Finite (non-linear) elasticity - small on large waves

In this TA we start looking at non-linear elasticity, a.k.a finite elasticity. As a first example, we'll consider a 2D plane-stress problem of an incompressible neo-Hookean material. The neo-Hookean energy functional for plane-stress is

$$u(\boldsymbol{F}) = \frac{\mu}{2}\left[\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F}) + (\det\boldsymbol{F})^{-2} - 3\right] \ , \tag{1}$$

where $\boldsymbol{F}$ is the two-dimensional deformation gradient tensor. We do not derive this functional here, but rather see what are the implications of this form. Note that $\det\boldsymbol{F}$ appears in the elastic energy functional due to the incompressibility condition.

1 The first Piola-Kirchhoff stress tensor

To calculate $\boldsymbol{P} \equiv \frac{\partial u}{\partial \boldsymbol{F}}$ note that

$$\frac{\partial\,\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F})}{\partial\boldsymbol{F}} = 2\boldsymbol{F}, \quad \frac{\partial(\det\boldsymbol{F})^{-2}}{\partial\boldsymbol{F}} = -2(\det\boldsymbol{F})^{-3}\left(\mathrm{tr}\,\boldsymbol{F}\ \boldsymbol{I} - \boldsymbol{F}^T\right) \ , \tag{2}$$

where the latter can be easily obtained using the identity (valid in 2D only)

$$\det\boldsymbol{F} = \frac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{F})^2 - \mathrm{tr}\,\boldsymbol{F}^2\right] \ . \tag{3}$$

Therefore, we have

$$\boldsymbol{P} = \frac{\partial u}{\partial\boldsymbol{F}} = \mu\left[\boldsymbol{F} - (\det\boldsymbol{F})^{-3}\left(\mathrm{tr}\,\boldsymbol{F}\ \boldsymbol{I} - \boldsymbol{F}^T\right)\right] \ , \tag{4}$$

$$\boldsymbol{P} = \mu\left[\begin{pmatrix} \partial_X\phi_x & \partial_Y\phi_x \\ \partial_X\phi_y & \partial_Y\phi_y \end{pmatrix} - J^{-3}\begin{pmatrix} \partial_Y\phi_y & -\partial_X\phi_y \\ -\partial_Y\phi_x & \partial_X\phi_x \end{pmatrix}\right] \tag{5}$$

2 Linearized energy functional

Before going fully non-linear, let's examine the linearized version our equations to see if we get something that we recognize. Assume for simplicity that the axes are chosen in parallel to the the principal stretches, i.e.

$$\boldsymbol{F} = \begin{pmatrix} 1+\varepsilon_x & 0 \\ 0 & 1+\varepsilon_y \end{pmatrix} \ .$$

The energy density is then

$$u = \frac{\mu}{2}\left[(1+\varepsilon_x)^2 + (1+\varepsilon_y)^2 + \frac{1}{(1+\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y)^2} - 3\right] \ . \tag{6}$$

Expanding to second order in the ε_i (we need second order because we develop the energy, which has quadratic terms in the stretch) we have

$$(1+\varepsilon_x)^2 = 1 + 2\varepsilon_x + \varepsilon_x^2 \tag{7}$$

$$(1+\varepsilon_y)^2 = 1 + 2\varepsilon_y + \varepsilon_y^2 \tag{8}$$

$$\frac{1}{(1+\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y)^2} = 1 - 2(\varepsilon_x+\varepsilon_y) + 3(\varepsilon_x^2+\varepsilon_y^2) + 4\varepsilon_x\varepsilon_y + \mathcal{O}(\varepsilon^3) \tag{9}$$

All in all we get

$$u = \mu\left(\varepsilon_x^2 + \varepsilon_y^2 + (\varepsilon_x + \varepsilon_y)^2\right) = \mu \operatorname{tr} \boldsymbol{\varepsilon}^2 + \mu \operatorname{tr}^2 \boldsymbol{\varepsilon} \ . \tag{10}$$

So we see that the linear form of the energy functional is the familiar and expected form $u = \frac{1}{2}(2\tilde{\mu} \operatorname{tr} \boldsymbol{\varepsilon}^2 + \tilde{\lambda} \operatorname{tr}(\boldsymbol{\varepsilon})^2)$. This also means that our material has $\tilde{\lambda} = 2\tilde{\mu}$ which implies

$$\tilde{\nu} = \frac{\lambda}{2(\lambda + \mu)} = \frac{1}{3} \ . \tag{11}$$

This value of ν should come as a surprise because we started with an incompressible material, so we should expect to have $\nu = \frac{1}{2}$. What went wrong? Keep in mind that the energy functional (1) is the result of the reduction of a set of 3D equations to 2D. We have done this in detail in the linear case, and we all remember well that the elastic constants are not the same as the 3D ones, but renormalized ones (see Eq. (5.60) in the lecture notes, or Sec. 4 in the TA session #4). The relation between the renormalized elastic constants to the real ones is

$$\tilde{\mu} = \mu, \qquad \tilde{\lambda} = \frac{2\nu\mu}{1-\nu} \ . \tag{12}$$

Rearranging the latter, we get

$$\nu = \frac{\tilde{\lambda}}{\tilde{\lambda} + 2\tilde{\mu}}$$

Plugging in our result $\tilde{\lambda} = 2\tilde{\mu}$ gives

$$\nu = \frac{2\tilde{\mu}}{2\tilde{\mu} + 2\tilde{\mu}} = \frac{1}{2} \ . \tag{13}$$

What a relief. The real Poisson ratio is 1/2, which means that the material is indeed incompressible. The fact that the "apparent" 2D Poisson's ration is different that 1/2 means that in-plane compressibility is allowed. This is because the material expands in the third direction, which is unaccounted for in the 2D description.

