1. An incompressible elastic-perfect-plastic cylindrical rod, of Young’s modulus $E$, yield stress $\sigma_Y \ll E$, length $L$ and cross section $A$ is compressed/pulled under uniaxial stress along its axis until its length is multiplied by a factor $\lambda$. How much work did the external loading perform? How much of it was dissipated? Work in the regime that $|\lambda - 1| \ll 1$, but plastic deformation does occur.

2. Consider the setting shown in Fig 1a: three elastic-perfect-plastic rods with cross sectional area $A$ are connected with pins that can transfer only axial forces but no torques, and a vertical force $F$ is exerted on them. The top pins are held at fixed positions to the ceiling (but not at a fixed angle). All rods have Young’s modulus $E$ and yield stress $\sigma_Y \ll E$. When $F = 0$ the system is stress-free. Assume small deformations.

(a) Denote the vertical displacement of the loading point by $\Delta$. Calculate and plot $\Delta(F)$ (choose some values for the parameters you need). What is the maximal force $F_E$ for which the response is elastic? What is the maximal force $F_U$ that can be applied?

(b) Calculate the residual strains and stresses if the force is removed after the displacement was $\Delta$.

(c) Suppose no force is applied, but the temperature is increased (or decreased) by $\Delta T$. Calculate the minimal temperature difference $\Delta T_E$ that causes plastic deformation (assume $\alpha_T, \sigma_Y, E$ are $T$-independent).

(d) Bonus: repeat (a) for the case where there are 5 bars, or better yet, $2n + 1$. The setup is shown in Figure 1b. Assume the system is symmetric with respect to horizontal reflection.

3. In class, we’ve found the elasto-plastic solution for a spherical shell. We now look at some interesting aspects of the results.
(a) Examine numerically Eq. (11.38) from the lecture notes. For the case that \( b = 10a \), plot \( c \) as a function of \( p \). Can you analytically explain what happens when \( p \to p_U \)? (hint: yes you can).

(b) For the case that \( p = \sigma_Y \), plot \( c/a \) as a function of \( b/a \). What is the asymptotic value of \( c \) when \( b/a \to \infty \)?

(c) Find the displacement field \( u_r(r) \) (from symmetry, \( \vec{u} \) is a function of \( r \) only and other components vanish). Is the stress/strain/displacement field continuous/differentiable across the elasto-plastic boundary?

Guidance: In the elastic region, there’s a particularly simple relation between \( u_r \) and some of the strain components. In the plastic region, the volumetric part of the deformation is still elastic - we still have \( \text{tr} \sigma = K \text{tr} \epsilon \), where \( K \) is the bulk modulus.

4. Continuing our TA session, consider an elastic-perfect-plastic 2D annulus with internal and external radii \( a, b \), subject to internal pressure \( p \) and zero outer pressure, under plane-stress conditions. Use the Tresca yield criterion, and preform the analysis that was done in class for the case of a spherical shell:

(a) Find the stress field \( \sigma_{ij}(r) \), the minimal internal pressure that induces plastic flow \( (p_E) \), the ultimate pressure for which the entire annulus is plastic \( p_U \), and give an equation that determines the radius of the elasto-plastic boundary \( c \).

(b) Show that your solution is valid only if

\[
1 + \frac{c^2}{b^2} - \log \frac{c^2}{a^2} \geq 0 .
\]  

(1)

What happens if this criterion is not satisfied? Why is this problem not present in plane strain conditions?

(c) Considering this, what is the condition on \( a/b \) that ensures that \( p_U \) exists? Give an equation that describes, for a given value of \( a/b \), the maximal possible value of \( c \). What is this value when \( b/a \to \infty \)?