Linear Elasticity III - Conformal mapping, elastic cavitation

1 Complex representation of scalar elasticity

We study a case of scalar elasticity, where $\boldsymbol{u} = u_z(x,y)\boldsymbol{e}_z$. We have seen in class that $\nabla^2 u_z = 0$, that is, u_z is a harmonic function. This means that we can write u_z as

$$u_z = 2\Re(\phi) = \phi(z) + \overline{\phi(z)}, \qquad z = x + iy \tag{1}$$

where ϕ is an analytic complex function. We will use the Cauchy-Riemman equations, that tell us that

$$\partial_x \phi = i\partial_y \phi = \phi' \tag{2}$$

$$\partial_x \overline{\phi} = \overline{\partial_x \phi} = -i\partial_y \overline{\phi} = \overline{\phi}' \tag{3}$$

The strains are

$$\epsilon_{yz} = \frac{1}{2}(\partial_y u_z + \partial_z u_y) = \frac{1}{2}\partial_y u_z \tag{4}$$

$$\epsilon_{xz} = \frac{1}{2}(\partial_z u_x + \partial_z u_x) = \frac{1}{2}\partial_x u_z \tag{5}$$

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = 0 \tag{6}$$

$$\tag{7}$$

and therefore the stresses are

$$\sigma_{xz} = \frac{\mu}{2}\partial_x u_z = \frac{\mu}{2}\left(\partial_x \phi + \partial_x \overline{\phi}\right) = \frac{\mu}{2}\left(\phi' + \overline{\phi}'\right) = \mu\Re(\phi') \tag{8}$$

$$\sigma_{yz} = \frac{\mu}{2}\partial_y u_z = \frac{\mu}{2}\left(\partial_y \phi + \partial_y \overline{\phi}\right) = -i\frac{\mu}{2}\left(\phi' - \overline{\phi}'\right) = -\mu\Im(\phi') \tag{9}$$

$$\Rightarrow \quad \mu\phi' = \sigma_{xz} - i\sigma_{yz} \tag{10}$$

and all other components vanish.

If our domain contains a hole, whose boundary is given by a curve that is parametrized by $x(s), y(s)$ with s being arc-length parametrization, then the normal to the hole is given by $\boldsymbol{n} = (\partial_s y, -\partial_s x)$. On the boundary we thus have

$$0 = \sigma_{zx} n_x + \sigma_{zy} n_y$$

$$= \sigma_{zx}\frac{\partial y}{\partial s} - \sigma_{zy}\frac{\partial x}{\partial s} \propto \partial_x u\, \partial_s y - \partial_y u\, \partial_s x$$

$$= \left(\partial_x \phi + \partial_x \overline{\phi}\right)\partial_s y - \left(\partial_y \phi + \partial_y \overline{\phi}\right)\partial_s x$$

$$= \left(i\partial_y \phi - i\partial_y \overline{\phi}\right)\partial_s y - \left(-i\partial_x \phi + i\partial_x \overline{\phi}\right)\partial_s x$$

$$= i\left(\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial s}\right) - i\left(\frac{\partial\overline{\phi}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\overline{\phi}}{\partial x}\frac{\partial x}{\partial s}\right)$$

$$= i\left(\frac{\partial\phi}{\partial s} - \frac{\partial\overline{\phi}}{\partial s}\right) = -2\frac{\partial\Im(\phi)}{\partial s} \tag{11}$$

So on the boundary $\mathfrak{I}(\phi)$ is constant. Since ϕ is only given up to an additive constant, we can choose $\mathfrak{I}(\phi) = 0$, or in other words $\phi = \overline{\phi}$ on the boundary. We see that solving for the displacement field is equivalent to finding an analytic function whose imaginary part is constant on the boundary.

