Non-Equilibrium Continuum Physics **TA session #6**

TA: Yohai Bar Sinai 20.04.2016

Linear Elasticity II - Waves and a simple composite material

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1 Elastic waves

I remind you that you have shown in class that the Navier-Lamé Equation ,

$$(\lambda+\mu)\nabla\left(\nabla\cdot\boldsymbol{u}\right)+\mu\nabla^2\boldsymbol{u}+\boldsymbol{b}=\rho\partial_{tt}\boldsymbol{u}\ , \tag{1}$$

is basically two uncoupled wave equations for dilatational and shear waves. They propagate at velocities

$$c_s=\sqrt{\frac{\mu}{\rho}}\ ,\qquad\qquad c_d=\sqrt{\frac{\lambda+2\mu}{\rho}}\ . \tag{2}$$

Thus, Eq. (1) can also be written as

$$(c_d^2-c_s^2)\nabla\left(\nabla\cdot\boldsymbol{u}\right)+c_s^2\nabla^2\boldsymbol{u}=\partial_{tt}\boldsymbol{u} \tag{3}$$

The two wave speeds differ by a significant factor. c_s is always smaller than c_d and their ratio is

$$\beta\equiv\frac{c_s}{c_d}=\sqrt{\frac{\mu}{\lambda+2\mu}}=\sqrt{\frac{1-2\nu}{2(1-\nu)}}\ . \tag{4}$$

For a typical value of $\nu=1/3$, this gives a ratio of $\frac{1}{2}$. This function is plotted in Fig. 1. Note that the ratio goes to 0 for $\nu\to\frac{1}{2}$. This is because incompressible materials ($\nu=\frac{1}{2}$) the dilatational velocity c_d diverges (as the bulk modulus K diverges). Seismographers use the difference in propagation velocity to determine the distance to an earthquake source, as is seen in Fig. 1.

1.1 Leftovers from Eran's lecture

In class, you have discussed the polarization of these two waves by writing

$$\boldsymbol{u}=g(\boldsymbol{x}\cdot\boldsymbol{n}-c\,t)\boldsymbol{a}\ , \tag{5}$$

where $\boldsymbol{n}$ is the propagation direction, $\boldsymbol{a}$ is the direction of the displacement and $|\boldsymbol{n}|=|\boldsymbol{a}|=1$. You have shown without proof that this implies

$$(c_d^2-c_s^2)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}+(c_s^2-c^2)\boldsymbol{a}=0\ . \tag{6}$$

Eran promised that I will show how to get from the former to the latter. This is done simply by applying the differential operators to $\boldsymbol{u}$ giving

$$\nabla g=\partial_i g=g'n_i=g'\boldsymbol{n}\ , \tag{7}$$

$$\nabla\boldsymbol{u}=\partial_j u_i=\partial_j(ga_i)=g'n_ja_i=g'\boldsymbol{a}\otimes\boldsymbol{n}\ , \tag{8}$$

$$\nabla\cdot\boldsymbol{u}=\mathrm{tr}\left(\nabla\boldsymbol{u}\right)=g'\boldsymbol{a}\cdot\boldsymbol{n}\ , \tag{9}$$

$$\nabla\left(\nabla\cdot\boldsymbol{u}\right)=\partial_i(g'\boldsymbol{a}\cdot\boldsymbol{n})=g''(\boldsymbol{a}\cdot\boldsymbol{n})n_i=g''(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}\ , \tag{10}$$

$$\nabla^2\boldsymbol{u}=\nabla\cdot\nabla\boldsymbol{u}=\partial_j(g'n_ja_i)=g''n_jn_ja_i=g''\boldsymbol{a}\ , \tag{11}$$

$$\partial_{tt}\boldsymbol{u}=c^2g''\boldsymbol{a}\ . \tag{12}$$

Plugging Eqs. (10)-(12) into (3) gives immediately Eq. (6).

