Kinematics, Strain and stress

1 An Eularian strain measure

In this section we'll introduce the Eularian analogue of the Green-Lagrange strain tensor defined in class. We'll then see in a concrete example how the two behave.

You are reminded that the Green-Lagrange strain tensor $\boldsymbol{E}$ was defined as a measure of the change in lengths squared. We examined a line element (dX_1, dX_2, dX_3), of total length $d\ell$, which changed under the deformation to be (dx_1, dx_2, dx_3), of total length $d\ell'$. $\boldsymbol{E}$ was defined by the relation

$$(d\ell')^2 - (d\ell)^2 = 2\, d\boldsymbol{X}^T \boldsymbol{E} d\boldsymbol{X} \ , \tag{1}$$

that is, $\boldsymbol{E}$ is defined with respect to the material (=reference =undeformed) coordinates. It was shown in class that $\boldsymbol{E}$ can be simply expressed in terms of the deformation gradient:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \ . \tag{2}$$

We now ask, what is the equivalent strain measure, in terms of the spatial (=laboratory =deformed) coordinates? We use the relation $d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$ to write

$$(d\ell')^2 - (d\ell)^2 = 2\, d\boldsymbol{X}^T\, \boldsymbol{E}\, d\boldsymbol{X} = 2\, d\boldsymbol{x}^T\, \boldsymbol{F}^{-T}\, \boldsymbol{E}\, \boldsymbol{F}^{-1}\, d\boldsymbol{x} \ , \tag{3}$$

and see that we can define the Eularian analogue of $\boldsymbol{E}$ as $\boldsymbol{e} = \boldsymbol{F}^{-T}\, \boldsymbol{E}\, \boldsymbol{F}^{-1}$. $\boldsymbol{e}$ is called the *Euler-Almansi strain tensor*. Explicitly, it is given by

$$\boldsymbol{e} = \boldsymbol{F}^{-T}\, \boldsymbol{E}\, \boldsymbol{F}^{-1} = \frac{1}{2}\boldsymbol{F}^{-T}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right)\boldsymbol{F}^{-1} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) \ . \tag{4}$$

It is easy to see that $\boldsymbol{e}$ is symmetric. In class we've shown that if λ_i are the principal stretches, then $\boldsymbol{E}$ can be written as

$$\boldsymbol{E} = \frac{1}{2}\left(\lambda_i^2 - 1\right)\boldsymbol{M}_i \otimes \boldsymbol{M}_i \ . \tag{5}$$

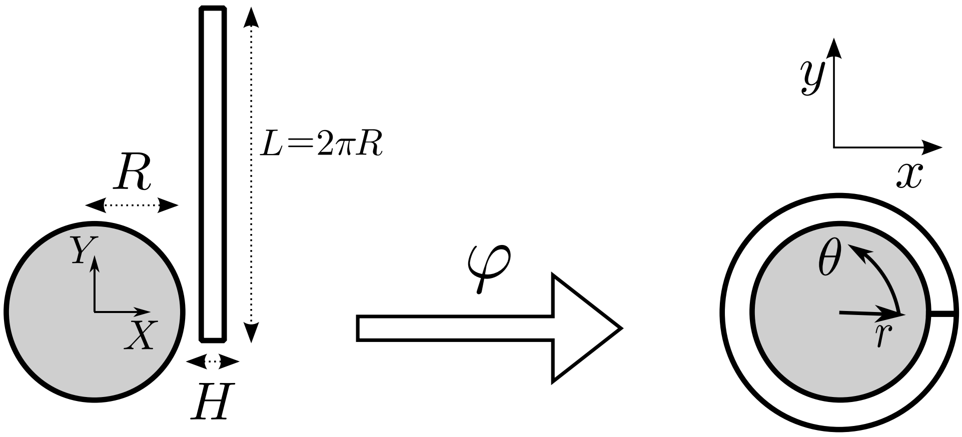
where $\boldsymbol{M}_i$ are the directions (in $\boldsymbol{X}$ coordinates) of the principal stretches (that's Eq. (3.24) from the lecture notes). In much the same way, $\boldsymbol{e}$ is

$$\boldsymbol{e} = \frac{1}{2}\left(1 - \lambda_i^{-2}\right)\boldsymbol{m}_i \otimes \boldsymbol{m}_i \ . \tag{6}$$

where $\boldsymbol{m}_i$ are the directions (in $\boldsymbol{x}$ coordinates) of the principal stretches.

As a side note, we remark that the process of taking a Lagrangian quantity and writing its Eularian analogue is called "push-forward". The inverse operation, of calculating the Lagrangian analogue of an Eulerian quantity, is called "pull-back". These are fundamental concepts in differential geometry. As above, it is generally true that pushing forward a (covariant) Lagrangian tensor is done by multiplying form the left and right by $\boldsymbol{F}^{-T}$ and $\boldsymbol{F}^{-1}$, respectively. If you're interested in the relation between differential geometry and elasticity, have a look at Elasticity & Geometry by Basil Audoly and Yves Pomeau. Also, An introduction to differential geometry with applications to elasticity by Philippe Ciarlet is a good introduction to both elasticity and differential geometry. In addition, you might want to have a chat with Efi Efrati who's been working on this for years. Lastly, this upcoming Sunday Eran Sharon is giving the Clore Seminar on this subject. He's a very good speaker and it will be worth your while (March 27 at 13:00. Free food at 12:45. Just sayin').