2 Conformal mapping: Inglis crack

(Refernce: Marder & Fineberg, Physics Reports 1999)

Complex treatment of 2D elasticity is very useful because Laplace's equation is conformally invariant, so one can use conformal mappings to deform the region over which we need to solve the equation into a more convenient geometry. Here we'll see an application of this method, which is called the Inglis (mode III) problem. In 1913 Charles Inglis solved the general problem of an elliptic hole in an infinite plate subject to distant loading. His solution turned out to be one of the cornerstones of fracture mechanics, and was later used and generalized by the works of Griffith, Irwin, and others.

So let's look at an infinite plane with an elliptic hole, subject to antiplane shear σ_∞ at $y \to \pm\infty$. As working with ellipses is unpleasant, we want to find a conformal mapping that maps the region outside the ellipse to a region outside a circle. Luckily, such a mapping is well known, and is given by

$$z = f(\omega) = R\left(\omega + \frac{\rho}{\omega}\right) \tag{12}$$

$$\omega = f^{-1}(z) = \frac{z}{2R} + \sqrt{\left(\frac{z}{2R}\right)^2 - \rho} \tag{13}$$

f maps the unit circle in the ω-plane to an ellipse with axes $R(1 \pm \rho)$ in the z-plane. $0 \le \rho \le 1$ is a parameter that measures the ellipse's eccentricity - when $\rho = 0$ the ellipse is a circle, while for $\rho = 1$ it is a 1D crack of length R. The conformal mapping is shown in Fig. (1).

The crux of the conformal mapping technique is that while in the real coordinates the geometry is elliptic (and thus complicated), in the ω-plane the domain is a circle (simple!), and therefore we want to reformulate the problem in terms of ω. That is, we want to describe ϕ as a function of ω, by the mapping $\phi(\omega) = \phi(\omega(z))$.

On the hole's boundary, which is the unit circle in ω-plane, we have

$$\phi(\omega) = \overline{\phi(\omega)} = \overline{\phi}(\overline{\omega}) = \overline{\phi}(1/\omega) , \tag{14}$$

because on the unit circle $\overline{\omega} = 1/\omega$. The property (14) can be analytically extended to all the ω-plane. What are the singularities of ϕ? Outside the hole, it must be completely regular, except at infinity where it diverges as $\phi \sim z$. This is because Eq. (10) tells us that far from the hole we have $\partial_z\phi = -i\sigma/\mu$, and therefore we conclude that

$$\phi \approx -i\frac{\sigma_\infty}{\mu}z \approx -i\frac{\sigma_\infty}{\mu}R\omega, \qquad \text{for } \omega, z \to \infty \tag{15}$$

Using the analytical continuation of Eq. (14), we get that

$$\overline{\phi}(1/\omega) \approx -i\frac{\sigma_\infty}{\mu}R\omega, \qquad \text{for } \omega \to \infty , \tag{16}$$
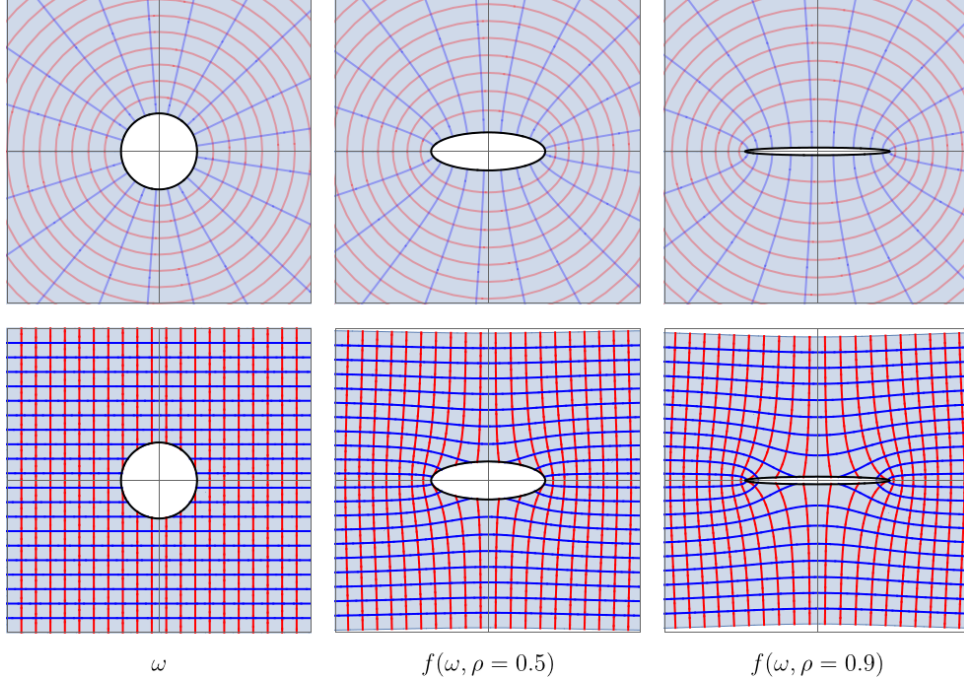
Figure 1: The conformal mapping with shows polar lines (top row) and the Cartesian lines (bottom row). Note that the lines remain perpendicular after the mapping.