The two waves are independent in the bulk. However, on the boundary of a body the traction-free condition $\sigma_{ij}n_j = 0$ couples between the two modes, and more modes arise with a distinct propagation velocity. These are called Rayleigh waves, and are very interesting.

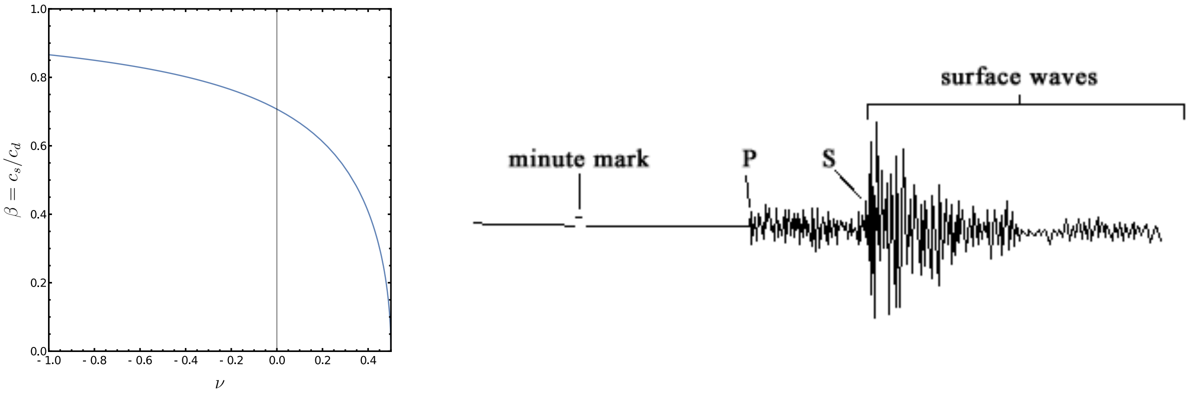


Figure 1: Left: c_s/c_d as a function of Poisson's ratio (Eq. (4)). Right: Seismograph reading of an earthquake. One can clearly see a P-wave (longitudinal) and an S-wave (transverse) arriving at different times. Later, surface waves are visible. The time difference can be used to obtain the distance from the earthquake source.

1.2 Rayleigh waves

So let's see exactly how this works. We want to look at surface waves which propagate, say, in the x-direction. To this end, consider a material that fills the lower half-space $z < 0$, and assume that

$$\boldsymbol{u} = \boldsymbol{f}(z)e^{ikx-i\omega t} \ . \tag{13}$$

If $\boldsymbol{u}$ satisfies the wave equation $(\frac{1}{c_i^2}\partial_{tt} - \nabla^2)\boldsymbol{u} = 0$, with $c_i = c_s$ or c_d, we have

$$\partial_{zz}\boldsymbol{f} = \left(k^2 - \frac{\omega^2}{c^2}\right)\boldsymbol{f}$$

If $k^2 > \frac{\omega^2}{c^2}$ this gives a damped wave in the bulk. We denote

$$\boldsymbol{f}(z) = \overrightarrow{\text{const}} \times e^{\eta_i z}, \qquad \eta_i = \sqrt{k^2 - \frac{\omega^2}{c_i^2}}, \quad i = s, d \ \ .$$

As stated above, Rayleigh waves are modes which mix dilational and shear waves. We therefore guess the ansatz

$$\boldsymbol{u} = \boldsymbol{u}^{(d)} + \boldsymbol{u}^{(s)} \ , \tag{14}$$

$$\boldsymbol{u}^{(i)} = \left(u_x^{(i)}\hat{\boldsymbol{x}} + u_z^{(i)}\hat{\boldsymbol{z}}\right) e^{\eta_i z+ikx-i\omega t} \ , \tag{15}$$

where $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ are dilatational and shear waves, and $u_j^{(i)}$ are constants. That is, each of $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ satisfies its own wave equation,

$$(\partial_{tt} - c_s^2 \nabla^2)\boldsymbol{u}^{(s)} = 0 \qquad (\partial_{tt} - c_d^2 \nabla^2)\boldsymbol{u}^{(d)} = 0 \ . \tag{16}$$

They both oscillate with the same frequency ω (the ω of Eq. (13)). Of course, the $\boldsymbol{u}^{(i)}$ are not exactly bulk modes, because they decay exponentially with z, each over over a different length-scale η_i.

