Dynamic fracture

1 Asymptotic near tip fields for in-plane dynamic cracks

We are looking for an asymptotic solution for the Lamè equation

$$(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})+\mu\nabla^2\boldsymbol{u}=\rho\ddot{\boldsymbol{u}}. \tag{1}$$

near a crack tip moving at a velocity v. Using the Helmholtz decomposition

$$\boldsymbol{u}=\nabla\phi+\nabla\times\boldsymbol{\psi} \tag{2}$$

for plane stress we obtain

$$c_d^2\nabla^2\phi=\ddot{\phi},\quad c_s^2\nabla^2\psi=\ddot{\psi}, \tag{3}$$

where ψ is the z component of $\boldsymbol{\psi}$ (the other two components vanish). For a steadily moving crack we are looking for a translational invariant solution, for which Eqs. (3) reduce to

$$\alpha_d^2\partial_x^2\phi+\partial_y^2\phi=0,\quad \alpha_s^2\partial_x^2\psi+\partial_y^2\psi=0\ , \tag{4}$$

with $\alpha_d^2\equiv 1-v^2/c_d^2$ and $\alpha_s^2\equiv 1-v^2/c_s^2$. These equations can be rewritten as

$$\alpha_d^2\left(\partial_x^2\phi+\partial_{\alpha_d y}^2\phi\right)=0,\quad \alpha_s^2\left(\partial_x^2\psi+\partial_{\alpha_s y}^2\psi\right)=0\ . \tag{5}$$

These are two Laplace's equations in the variables $\zeta_d=x+i\alpha_d y$ and $\zeta_s=x+i\alpha_s y$, which are coupled on the boundaries (remember the TA about Rayleigh waves?). The general solution is readily given as

$$\phi(r,\theta)=\Re\{F(\zeta_d)\},\quad \psi(r,\theta)=\Im\{G(\zeta_s)\}\ , \tag{6}$$

where F and G are analytic functions (recall that Laplace's equation is solved by a real or imaginary part of an analytic function).

In mode I the displacement potentials ϕ and ψ have the symmetries

$$\phi(r,\theta)=\phi(r,-\theta),\quad \psi(r,\theta)=-\psi(r,-\theta)\ . \tag{7}$$

For these symmetries to hold, we need to demand that F and G will have the following symmetries:

$$F(\bar{\zeta})=\overline{F(\zeta)},\quad G(\bar{\zeta})=\overline{G(\zeta)}\ . \tag{8}$$

To see that these requirements ensure the symmetries of Eq. (7), note that Eqs. (8) simply imply that the coefficients of the Laurent expansions of F and G (the complex generalization of the Taylor expansion of real functions) are all real. Writing now $\zeta_d=r_d e^{i\theta_d}$ and $\zeta_s=r_s e^{i\theta_s}$ we see that ϕ in Eq. (6) picks up the cosines and ψ picks up the sines, as required.

To determine F and G we should impose the free boundary conditions on the crack faces

$$\sigma_{yy}(r,\theta=\pm\pi)=\sigma_{xy}(r,\theta=\pm\pi)=0\ . \tag{9}$$

These components are written in terms of ϕ and ψ as

$$\begin{aligned}\sigma_{yy}(r,\theta)&=\mu\left[\frac{c_d^2}{c_s^2}\nabla^2\phi-2\partial_{xx}\phi-2\partial_{xy}\psi\right]\\ \sigma_{xy}(r,\theta)&=\mu\left[2\partial_{xy}\phi+\partial_{yy}\psi-\partial_{xx}\psi\right]\ .\end{aligned} \tag{10}$$

The second derivatives of ϕ and ψ, in terms of F and G, are

$$\begin{aligned}\partial_{xx}\phi&=\Re[F''(\zeta_d)], & \partial_{xx}\psi&=\Im[G''(\zeta_s)],\\ \partial_{yy}\phi&=\Re[-\alpha_d^2F''(\zeta_d)], & \partial_{yy}\psi&=\Im[-\alpha_s^2G''(\zeta_s)],\\ \partial_{xy}\phi&=\Re[i\alpha_dF''(\zeta_d)], & \partial_{xy}\psi&=\Im[i\alpha_sG''(\zeta_s)]\end{aligned} \tag{11}$$

Substituting into Eqs. (10) we obtain

$$\begin{aligned}\sigma_{yy}(r,\theta)&=-\mu\Re\left[(1+\alpha_s^2)F''(\zeta_d)+2\alpha_sG''(\zeta_s)\right]\\ \sigma_{xy}(r,\theta)&=-\mu\Im\left[2\alpha_dF''(\zeta_d)+(1+\alpha_s^2)G''(\zeta_s)\right]\ .\end{aligned} \tag{12}$$