3 The equations of motion in our system

We remind ourselves that in the material coordinates the equations of motion read (see TA session #3).

$$\rho_0 \dot{\boldsymbol{V}} = \nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} \tag{14}$$

Plugging in our expression for $\boldsymbol{P}$, Eq. (5), we get

$$\frac{\rho_0}{\mu}\ddot{\phi}_x = \nabla^2 \phi_x - \frac{\partial \phi_y}{\partial Y}\frac{\partial J^{-3}}{\partial X} + \frac{\partial \phi_y}{\partial X}\frac{\partial J^{-3}}{\partial Y} \tag{15}$$

$$\frac{\rho_0}{\mu}\ddot{\phi}_y = \nabla^2 \phi_y - \frac{\partial \phi_x}{\partial X}\frac{\partial J^{-3}}{\partial Y} + \frac{\partial \phi_x}{\partial Y}\frac{\partial J^{-3}}{\partial X} \tag{16}$$

4 Small-on-Large waves

Consider then a homogeneously deformed body with principal stretches λ_x and λ_y. On this stretched state we superimpose a small displacement $\mathbf{\Delta}(\boldsymbol{X},t)$. The deformation $\boldsymbol{\varphi}(\boldsymbol{X},t)$ is thus

$$\varphi_X(\boldsymbol{X},t) = \lambda_x X + \Delta_X(\boldsymbol{X},t) \ , \tag{17}$$
$$\varphi_Y(\boldsymbol{X},t) = \lambda_y Y + \Delta_Y(\boldsymbol{X},t) \ . \tag{18}$$

Note that the homogeneous solution $\mathbf{\Delta} = 0$ satisfies the equations of motion. The motion gradient reads

$$\boldsymbol{F} = \begin{pmatrix} \lambda_x + \partial_X \Delta_X & \partial_Y \Delta_X \\ \partial_X \Delta_Y & \lambda_y + \partial_Y \Delta_Y \end{pmatrix} \tag{19}$$

We want to look at small perturbations on the stretched state, that is, we want to expand the equations of motion to first order in $\mathbf{\Delta}(\boldsymbol{X},t)$. First, we calculate

$$\det \boldsymbol{F} \simeq (\lambda_x + \partial_X \Delta_X)(\lambda_y + \partial_Y \Delta_Y) \approx \lambda_x \lambda_y \left(1 + \frac{\partial_X \Delta_X}{\lambda_x} + \frac{\partial_Y \Delta_Y}{\lambda_y}\right) + \mathcal{O}\left(\mathbf{\Delta}^2\right) \tag{20}$$

$$(\det \boldsymbol{F})^{-3} \simeq \frac{1}{\lambda_x^3 \lambda_y^3}\left(1 - 3\frac{\partial_X \Delta_X}{\lambda_x} - 3\frac{\partial_Y \Delta_Y}{\lambda_y}\right) + \mathcal{O}\left(\mathbf{\Delta}^2\right) \ . \tag{21}$$

The equations of motions are thus, to linear order,

$$\begin{aligned} \nabla^2 \Delta_X + 3\frac{\partial_{XX}\Delta_X}{\lambda_x^4 \lambda_y^2} + 3\frac{\partial_{XY}\Delta_Y}{\lambda_x^3 \lambda_y^3} &= \frac{\rho}{\mu}\ddot{\Delta}_X = c_s^{-2}\ddot{\Delta}_X \ , \\ \nabla^2 \Delta_Y + 3\frac{\partial_{YY}\Delta_Y}{\lambda_x^2 \lambda_y^4} + 3\frac{\partial_{XY}\Delta_X}{\lambda_x^3 \lambda_y^3} &= \frac{\rho}{\mu}\ddot{\Delta}_Y = c_s^{-2}\ddot{\Delta}_Y \ , \end{aligned} \tag{22}$$

where $c_s \equiv \sqrt{\frac{\mu}{\rho}}$. Assume then a solution in the form of plane waves

$$\begin{aligned} \Delta_X(\boldsymbol{X},t) &= a_X e^{i(\boldsymbol{N}\cdot\boldsymbol{X} - ct)} \ , \\ \Delta_Y(\boldsymbol{X},t) &= a_Y e^{i(\boldsymbol{N}\cdot\boldsymbol{X} - ct)} \ , \end{aligned} \tag{23}$$

where $\boldsymbol{N} = (\cos\theta, \sin\theta)$ is the direction of propagation in the undeformed coordinates and c is the (yet unknown) speed. What kind of waves are there in the system? what is (are) the wavespeed(s)?