Following the discussion about the different polarizations of the different types of waves, note that we should demand

$$\vec{\nabla} \cdot \boldsymbol{u}^{(s)} = \vec{\nabla} \times \boldsymbol{u}^{(d)} = 0 \ . \tag{17}$$

Plugging the ansatz into equation (17) yields

$$\frac{\partial u_x^{(s)}}{\partial x} + \frac{\partial u_z^{(s)}}{\partial z} = \left(iku_x^{(s)} + \eta_s u_z^{(s)}\right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{u_z^{(s)}}{u_x^{(s)}} = -i\frac{k}{\eta_s} \ , \tag{18}$$

$$\frac{\partial u_x^{(d)}}{\partial z} - \frac{\partial u_z^{(d)}}{\partial x} = \left(\eta_d u_x^{(d)} - iku_z^{(d)}\right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{u_z^{(d)}}{u_x^{(d)}} = -i\frac{\eta_d}{k} \ . \tag{19}$$

So we write

$$\boldsymbol{u}^{(s)} = A\left(\eta_s \hat{\boldsymbol{x}} - ik\hat{\boldsymbol{z}}\right) e^{\eta_s z + ikx - i\omega t} \qquad A \in \mathbb{C} \ , \tag{20}$$

$$\boldsymbol{u}^{(d)} = B\left(ik\hat{\boldsymbol{x}} + \eta_d \hat{\boldsymbol{z}}\right) e^{\eta_d z + ikx - i\omega t} \qquad B \in \mathbb{C} \ , \tag{21}$$

We now want to demand that the boundary is traction-free. That is, we want to impose $\sigma_{ij}\big|_{z=0} n_j = 0$, where n_j is the local normal to the deformed surface. In principle, $\hat{n}$ also changes because the surface deforms. However, since $\boldsymbol{\sigma}$ is already first-order in the deformation, we are allowed to take the zeroth order of $\hat{n}$, that is, we can take $\hat{n} = \hat{z}$. Therefore, imposing the traction-free boundary conditions means $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ on $z = 0$. This translates via Hooke's law to

$$\sigma_{xz} = 2\mu\epsilon_{xz} \qquad \propto \qquad \partial_z u_x + \partial_x u_z = 0 \tag{22}$$

$$\begin{aligned} \sigma_{zz} &= (2\mu + \lambda)\epsilon_{zz} + \lambda\epsilon_{xx} = (2\mu + \lambda)\partial_z u_z + \lambda\partial_x u_x \\ &= \rho^{-1}\left[c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x\right] = 0 \end{aligned} \tag{23}$$

We now plug Eqs. (20)-(21) into (22)-(23). This is some uninteresting but necessary algebra. Eq. (22) is relatively simple:

$$0 = \partial_z u_x + \partial_x u_z = \Big(\eta_s^2 A + i\eta_d k B\Big) + \Big(k^2 A + i\eta_d k B\Big) = \Big(\eta_s^2 + k^2\Big)A + 2i\eta_d k B \tag{24}$$

Eq. (23) requires some simplification on order to be sensible:

$$0 = c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x \tag{25}$$

$$= c_d^2\Big(- ik\eta_s A + \eta_d^2 B\Big) - \left(c_d^2 - 2c_s^2\right)\Big(ik\eta_s A - k^2 B\Big) \tag{26}$$

$$= -2ik\eta_s A + \left(\frac{c_d^2}{c_s^2}\left(\eta_d^2 - k^2\right) + 2k^2\right) B \tag{27}$$