Note that the polar coordinates r,θ are related to $r_{d,s},\theta_{d,s}$ according to

$$\begin{aligned}r_d&=r\sqrt{1-(v\sin\theta/c_d)^2}, & \tan\theta_d&=\alpha_d\tan\theta,\\ r_s&=r\sqrt{1-(v\sin\theta/c_s)^2}, & \tan\theta_s&=\alpha_s\tan\theta\ .\end{aligned}$$

We are now ready to solve the problem in a power series. The leading term in the expansion of σ is a square root, i.e.

$$F''(\zeta)=a\zeta^{-1/2},\quad G''(\zeta)=b\zeta^{-1/2}\ , \tag{13}$$

where a/b will be determined by the boundary conditions of Eq. (9). Substituting Eq. (13) into Eqs. (12), we obtain

$$\begin{aligned}\sigma_{yy}(r,\theta)&=-\mu\left[(1+\alpha_s^2)ar_d^{-1/2}\cos(\theta_d/2)+2\alpha_sbr_s^{-1/2}\cos(\theta_s/2)\right]\\ \sigma_{xy}(r,\theta)&=\mu\left[2\alpha_dar_d^{-1/2}\sin(\theta_d/2)+(1+\alpha_s^2)br_s^{-1/2}\sin(\theta_s/2)\right]\ .\end{aligned} \tag{14}$$

Noting that

$$r_{d,s}(\theta=\pm\pi)=r,\quad \theta_{d,s}(\theta=\pm\pi)=\pm\pi\ , \tag{15}$$

we observe that in Eqs. (12) $\sigma_{yy}(r,\pm\pi)$ vanishes identically for any a and b and that $\sigma_{xy}(r,\pm\pi)$ vanishes only if

$$a=(1+\alpha_s^2)A,\quad b=-2\alpha_dA\ . \tag{16}$$

A is some undetermined prefactor that will be related to the stress intensity factor. To proceed, by double integration (omitting terms that do not affect stress/strain) we readily obtain

$$F(\zeta)=(1+\alpha_s^2)\frac{4A\zeta^{3/2}}{3},\quad G(\zeta)=-2\alpha_d\frac{4A\zeta^{3/2}}{3}\ . \tag{17}$$

The stresses are then obtained by replacing Eq. (16) into (14):

$$\begin{aligned}
\sigma_{yy}(r,\theta) &= -\mu A\left[(1+\alpha_s^2)^2 r_d^{-1/2}\cos(\theta_d/2) - 4\alpha_s\alpha_d r_s^{-1/2}\cos(\theta_s/2)\right] \\
\sigma_{xy}(r,\theta) &= \quad \mu A\left[2\alpha_d(1+\alpha_s)^2 r_d^{-1/2}\sin(\theta_d/2) - 2(1+\alpha_s^2)\alpha_d r_s^{-1/2}\sin(\theta_s/2)\right] \ .
\end{aligned} \tag{18}$$

The stress intensity factor is defined as

$$K_I = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{yy}(r,\theta{=}0;v) \tag{19}$$

and hence

$$A = \frac{K_I}{\mu D(v)\sqrt{2\pi}} \ , \tag{20}$$

where

$$D(v) = 4\alpha_s\alpha_d - (1+\alpha_s^2)^2 \tag{21}$$

is the Rayleigh function (the very same function as Eq. (29) from the TA#6 about Rayleigh waves. It vanishes at $v = c_R$).

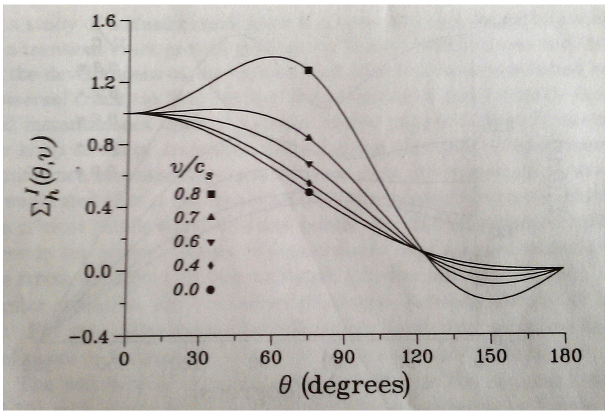
Collecting everything, we end up with

$$\begin{aligned}
u_x(r,\theta) =& \frac{2K_I}{\mu\sqrt{2\pi}D(v)}\left[(1+\alpha_s^2)r_d^{1/2}\cos\left(\frac{\theta_d}{2}\right) - 2\alpha_d\alpha_s r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right)\right], \\
u_y(r,\theta) =& -\frac{2K_I\alpha_d}{\mu\sqrt{2\pi}D(v)}\left[(1+\alpha_s^2)r_d^{1/2}\sin\left(\frac{\theta_d}{2}\right) - 2r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right)\right] \ .
\end{aligned} \tag{22}$$