or equivalently,

$$\phi(\omega) \approx i\frac{\sigma_\infty R}{\mu\omega}, \qquad \text{for } \omega \to 0 \ , \tag{17}$$

and there are no other singularities inside the unit circle. Having determined all the possible singularities of ϕ, it is determined up to an additive constant. It must be

$$\phi(\omega) = -i\frac{\sigma_\infty R}{\mu}\left(\omega - \frac{1}{\omega}\right) + const \tag{18}$$

As discussed above, another way of finding ϕ is by to find a function whose real part vanishes on the boundary on the hole, i.e. on the unit circle. A natural candidate would be $\Im(\omega)$, but sadly this is not an analytic function. However, we note again that *on the unit circle* we have $\overline{\omega} = 1/\omega$. Therefore, the function $\Im(\omega)$ can be written as

$$\Im(\omega) = \frac{\omega - \overline{\omega}}{2i} = \frac{\omega - 1/\omega}{2i} \qquad \text{on } |\omega| = 1 \ . \tag{19}$$

So $\omega - 1/\omega$ is clearly a function whose imaginary part vanishes along the unit circle. Therefore, it is exactly the function we're looking for, up to a global multiplicative factor which is set by the conditions at $\omega \to \infty$.

We can now calculate the displacement field in the "real" coordinate z by joining Eqs. (18) and (13):

$$u_z = 2\Re\left\{-i\frac{\sigma_\infty R}{\mu}\left(\zeta + \sqrt{\zeta^2 - \rho} - \frac{1}{\zeta + \sqrt{\zeta^2 - \rho}}\right)\right\} \qquad \text{where } \zeta \equiv \frac{z}{2R} \ . \tag{20}$$

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What is the stress at the tip of the ellipse? We can differentiate $u_z(z)$ of Eq. (20) explicitly, but this gives a nasty expression that is very difficult to understand. It is simpler to use the conformal property of the mapping:

$$\partial_z \phi(z) = \partial_z \phi(\omega(z)) = \phi'(\omega) \frac{\partial \omega}{\partial z} \tag{21}$$

$$\phi'(\omega) = -i \frac{\sigma R}{\mu} \left(1 + \frac{1}{\omega^2} \right) \tag{22}$$

$$\frac{\partial \omega}{\partial z} = \left(\frac{\partial z}{\partial w} \right)^{-1} = \frac{1}{f'(\omega)} \tag{23}$$

$$f'(\omega) = R(1 - \frac{\rho}{\omega^2}) \tag{24}$$

$$\phi'(z) = \frac{-i \frac{\sigma R}{\mu} \left(1 + \frac{1}{\omega^2} \right)}{R(1 - \frac{\rho}{\omega^2})} = -\frac{i\sigma}{\mu} \frac{\omega(z)^2 + 1}{\omega(z)^2 - \rho} . \tag{25}$$

Note that in the last equation ω is a function of z.