But since $(k^2-\eta_i^2)c_i^2=\omega^2$ is the same for both i's, we can replace $\left(\eta_d^2-k^2\right)c_d^2$ in the last equation by $\left(\eta_s^2-k^2\right)c_s^2$ and get

$$-2ik\eta_s A+\left(\eta_s^2+k^2\right)B \tag{28}$$

Eq. (24) together with (28) form a linear set of equations:

$$\begin{pmatrix} k^2+\eta_s^2 & 2ik\eta_d \\ -2i\eta_s k & k^2+\eta_s^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=0$$

The condition for a non-trivial solution to exist is det=0, that is $\left(k^2+\eta_s^2\right)^2=4k^2\eta_s\eta_d$. Plugging in $\eta_i^2=k^2-\left(\frac{\omega}{c_i}\right)^2$ and squaring, this gives

$$\left(2k^2-\frac{\omega^2}{c_s^2}\right)^4=16\,k^4\left(k^2-\frac{\omega^2}{c_s^2}\right)\left(k^2-\frac{\omega^2}{c_d^2}\right)\ . \tag{29}$$

This is the dispersion relation for Rayleigh waves (sometimes this equation is called the Rayleigh equation). It is a very nice and simple dispersion relation because...it is linear! Huh! you didn't see that coming now, did you? Divide both sides by k^8 to get

$$\left(2-\left(\frac{\omega}{kc_s}\right)^2\right)^4=16\left(1-\left(\frac{\omega}{kc_s}\right)^2\right)\left(1-\left(\frac{\omega}{kc_s}\right)^2\left(\frac{c_s}{c_d}\right)^2\right)\ .$$

Denoting the dimensionless phase velocity $z=\frac{\omega}{kc_s}=\frac{c_{ph}}{c_s}$ and remembering our definition $\beta=\frac{c_s}{c_d}$ (cf. Eq. (4)), this turns to

$$\left(2-z^2\right)^4-16\left(1-z^2\right)\left(1-\beta^2z^2\right)=0\ ,$$

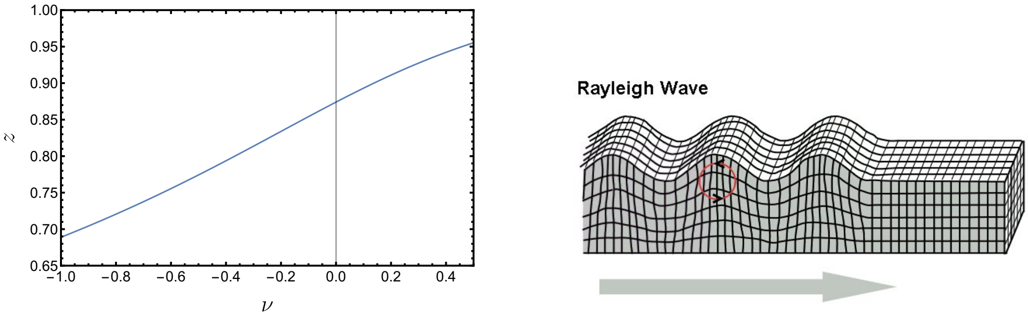
or,

$$z^6-8z^4+8\left(3-2\beta^2\right)z^2+16\left(\beta^2-1\right)=0$$

So knowing β, which is a material parameter that equals $\sqrt{\frac{1-2\nu}{2(1-\nu)}}$ gives the (physically unique) solution for z and thus completely defines the **linear** dispersion relation $\omega=zc_sk$. the solution is shown in Fig. 2a, and it is seen that the wave speed, zc_s, is somewhat slower than c_s.

1.2.1 Some remarks regarding Rayleigh waves

- Dilational and shear waves travel at two different speeds. Nevertheless, Rayleigh waves couple the two (!) to create a different mode that travels at a third speed (!!), and all this is within a linear theory (!!!).
- The coupling comes from the traction-free boundary condition.
- A single Rayleigh mode with k,ω is a combination of two evanescent bulk modes with the same ω, but different k.