1.1 Properties of the solution

1. In the limit of small velocities, $v \ll c_s$, we want to recover the static field presented in the lecture notes. However, this is not so simple. Look at the numerator of u_x in Eq. (22): when $v \to 0$, we have $\alpha_i \to 1$ and $r_i \to r$ so the the numerator goes to zero. Luckily, the Rayleigh function $D(v)$ also goes to zero, and they do so at the same rate of v^2 (it is easily that both $D(v)$ and the numerator of u_x go as v^2 to leading order in v, since they depend on v only through functions of v^2).

2. From the stress field, we can calculate the hoop stress $\sigma_{\theta\theta}$. It turns out that it depends on velocity in an interesting manner, as seen in this figure (from Freund 1990):



One sees that for $v \sim 0.6c_s$ the maximal hoop stress occurs for $\theta = 0$ which is what we would expect. However, at higher velocities there seems to be a "first order phase transition" where the maximum jumps to a finite angle, around 60°. This was first noticed by Yoffe (1951) and was suggested as a mechanism for the branching instability. the angle between bifurcating cracks is generically much less than 60° and anyhow we already now that this theory cannot tell us how things break because the theory fails at the process zone.

3. Explicit calculation of the J-integral gives:

$$G(v) = \frac{v^2 \alpha_d K_I^2}{2\mu c_s^2 D(v)} \equiv A_I(v) \frac{K_I^2}{E} \ . \tag{23}$$

   The function $A_I(v)$ is universal, i.e. independent of the geometry and loading of a given crack problem. As before, all of the details of a given problem are introduced by a single non-universal number, K_I.

4. In the limit $v \to 0$ we have $A_I(v) \to 1$, and then Eq. (23) coincides with the quasi-static problem that we solved in class.

5. Since $D(c_R) = 0$, $A_I(v)$ clearly diverges as v approaches the Rayleigh velocity. What happens with G? Since energy balance implies $G(v) = \Gamma(v)$ for all velocities, and since $\Gamma(v)$ is obviously finite, we see from this analysis that K_I should be (and indeed is) velocity dependent. This compensates for this divergence so as to allow the equality for all velocities.

2 The J-integral

There is one important class of situations in which G (and J) are independent of the path (contour) $\mathcal{C}$ for *any* $\mathcal{C}$, irrespective of whether it is inside or outside the universal singular region (= K-field). This happens for steady state crack propagation, i.e. when v is time-independent. Let us prove it.

2.1 Momentum-energy balance as a continuity equation

Before we start, we note that since the J-integral measures energy flux into the crack, we want to present our equation of motion as a continuity equation for the elastic and kinetic energy (note: we already did that in the Q&A session, but there are no ntes from that session so here it is again). The generic way to do it is to multiply the equation of motion (30) by $\dot{u}_i$ and sum over the index i. In linear elasticity this reads

$$\rho \ddot{u}_i \dot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} \dot{u}_i \tag{24}$$

The right-hand-side is exactly the time derivative of the kinetic energy T:

$$\rho \ddot{u}_i \dot{u}_i = \partial_t \left(\frac{1}{2} \rho \dot{u}_i \dot{u}_i \right) = \frac{\partial T}{\partial t} \tag{25}$$

