then we have succeeded in teaching the first part of the course.

1. (8 pts) Consider a 3D rectangular box, subject to uniaxial stress σ_0 in the z direction, as shown in Fig. 1. The faces in the x, y directions are traction-free. The rest-lengths of the boxes sides are a, b. Calculate the slope of the dashed line as a function of σ_0, E and ν. When $\sigma_0 \geq E$ something weird happens. Is linear elasticity wrong?

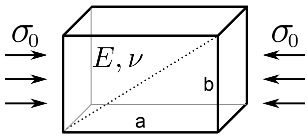

Figure 1: Box under uniaxial compression.

2. (8 pts) A rod of radius R and length L is pointed along the z direction, and is held between two rigid walls at $z = 0$ and $z = L$. It is free of constraints in the other two dimensions. The rod is then uniformly heated by some amount ΔT. Without solving the problem completely (although you may do it, if you have nothing better to do) estimate how the

   (a) stresses in the rod,

   (b) strains in the rod,

   (c) forces on the walls,

   (d) elastic energy stored in the rod

   scale with R, L and ΔT.

3. (8 pts) Consider an infinite 2D material, from which a circular hole is taken out. The material is now heated by some amount. Will the hole shrink or expand?

4. (12 pts) Consider a static infinite 3D material with a given arbitrary distribution of temperature $T(x, y, z)$, that decays at infinity: $T(\vec{r}) \to T_\infty$, as $|\vec{r}| \to \infty$. Before reading further it might by nice to try to estimate: if the temperature variation is localized, how does the displacement field decay at large $\boldsymbol{r}$? And the strain field?

   Here's a nice way to gain intuition as to what temperature gradients do in thermo-elasticity: Prove that the displacement field is curl-free, i.e. is of the form $\vec{u} = \nabla\phi$, and that ϕ satisfies Poisson's equation $\nabla^2\phi = T$. Assuming you already have some intuition about electrostatics, this should help you gain intuition about thermo-elasticity.

   Guidance: Begin with Navier-Lamé equation $(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = K\alpha\nabla T$. You can guess the correct form for u, and if it works then you're done because the solution is unique. Some vector-analysis identities might prove useful.

5. (64 pts) In 1993, Yuse & Sano published a remarkable paper regarding instabilities of thermally induced fracture (Yuse & Sano, Nature (362) 1993). They consider a strip of material which is pulled out of an oven **at a constant velocity** and cools down as it moves. The gradients of the temperature field induce fracture, as is seen in Figure 2.

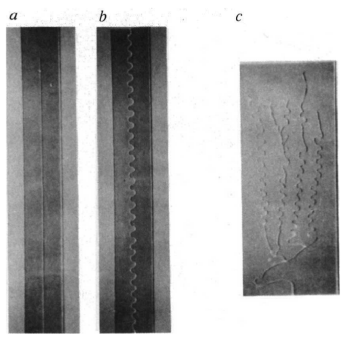

FIG. 2 Typical examples of three different types of cracks: *a*, a straight crack, *b*, oscillating crack, *c*, branched crack. Sample size (*a* and *b*) $10 \times 100 \times 0.11$ mm^3, (*c*) $24 \times 60 \times 0.13$ mm^3.

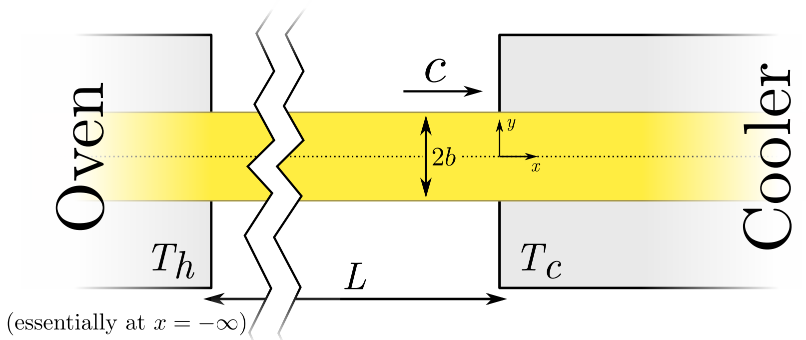

Figure 2: Left: Thermally induced fracture (from the paper). Right: the simplified model. Note the position of the origin of axes, and that the plot **is not to scale**: we assume $L \gg b$.

To model the phenomenon, consider an infinite (in the x direction) 2D strip of width $2b$. The strip is subject to a y-independent temperature distribution $T(x)$, and is free of tractions at its boundaries $y = \pm b$. Fracture will be considered later in this course. For now, we'll limit ourselves to finding an expression for the stretching component σ_{yy} along the strip's symmetry axis $y = 0$. This is the driving force that induces fracture.

(a) We begin by finding the temperature distribution. Write the heat diffusion equation in both the material ($\boldsymbol{X}$) and laboratory ($\boldsymbol{x}$) coordinates, and solve it in the laboratory coordinates for our problem. Assume that the cooler and the oven are strong enough such that $T(x>0)=T_c$, and $T(x<-L)=T_h$. Assume also that L is much larger than any other length scale of the system. The heat diffusion constant D is of course given. Remember that you can always shift T by a global constant to get a simpler expression.

(b) Show that the equations of plane-stress combined with static thermo-elasticity are

$$\begin{aligned}\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] + \frac{1}{3}\alpha_T \Delta T,\\ \varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] + \frac{1}{3}\alpha_T \Delta T,\\ \varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}\end{aligned} \tag{1}$$

Guidance: Start with the known Hooke's law derived in class, $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, T) = \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon} - \alpha K T \boldsymbol{I}$, invert it to the compliance form $\boldsymbol{\varepsilon}(\boldsymbol{\sigma}, T)$, and use the relations between λ, μ, K to E, ν (which are summarized in a nice table in Wikipedia).

(c) Prove the compatibility relation

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}\ , \tag{2}$$

and use it together with the definition of the Airy potential χ and Eqs. (1) to show that χ satisfies the equation

$$\nabla^2\nabla^2\chi = -\frac{1}{3}E\alpha_T\nabla^2 T \tag{3}$$

What is the symmetry of χ with respect to y? What are the boundary conditions that χ satisfies?

(d) Solve Eq. (3) by Fourier transforming it in the x direction and imposing the boundary conditions. Express $\sigma_{yy}(x, y=0)$ in an integral form. You should obtain an expression of the form

$$\sigma_{yy}(x, y=0) = \int_{-\infty}^{\infty} T(x')\Psi(x-x')dx' \equiv T * \Psi\ , \tag{4}$$

where $*$ denotes convolution, and $\Psi(x)$ is the convolution kernel, for which you should have a closed expression (as an integral of something).

(e) Calculate numerically $\sigma_{yy}(x, y=0)$ for three cases: $b \ll D/c$ (very narrow strip), $b \approx D/c$ (intermediate) and $b \gg D/c$ (very wide strip). Is the scale of variation of σ determined by b or by D/c?

(f) BONUS: Try to solve (e) by guessing an ansatz of the form $\chi(x,y) = f(y)g(x)$ where g has the same x-dependence as $T(x)$. Plugging it into Eq. (3) should give you a differential equation on $f(y)$ which is solvable. The solution is drastically different from the one you obtained in (e), but is an exact solution of Eq. (3) in the region $x > 0$. How do you resolve this apparent contradiction?