(a) Numerical solution for z.

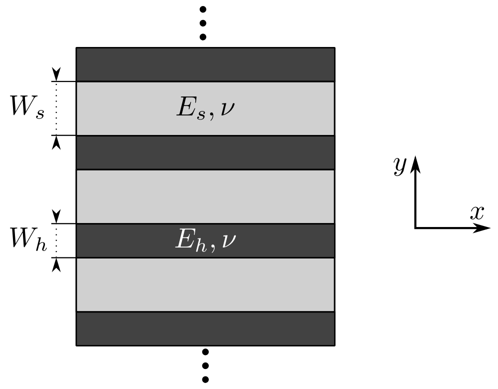
(b) A Rayleigh wave (from wikipedia).

Figure 2: Stuff about Rayleigh waves.

- The bulk modes are evanescent because the velocity of the Rayleigh mode is slower than c_s and c_d. This makes η_s, η_d real. Otherwise, the modes will not be localized on the surface.
- Rayleigh waves are surface waves.Therefore, their magnitude decreases only as $1/\sqrt{r}$ rather than the bulk $1/r$. In large earthquakes, some Rayleigh waves circle the earth a few times before dissipating!
- They are confined to propagate on the surface and decay exponentially with depth. Therefore, the amplitude of earthquake-generated Rayleigh waves is generally a decreasing function of the depth of the earthquake's hypocenter (origin/focus).
- The particle trajectories in a Rayleigh wave are elliptic, much like in ocean surface waves.

2 A simple model for a simple composite material

Composite materials are materials that have a microscopic structure. They are abundant in nature, and examples include bone, wood, dentin (the material your teeth are made of), and many more (graphic examples will be shown in class). In the last few decades there are also many man-made composite materials. The vast advancement in composite material technology is one of the most influential revolutions in modern technology, and it allows manufacturing materials that have desirable characteristics – light-weight, high strength, shape memory, etc. – which are orders of magnitude better than homogeneous materials. This is a fascinating topic which is the subject of huge and very active ongoing research. In this short exercise we'll examine some simple outcomes of a simple model of a simple composite material. This model was given as a question in the final exam of the 2012 course.



Consider a material that is composed of layers of two linear-elastic isotropic materials, one is hard and the other soft. The hard material has a Young's modulus E_h and the soft material has a Young's modulus of E_s. For simplicity, we'll assume that both materials have the same Poisson's ratio ν. The width of each layer is denoted by W_i, and the layers are glued perfectly to each other. We assume the material is infinite in all directions, has a periodic structure in the y direction and is translationally invariant in the x direction. We also assume plane-stress conditions in the z direction. We define the volume ratio of the hard material by

$$\phi = \frac{W_h}{W_h + W_s} \ . \tag{30}$$

Our goal is to calculate the coarse-grained Hooke's law of the composite system, that is, it's (linear) response to loading on length-scales much larger than the scale of the microscopic structure (in our case - W_s and W_h). To this end, we consider a small segment of the material over which the stress and strain field are constant *in each material.* We denote these fields by $\sigma^{(i)}$, $\varepsilon^{(i)}$ where $i = h, s$.