Now let's examine the solution. One thing we would like to know is where in space is the stress maximal. Clearly, ϕ' diverges for $w = \pm\sqrt{\rho}$, but remember that $\rho < 1$ and our domain is outside the unit circle, so this point is inside the hole. Some trivial algebra shows that the ϕ' is maximal for $\omega = \pm 1$, which are, not surprisingly, the closest points outside the unit circle to $\pm\rho$. When $\omega = \pm 1$ we have $z = \pm R(1 + \rho)$ - these are the horizontal tips of the ellipse. The stresses there are

$$\sigma_{xz} - i\sigma_{yz} = \mu\phi' = -\sigma \frac{2i}{1 - \rho} \tag{26}$$

$$\sigma_{yz} = \sigma_\infty \frac{2}{1 - \rho} = \frac{2\sigma_\infty}{1 + \rho} \frac{R}{r_c}, \qquad \sigma_{xz} = 0 , \tag{27}$$

where $r_c = R\frac{1-\rho}{1+\rho}$ is the radius of curvature at the tip.

The case $\rho = 0$ gives $r_c = 1$ and therefore $\sigma_{yz} = 2\sigma_\infty$, in accordance to what was done in class. In the opposite extremity, $\rho \to 1$ and $r_c \to 0$, and the stress field diverges (but the displacement doesn't). We see that the stress at the tip is inversely proportional to the radius of curvature there. An interesting consequence of this is that in order to arrest a crack from propagating one can drill a hole at its tip (!). This will reduce the radius of curvature at the tip and weaken the singularity.

The limiting case $\rho \to 1$ is of particular interest, as it describes a 1-dimensional cut in the material. It is known in the literature as Mode III crack. The power with which σ_{zy} diverges in the case $\rho = 1$ can be easily obtained. In this case we have

$$\phi = -i\frac{\sigma}{\mu} R \left(\zeta + \sqrt{-1 + \zeta^2} - \frac{1}{\zeta + \sqrt{-1 + \zeta^2}} \right) . \tag{28}$$

Plugging in $\zeta = 1 + \delta$ (where $\delta \in \mathbb{C}$) and keeping the leading order in δ gives

$$\phi = -i\frac{\sigma}{\mu} R\sqrt{8\delta} + o(\delta^{3/2}) \tag{29}$$

$$\sigma_{yz} \sim \frac{1}{\sqrt{\delta}} \tag{30}$$

The fact that near the crack tip the stress field diverges as the square root of the distance from the crack tip, and that the displacement field is regular, is of general applicability, and is true for static cracks in all loading configurations. The square-root divergence is a consequence of the branch-cut at the crack surface.

3 Elastic cavitation

Note: this section appears mostly in Eran's lecture notes.

Consider a spherical cavity of initial radius L inside an infinite elastic material loaded by a radially symmetric tensile stress $\boldsymbol{\sigma} = \sigma^\infty \boldsymbol{I}$ far away. The geometry of the problem clearly suggests that spherical coordinates should be used. So let's take the opportunity to discuss how to derive the equation of motion $\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\boldsymbol{u}}$ in curvilinear coordinates.

3.1 Divergence in spherical coordinates

What do we mean when we write a tensor $\boldsymbol{A}$ in Cartesian coordinates as

$$\boldsymbol{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} ? \tag{31}$$

This is a shorthand notation for $\boldsymbol{A} = \sum_{ij} A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ with $i, j \in \{x, y, z\}$ and $\boldsymbol{e}_i$ is the unit vector in the i direction. Writing the tensor in, say, spherical coordinates, means to write it in terms of the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$ which are *space-dependent*. Since spherical coordinates are orthonormal, we know that *locally* the transformation from Cartesian to spherical coordinates is given by a rotation,

$$[\boldsymbol{A}]_{r,\phi,\theta} = \boldsymbol{R}(\phi, \theta)^T [\boldsymbol{A}]_{x,y,z} \boldsymbol{R}(\phi, \theta) , \tag{32}$$

but you have to remember that the rotation matrix $\boldsymbol{R}$ is different in different points in space.