The right-hand-side of Eq. (24) can be manipulated as follows:

$$\frac{\partial}{\partial x_j}\sigma_{ij}\dot{u}_i = \frac{\partial}{\partial x_j}\left(\sigma_{ij}\dot{u}_i\right) - \sigma_{ij}\frac{\partial^2 u_i}{\partial t \partial x_j} = \frac{\partial}{\partial x_j}\left(\sigma_{ij}\dot{u}_i\right) - \sigma_{ij}\dot{\epsilon}_{ij} \ , \tag{26}$$

where the symmetry of σ_{ij} was used in the last transition. The first term is a divergence of what we will identify as the energy flux. The second term is exactly the time derivative of the internal energy $U = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$. To see this, note that

$$\sigma_{ij}\dot{\epsilon}_{ij} = C_{ijkl}\dot{\epsilon}_{ij}\epsilon_{kl} = \frac{\partial}{\partial t}\left(\frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl}\right) = \frac{\partial U}{\partial T} \tag{27}$$

where C_{ijkl} is the stiffness tensor and its symmetry was used. Combining all these together we obtain finally the continuity equation

$$\frac{\partial}{\partial t}(T+U) = \frac{\partial}{\partial x_j}\left(\sigma_{ij}\dot{u}_i\right) \ . \tag{28}$$

and thus $\sigma_{ij}\dot{u}_i$ is the elastic energy flux.

2.2 Path-independence of the J-integral

Now back to fracture. We remind you of the definition of the J integral,

$$G = \frac{1}{v}\int_{\mathcal{C}}\left[s_{ij}\frac{\partial u_i}{\partial t}n_j + (U+T)\,v n_x\right] ds \ . \tag{29}$$

The finite-elasticity analog of Eq. (24) that we use is

$$\frac{\partial s_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2} \ , \tag{30}$$

where s_{ij} is the first Piola-Kirchhoff stress tensor, ρ is the mass density in the reference (undeformed) configuration, u_i are the components of the displacement vector and x_j are the reference (undeformed) coordinates. The analog of Eq. (28) is

$$\frac{\partial}{\partial x_j}\left(s_{ij}\frac{\partial u_i}{\partial t}\right) = \frac{\partial}{\partial t}(U+T) \ . \tag{31}$$

The crucial point is that for steady state propagation we have

$$\partial_t = -v\partial_x \ , \tag{32}$$

for which Eq. (29) is transformed to

$$G = \int_{\mathcal{C}}\left[(U+T)\,n_x - s_{ij}\frac{\partial u_i}{\partial x}n_j\right] ds \ . \tag{33}$$

We should prove that the latter is independent of $\mathcal{C}$. For that aim, consider two contours that start on one side of the crack surface, encircle the crack tip and end on the other side, as in in Fig. 1. Now construct a *closed path* $\partial\Omega$ by going from A $\rightarrow$ ① $\rightarrow$ B

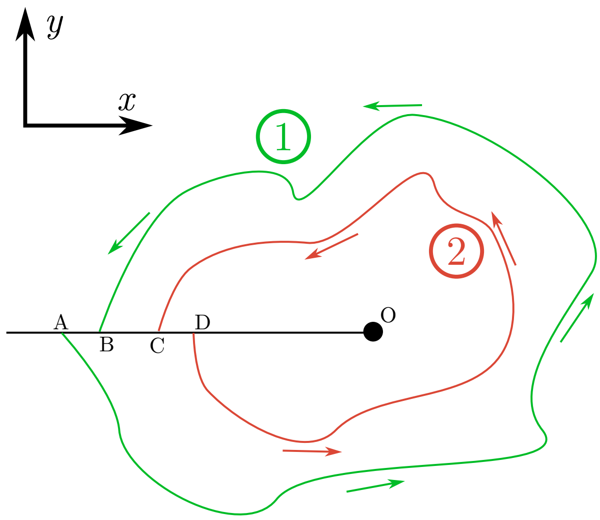


Figure 1: Two contours that start on one side of the crack surface and end on the other side.

$\rightarrow$ C $\rightarrow$ ② $\rightarrow$ D $\rightarrow$ A, where the two segments B $\rightarrow$ C and D $\rightarrow$ A lie along the two crack faces. Note that we traverse contour ② in the opposite direction.

Denoting the area enclosed in $\partial\Omega$ by Ω, the proof is as follows

$$\begin{aligned} &\int_{\partial\Omega} (U+T)\, n_x ds = \int_{\Omega} \frac{\partial}{\partial x}(U+T)\, dA = -v^{-1}\int_{\Omega}\frac{\partial}{\partial t}(U+T)\, dA \\ &= -v^{-1}\int_{\Omega}\frac{\partial}{\partial x_j}\left(s_{ij}\frac{\partial u_i}{\partial t}\right) dA = \int_{\Omega}\frac{\partial}{\partial x_j}\left(s_{ij}\frac{\partial u_i}{\partial x}\right) dA = \int_{\partial\Omega} s_{ij}\frac{\partial u_i}{\partial x} n_j ds \ , \end{aligned} \tag{34}$$

This implies

$$\int_{\partial\Omega}\left[(U+T)\, n_x - s_{ij}\frac{\partial u_i}{\partial x} n_j\right] ds = 0 \ , \tag{35}$$

which completes the proof. To see why, recall that the two segments that are lie along the crack faces do not contribute to the integral of Eq. (35) because there we have $n_x = 0$ and $s_{ij}n_j = 0$ (due to the traction-free boundary conditions). This implies that the sum of the two line integrals over contours ① and ② vanishes, and since ② was traversed in the opposite direction, the integral over ① and ② are equal. Since the contours were genera, we can conclude that for every path that starts at the lower crack face, encircles the tip in the counterclockwise direction and ends at the upper crack face, the value of the integral is the same.