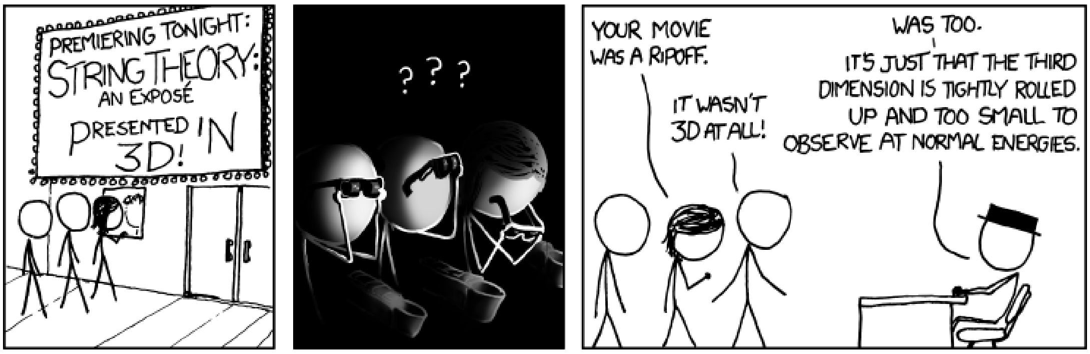
Each of these fields follows Hooke's law with the relevant elastic constants:

$$\begin{pmatrix} {\varepsilon_{xx}}^{(h,s)} \\ {\varepsilon_{yy}}^{(h,s)} \\ {\varepsilon_{xy}}^{(h,s)} \end{pmatrix} = \frac{1}{E^{(h,s)}} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} {\sigma_{xx}}^{(h,s)} \\ {\sigma_{yy}}^{(h,s)} \\ {\sigma_{xy}}^{(h,s)} \end{pmatrix} \tag{31}$$

$$\begin{pmatrix} {\sigma_{xx}}^{(h,s)} \\ {\sigma_{yy}}^{(h,s)} \\ {\sigma_{xy}}^{(h,s)} \end{pmatrix} = \frac{E^{(h,s)}}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{pmatrix} {\varepsilon_{xx}}^{(h,s)} \\ {\varepsilon_{yy}}^{(h,s)} \\ {\varepsilon_{xy}}^{(h,s)} \end{pmatrix} \ . \tag{32}$$

2.1 Boundary conditions at the interfaces

We now need to determine are the boundary conditions at the interfaces between the different layers. Clearly, the displacement field $\boldsymbol{u}$ must be continuous across the interface, because we assumed perfect bonding between the materials, which means no relative displacement. But what does that tell us about $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$? Which components are continuous across the interface and which experience a jump? How is the jump determined? If you are reading this before the class, I strongly recommend that you stop here for a minute and try to answer this question on your own. If you need a pause, here's an XKCD strip about string theory:



Now that you've answered the question here's the correct answer. Let's say that some of the interfaces lies on $y = 0$. As said before, $\boldsymbol{u}$ is continuous across the interface and therefore

$$u_i^{(s)}(x, y = 0) = u_i^{(h)}(x, y = 0) \tag{33}$$

for all x. Specifically, this means that $\partial_x u_i$ is continuous across the interface. From this we conclude immediately that $\varepsilon_{xx}^{(s)} = \varepsilon_{xx}^{(h)}$.

Next, consider force balance. Think of a small volume element of length L and infinitesimal height, which is half in the soft region and half in the hard region. The vertical forces applied to it (per unit thickness in the z direction) sum up to

$$L\left(\sigma_{yy}^{(h)} - \sigma_{yy}^{(s)}\right)\hat{y} + L\left(\sigma_{xy}^{(h)} - \sigma_{xy}^{(s)}\right)\hat{x} \tag{34}$$

Since the situation is static, we conclude that σ_{iy} is continuous across the interface. To summarize:

- $\boldsymbol{u}$, σ_{yy}, σ_{xy}, ε_{xx} are continuous across the interface.
- σ_{xx}, ε_{yy}, ε_{xy} experience a jump across the interface. The jump can be calculated from Hooke's law.