Plugging in the ansatz (23) into the equations of motion (22) we get

$$a_X + 3\frac{\cos^2\theta}{\lambda_x^4\lambda_y^2}a_X + 3\frac{\sin\theta\cos\theta}{\lambda_x^3\lambda_y^3}a_Y - \frac{c^2}{c_s^2}a_x = 0 \tag{24}$$

$$a_y + 3\frac{\sin^2\theta}{\lambda_x^4\lambda_y^2}a_Y + 3\frac{\sin\theta\cos\theta}{\lambda_x^3\lambda_y^3}a_X - \frac{c^2}{c_s^2}a_y = 0 \tag{25}$$

which is more concisely written as

$$\underbrace{\begin{pmatrix} 1 + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} - \frac{c^2}{c_s^2} & \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} \\ \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} & 1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} - \frac{c^2}{c_s^2} \end{pmatrix}}_{\equiv \boldsymbol{M}} \begin{pmatrix} a_X \\ a_Y \end{pmatrix} = 0 \tag{26}$$

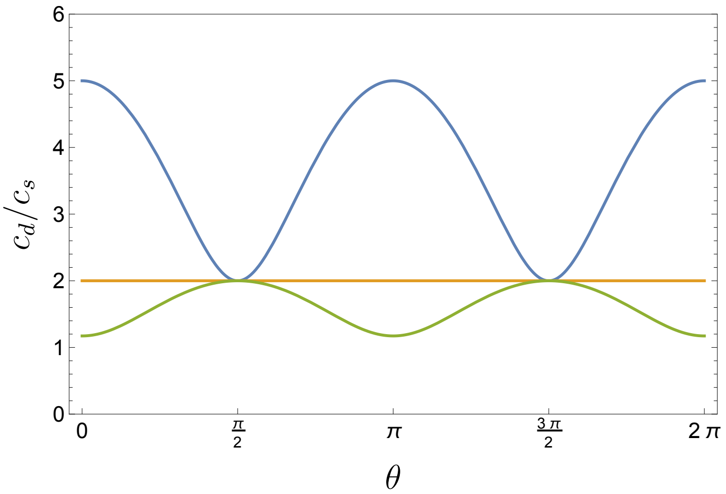

Figure 1: The longitudinal wavespeed (in units of c_s) as a function of the propagation direction θ for $\lambda = \frac{1}{2}, 1, 2$ (blue, orange, green, respectively).

Similarly to what we've done with Rayleigh waves, solutions are obtained when the determinant vanishes. This condition reads

$$\det \boldsymbol{M} = \left(1 - \frac{c^2}{c_s^2}\right)\left(1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} - \frac{c^2}{c_s^2}\right) = 0 \tag{27}$$

So you immediately see that there are two families of solutions,

$$c = \pm c_s \,, \qquad \text{and} \qquad c = \pm c_s\sqrt{1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2}} \,. \tag{28}$$

The first family are shear-like waves and their velocity is independent on direction. In order to see that they are shear waves, note that the amplitudes a_X, a_Y can be obtained, up to a multiplicative factor, by the kernel of the matrix $\boldsymbol{M}(c = c_s)$, which is

$$(a_X, a_Y) \in \ker \begin{pmatrix} \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} & \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} \\ \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} & \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} \end{pmatrix} \propto (-\lambda_x\sin\theta, \lambda_y\cos\theta) \tag{29}$$

These waves are "almost transverse" because $(a_X, a_Y)\cdot\boldsymbol{N} \propto (\lambda_x - \lambda_y)\sin(2\theta)$. Therefore, they are purely transverse for $\theta = 0, \frac{\pi}{2}$ (i.e. in the X or Y directions) or when $\lambda_x = \lambda_y$. Note that this also means that the shape of the waves will depend on the direction of propagation.

The other family of solutions has a direction-dependent velocity, which is an interesting situation which is not uncommon of anisotropic systems. Following the same logic as above, the amplitudes of these waves is, up to a multiplicative factor

$$(a_X, a_Y) \propto (\lambda_y\cos\theta, \lambda_x\sin\theta) \tag{30}$$

such that $(a_X, a_Y)\times\boldsymbol{N} \propto (\lambda_x - \lambda_y)\sin(2\theta)$ and again these waves are purely longitudinal for waves propagating in the X or Y direction, or for $\lambda_x = \lambda_y$.

4.1 Example

Consider a uniaxial pre-stress (applied λ_y), for which we have (due to incompressibility)

$$\lambda \equiv \lambda_x = \lambda_y^{-1/2} \ . \tag{31}$$

With this setup, the longitudinal wavespeed will be

$$c = \pm\sqrt{1 + \frac{3}{\lambda^3}\left(1 + (\lambda^3 - 1)\sin^2\theta\right)} \tag{32}$$

This function is plotted in Fig. 1.

Note that if you'd go to the lab and measure the wave speeds, you'll find different results, because these wave speeds are given in the *material coordinates*, and not in the deformed (lab) coordinates. Also, the absence of anisotropy in the shear wave-speed is a special case specific to this constitutive law and not a general feature of finite elasticity.