When you calculate derivative of the tensor in curvilinear coordinates you need to keep track of the fact that not only the components of the tensor change in space, but also the unit vectors themselves change. This amounts to differentiating Eq. (32) and remembering to differentiate both copies of $\boldsymbol{R}(\phi, \theta)$, because ϕ and θ are space-dependent. Doing this properly is a long and technical calculation which we will not do here, but you should be able in principle to do it, and you should definitely understand it's algebraic structure. The bottom line is that the divergence of a tensor in spherical coordinates is

$$\nabla \cdot \boldsymbol{A} =$$
$$\left[\frac{\partial A_{rr}}{\partial r} + 2\frac{A_{rr}}{r} + \frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta} + \frac{\cot\theta}{r} A_{\theta r} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi r}}{\partial \phi} - \frac{1}{r}(A_{\theta\theta} + A_{\phi\phi}) \right] \boldsymbol{e}_r$$
$$+ \left[\frac{\partial A_{r\theta}}{\partial r} + 2\frac{A_{r\theta}}{r} + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\cot\theta}{r} A_{\theta\theta} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\theta}}{\partial \phi} + \frac{A_{\theta r}}{r} - \frac{\cot\theta}{r}A_{\phi\phi} \right] \boldsymbol{e}_\theta \tag{33}$$
$$+ \left[\frac{\partial A_{r\phi}}{\partial r} + 2\frac{A_{r\phi}}{r} + \frac{\sin\theta}{r}\frac{\partial A_{\theta\phi}}{\partial \theta} + \frac{\cos\theta}{r} A_{\theta\phi} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\phi}}{\partial \phi} + \frac{1}{r}(A_{\phi r} + A_{\phi\theta}) \right] \boldsymbol{e}_\phi$$

You can find similar expressions for other differential operators (Laplacian, gradient, material derivative, etc.) for both spherical and cylindrical coordinates on this page in wikipedia.

3.2   Returning to our problem

The symmetry of the problem suggests that all quantities are functions of r alone and that $\sigma_{\phi\phi} = \sigma_{\theta\theta}$. Also, The θ and ϕ components of force balance equation clearly vanish and he radial component reads is greatly simplified to read

$$\partial_r \sigma_{rr} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \ . \tag{34}$$

Integrating this equation from the deformed radius of the cavity ℓ to r we obtain

$$\sigma_{rr}(r) = -2 \int_\ell^r \frac{\sigma_{rr}(\tilde{r}) - \sigma_{\theta\theta}(\tilde{r})}{\tilde{r}} d\tilde{r} \ , \tag{35}$$

where we used the traction-free boundary condition $\sigma_{rr}(r = \ell) = 0$ and $\tilde{r}$ is a dummy integration variable. Denote then $r' = \tilde{r}/\ell$ and focus on $r \to \infty$, we obtain

$$\sigma^\infty = \sigma_{rr}(\infty) = -2 \int_1^\infty \frac{g(r', L/\ell)}{r'} dr' \ , \tag{36}$$

where $\sigma_{rr} - \sigma_{\theta\theta} = g(r', L/\ell)$ is a property of the solution (which involves also the constitutive relation). Note that Eq. (36) is an equation that connects σ^∞ to the deformed radius ℓ. That is, given σ^∞ one can solve for ℓ or vice versa.