2.2 Coarse graining

Now comes a crucial part of the intellectual path that we try to follow. We want to describe the large-scale/macroscopic/coarse-grained fields $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ in terms of the small-scale/microscopic elastic fields of the constituent materials $\sigma^{(i)}, \varepsilon^{(i)}$. The final goal is to find the macroscopic Hooke's law, i.e. the linear relation

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \boldsymbol{C} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} \; . \tag{35}$$

In our case this step fairly easy: consider a large square which contains a lot of layers. The total force exerted on the x side is equal to the total force exerted by all the different layers:

$$\sigma_{xx} = \frac{W_h \sigma_{xx}^{(h)} + W_s \sigma_{xx}^{(s)}}{W_h + W_s} = \phi\, \sigma_{xx}^{(h)} + (1 - \phi)\sigma_{xx}^{(s)} \; . \tag{36}$$

The macroscopic strain in the x direction is $\varepsilon_{xx} = \varepsilon_{xx}^{(h)} = \varepsilon_{xx}^{(s)}$, which translates through the microscopic Hooke's law Eq. (31) to

$$\varepsilon_{xx} = \varepsilon_{xx}^{(h)} = \varepsilon_{xx}^{(s)} = \frac{\sigma_{xx}^{(h)} - \nu\sigma_{yy}^{(h)}}{E_h} = \frac{\sigma_{xx}^{(s)} - \nu\sigma_{yy}^{(s)}}{E_s} \ . \tag{37}$$

In the y direction things are exactly the other way around:

$$\sigma_{yy} = \sigma_{yy}^{(h)} = \sigma_{yy}^{(s)} = \frac{E^{(s)}}{1-\nu^2}\left(\varepsilon_{yy}^{(s)} + \nu\varepsilon_{xx}^{(s)}\right) = \frac{E^{(h)}}{1-\nu^2}\left(\varepsilon_{yy}^{(h)} + \nu\varepsilon_{xx}^{(h)}\right) \tag{38}$$

$$\varepsilon_{yy} = \phi\ \varepsilon_{yy}^{(h)} + (1-\phi)\varepsilon_{yy}^{(s)} \tag{39}$$

We now want to write the macroscopic Hooke's law and we'll begin with the diagonal part. Before we start, let's count to see that we already have all the ingredients. We use the fact that $\varepsilon_{xx} = \varepsilon_{xx}^{(h,s)}$ and $\sigma_{yy} = \sigma_{yy}^{(h,s)}$ to summarize our equations as

$$\sigma_{xx} = \phi\,\sigma_{xx}^{(h)} + (1-\phi)\sigma_{xx}^{(s)} \tag{40}$$

$$\sigma_{yy} = \frac{E^{(s)}}{1-\nu^2}\left(\varepsilon_{yy}^{(s)} + \nu\varepsilon_{xx}\right) = \frac{E^{(h)}}{1-\nu^2}\left(\varepsilon_{yy}^{(h)} + \nu\varepsilon_{xx}\right) \tag{41}$$

$$\varepsilon_{yy} = \phi\ \varepsilon_{yy}^{(h)} + (1-\phi)\varepsilon_{yy}^{(s)} \tag{42}$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}^{(h)} - \nu\sigma_{yy}}{E_h} = \frac{\sigma_{xx}^{(s)} - \nu\sigma_{yy}}{E_s} \ . \tag{43}$$

These are linear 6 equations, and we want to eliminate 4 variables: $\sigma_{xx}^{(h,s)}$ and $\varepsilon_{yy}^{(h,s)}$. Thus, we expect to finish with two linear equations, which we will be able to put in the desired form

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\end{pmatrix} = \begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\end{pmatrix} \tag{44}$$

The full solution is therefore obtained by simple algebra, which we will not do because it's not very interesting.

For the shear part of Hooke's law we have

$$\partial_x u_y = \partial_x u_y^{(s)} = \partial_x u_y^{(h)} \ . \tag{45}$$

Therefore,

$$\frac{E_s\left(\partial_x u_y + \partial_y u_x^{(s)}\right)}{2(1+\nu)} = \frac{E_h\left(\partial_x u_y + \partial_y u_x^{(h)}\right)}{2(1+\nu)} = \sigma_{xy} \ . \tag{46}$$

Inverting, we get

$$\partial_y u_x^{(i)} = \frac{2(1-\nu)}{E^{(i)}}\sigma_{xy} - \partial_x u_y \ . \tag{47}$$