In particular, this result is valid for quasi-static cracks. Though the ideas about configurational forces were developed by Eshelby in the early 1950's, the integral discussed above in the context of fracture theory was independently derived by Cherepanov (1967) and Rice (1968) for quasi-static cracks and then extended to the dynamic case in the 1970's.

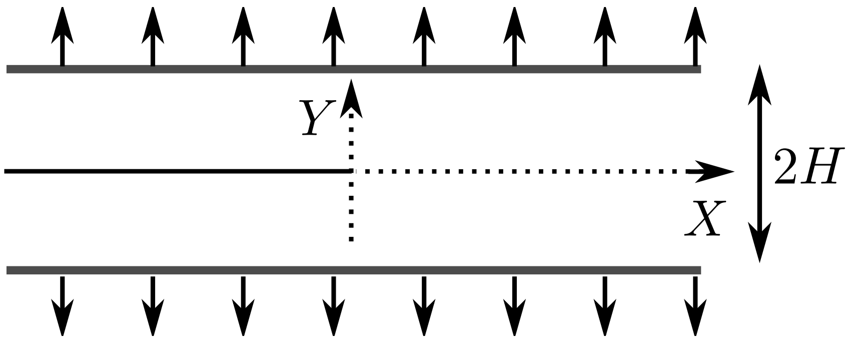


Figure 2: Strip with crack

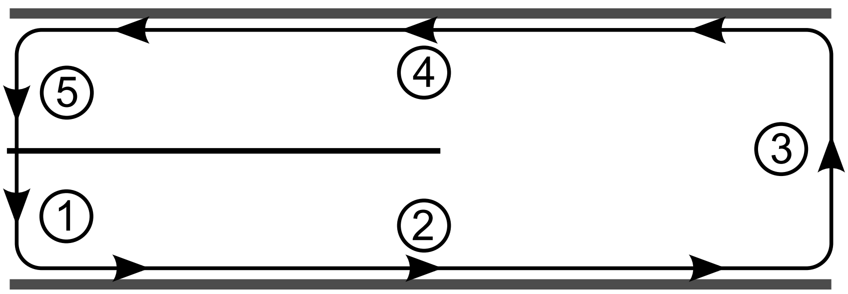


Figure 3: The integration contour.

2.3 Example of usage

Consider the system described in the Fig. 2: a plane-stress linear elastic strip (with Young's modulus E and Poisson's ratio ν) spanning $-\infty < x < \infty$ and $-H < y < H$, containing a semi-infinite crack propagating along $y = 0$ at a speed v. The strip is loaded as follows

$$u_y(x, y{=}\pm H, t) = \pm\delta \quad \text{and} \quad u_x(x, y{=}\pm H, t) = 0 \ . \tag{36}$$

Let's try to calculate the energy release rate into the crack tip region using the J-integral. In steady-state propagation we can use the fact that the integral is path-independent and we can choose a convenient contour, that is, a path that most of it will give simple contributions to the integral. The one we choose is shown in Fig. 3, where we take the lateral parts of the contour (i.e. segments ①, ③ and ⑤) to $x = \pm\infty$.

The integration is very easy along this path: Along the segments ① and ⑤ both the stress field and the velocity vanish (why?), therefore they do not contribute. Along segments ② and ④ n_x is zero, and u is constant so they too vanish. Thus, the only contribution comes from segment ③, and it equals $2UH$, because u goes to a constant and T vanishes.

The physical picture here is simple and neat: a piece of material at $x \to \infty$ is statically stretched with energy density U. After the crack passes, when the same piece of material approaches $x \to -\infty$ it releases all its energy and is completely stress free. Where did the energy go to? To the only place in the theory that can have dissipation, which is the process zone at the crack tip. Therefore the energy release rate at steady state, J, is exactly the amount of energy that is stored in a piece of material long before the crack touches it.