From our previous analysis we know that the existence of the cavity amplifies the (circumferential) stress at the surface as compared to the applied stress σ^∞ (for a cylindrical cavity we calculated the amplification factor to be 2 and for a sphere it is 3/2). If we keep on increasing the applied stress an ordinary material will simply break near the cavity surface. However, in soft materials something else can happen (the same can happen in an elasto-plastic material, to be discussed later). We can ask ourselves whether the cavity can grow (elastically!) without bound under the application of a finite stress at infinity. To mathematically formulate the question take the $\ell \to \infty$ limit in Eq. (36) and define

$$\sigma_c = -2 \lim_{\ell\to\infty} \int_1^\infty \frac{g(r', L/\ell)}{r'} dr' \ . \tag{37}$$

Therefore, if the integral above converges, then for any $\sigma^\infty > \sigma_c$ the cavity will grow indefinitely. Mathematically, we see this by the fact that no ℓ will provide a solution for Eq. (36) for $\sigma^\infty > \sigma_c$. The critical stress σ_c is called the cavitation threshold. σ_c is finite if $g(r', L/\ell) = \sigma_{rr} - \sigma_{\theta\theta} \to 0$ as $r \to \infty$, which is the typical situation.

Let us see how this works in a concrete example. Consider an incompressible elastic material. As above, the initial radius of the cavity is L and the radial coordinate is denoted as R. The deformed radius is ℓ and the coordinate of the deformed configuration is r. Volume conservation implies

$$\frac{4\pi}{3}\left(R^3 - L^3\right) = \frac{4\pi}{3}\left(r^3 - \ell^3\right) \quad \Longrightarrow \quad R(r) = (r^3 + L^3 - \ell^3)^{1/3} \ . \tag{38}$$

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The non-radial stretches take the form

$$\lambda_\phi = \lambda_\theta = \frac{r}{R} \ . \tag{39}$$

Incompressibility implies

$$\lambda_r \equiv \lambda \quad \Longrightarrow \quad \lambda_\phi = \lambda_\theta = \lambda^{-1/2} \tag{40}$$

which leads to

$$\lambda^{-1/2} = \frac{r}{R} \quad \Longrightarrow \quad \lambda = \left(\frac{R}{r}\right)^2 \ . \tag{41}$$

Finally, this leads to

$$\lambda = \left(\frac{r^3 + L^3 - \ell^3}{r^3}\right)^{2/3} = \left[\frac{r'^3 + (L/\ell)^3 - 1}{r'^3}\right]^{2/3} \ , \tag{42}$$

with $r' \equiv r/\ell$. Consider then the stress state. It is triaxial and contains only the diagonal components $(\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{\theta\theta})$, with $\sigma_{\phi\phi} = \sigma_{\theta\theta}$. However, since the material is incompressible we can superimpose on this stress state a hydrostatic stress tensor of the form $-\sigma_{\theta\theta}\boldsymbol{I}$ without affecting the deformation state, resulting in $(\sigma_{rr} - \sigma_{\theta\theta}, 0, 0)$, which is a uniaxial stress state in the radial direction. Therefore, the constitutive relation takes the form $\sigma_{rr} - \sigma_{\theta\theta} = g(\lambda_r)$. Focus then on a neo-Hookean material for which $g(\lambda) = \mu(\lambda^2 - \lambda^{-1})$ and evaluate the integral in Eq. (37)

$$\sigma_c = -2 \lim_{\ell \to \infty} \int_1^\infty \frac{g[\lambda(r', L/\ell)]}{r'} dr' = -2\mu \int_1^\infty \left[\frac{(1 - r'^{-3})^{4/3} - (1 - r'^{-3})^{-2/3}}{r'}\right] dr' \ . \tag{43}$$

This integral can be readily evaluated (just use $x \equiv 1 - r'^{-3}$ and $dx = 3r'^{-4}dr'$), yielding

$$\sigma_c = \frac{5\mu}{2} \ . \tag{44}$$

This result, which was verified experimentally (see, for example, J. Appl. Phys. **40**, 2520 (1969)), clearly demonstrates the striking difference between ordinary and "soft" solids. The *ideal* strength of ordinary solids is about $\mu/10$. The actual strength is *much* smaller (see later in the course). However, "soft" solids can sustain stresses larger than μ without breaking (though, as we have just shown, they can experience unique instabilities such as elastic cavitation).