We now use

$$\partial_y u_x = (1-\phi)\partial_y u_x^{(s)} + \phi\,\partial_y u_x^{(h)} \tag{48}$$

to obtain

$$\varepsilon_{xy} = \frac{1}{2}(\partial_x u_y + \partial_y u_x) = (1+\nu)\left[\frac{(1-\phi)}{E_s} + \frac{\phi}{E_h}\right]\sigma_{xy} \equiv \frac{\sigma_{xy}}{\mu_{\text{eff}}} \ . \tag{49}$$

2.3 Discussion

As you see we did not write the full expressions for the coarse-grained Hooke's law (as it is a bit ugly). The point was to show the structure of the solution. Nonetheless, some interesting cases can be easily obtained without having to write the full expression.

First of all, we note that the resulting coarse-grained material is, unsurprisingly, anisotropic. We see that the symmetry $x \leftrightarrow y$ is clearly broken. However, plugging in $E_s = E_h$, the isotropic case is immediately recovered. Also, the cases $\phi \to 1$ or $\phi \to 0$ easily recover a homogeneous isotropic material.

Second, let's consider the (very realistic) case of $E_h \gg E_s$. We are only interested in approximate trends, so we'll assume ϕ is not to close to 0 or to 1, and omit all pre-factors of the form $\phi, \phi/(1-\phi)$ and so on. We rewrite Eqs. (40) and (43) as

$$\sigma_{xx}^{(h)} = \frac{\sigma_{xx} - (1-\phi)\sigma_{xx}^{(s)}}{\phi} \tag{50}$$

$$\sigma_{xx}^{(s)} = \frac{\sigma_{xx} + \phi\,\nu\left(\frac{E_h}{E_s} - 1\right)\sigma_{yy}}{\phi\,\frac{E_h}{E_s} + (1-\phi)}\;. \tag{51}$$

Expanding these to order in E_s/E_h, we get

$$\sigma_{xx}^{(s)} \approx \frac{E_s}{E_h}\sigma_{xx} + \sigma_{yy} \tag{52}$$

$$\sigma_{xx}^{(h)} \approx \sigma_{xx} + \sigma_{yy} \tag{53}$$

Plugging this into Hooke's law we get

$$\varepsilon_{xx} \approx \frac{\frac{E_s}{E_h}\sigma_{xx} + \sigma_{yy} - \nu\sigma_{yy}}{E_s} \approx \frac{\sigma_{xx}}{E_h} + \frac{\sigma_{yy}}{E_s}\;. \tag{54}$$

Similarly, plugging (52)-(53) into (42),

$$\begin{aligned}\varepsilon_{yy} &\approx \phi\frac{\sigma_{yy} - \nu\sigma_{xx}^{(h)}}{E_h} + (1-\phi)\frac{\sigma_{yy} - \nu\sigma_{xx}^{(s)}}{E_s} \\ &\approx \frac{\sigma_{yy} - (\sigma_{xx} + \sigma_{yy})}{E_h} + \frac{\sigma_{yy} - \left(\frac{E_s}{E_h}\sigma_{xx} + \sigma_{yy}\right)}{E_s} \approx \frac{\sigma_{yy}}{E_s} + \frac{\sigma_{xx}}{E_s}\end{aligned} \tag{55}$$

In addition, Eq. (49) shows that $\mu_{\text{eff}} \sim E_s$. We conclude that the energy functional takes the approximate form

$$u = \frac{1}{2}\left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + 2\sigma_{xy}\epsilon_{xy}\right) \approx \frac{\sigma_{xx}^2}{E_h} + \frac{\sigma_{yy}^2}{E_s} + \frac{\sigma_{xx}\sigma_{yy}}{E_s} + \frac{\sigma_{xy}^2}{E_s} \tag{56}$$

Thus, all the responses except the one in the xx direction are dominated by the soft material.