1.1 Application of strain measures, rotation invariance



In this exercise, we'll look at the differences between three strain tensors: the Green-Lagrange tensor $\boldsymbol{E}$, the Cauchy (linearized) tensor $\boldsymbol{\varepsilon}$, and the Euler-Almansi tensor $\boldsymbol{e}$. To this end, consider a thin rod of length $L = 2\pi R$ which is wrapped around a circle or radius R, like in the figure. By "thin" we mean $L \ll H$. The *motion* $\boldsymbol{\varphi}(X, Y)$ is defined by

$$x(X,Y) = X\cos\left(2\pi\frac{Y}{L}\right)\ ,$$
$$y(X,Y) = X\sin\left(2\pi\frac{Y}{L}\right)\ , \tag{7}$$
$$\boldsymbol{\varphi}(X,Y) = \begin{pmatrix} X\cos\left(2\pi\frac{Y}{L}\right) \\ X\sin\left(2\pi\frac{Y}{L}\right) \end{pmatrix}\ .$$

This might be more intuitive if we introduce the shorthand notations

$$r = X\ , \qquad\qquad \theta = 2\pi\,\frac{Y}{L} = \frac{Y}{R}\ , \tag{8}$$

and then the motion takes the form

$$\phi(X,Y) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}\ , \tag{9}$$

but we stress that r, θ here are only shorthand notations and *not coordinates* - we are strictly working in Cartesian coordinates.

Note that what we do here is not the usual scenario in this kind of problems. Usually, the motion is not given but rather has to be solved for. We usually know only the boundary conditions – the displacements or forces applied on the body's surface – and the motion in the bulk is calculated by solving the relevant continuum equations. However, since we didn't learn about these equations yet, we specify the motion at the outset. The purpose here is only to see the differences between the strain measures.

The deformation gradient is given by

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}} \begin{pmatrix} x(X,Y) \\ y(X,Y) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -h\sin\theta \\ \sin\theta & h\cos\theta \end{pmatrix} \tag{10}$$

where the notation $h = X/R$ is introduced. Note that $h-1 = \frac{X-R}{R} \approx H/L \ll 1$ is a small number. The strain tensors are given by

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\left(h^2-1\right) \end{pmatrix} \tag{11}$$

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) = \frac{1}{2}\left(1-h^{-2}\right)\begin{pmatrix} \sin^2(\theta) & -\cos(\theta)\sin(\theta) \\ -\cos(\theta)\sin(\theta) & \cos^2(\theta) \end{pmatrix} \tag{12}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{F} + \boldsymbol{F}^T - 2\boldsymbol{I}\right) = \begin{pmatrix} \cos(\theta)-1 & -\frac{1}{2}(h-1)\sin(\theta) \\ -\frac{1}{2}(h-1)\sin(\theta) & h\cos(\theta)-1 \end{pmatrix} \tag{13}$$

Note that in the expression for $\boldsymbol{E}$, h is shorthand for X/R, and θ is shorthand for $2\pi Y/R$ (incidentally, $\boldsymbol{E}$ is independent of θ). However, in the expression for $\boldsymbol{e}$, h is shorthand for $\sqrt{x^2+y^2}/R$, and θ stands for $\tan^{-1}(y/x)$. This is an important distinction – $\boldsymbol{E}$ is given in terms of X and Y, and $\boldsymbol{e}$ in terms of x and y!

Now let's analyze these expressions. We see that $\boldsymbol{E}$ is diagonal in the $\boldsymbol{X}$ coordinates and is independent of Y (or θ). $\boldsymbol{e}$ is a bit more complicated, but actually it can be written in a much simpler manner. If we define the rotation matrix $\boldsymbol{R}$ as

$$\boldsymbol{R} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} , \tag{14}$$

then we can write $\boldsymbol{e}$ as

$$\boldsymbol{e} = \boldsymbol{R}^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\left(1-h^{-2}\right) \end{pmatrix} \boldsymbol{R} . \tag{15}$$

So we see that like $\boldsymbol{E}$, $\boldsymbol{e}$ has one eigenvalue which is always zero, and the other eigenvalue is independent of θ (but not of y!). However, the principal directions in the spatial coordinates are rotating with θ, which is not surprising. The principal directions are indeed the polar and radial directions.

As for $\boldsymbol{\varepsilon}$, things are much less neat. Its eigenvalues are

$$\varepsilon_1 = \frac{1}{2}\left((1+h)(\cos\theta - 1)\right) , \qquad \varepsilon_2 = \frac{1}{2}\left(h - 3 + (1+h)\cos\theta\right) , \tag{16}$$

and the principal directions are also a mess.

It seems that $\boldsymbol{E}$ and $\boldsymbol{e}$ are somehow "the same" in some sense, but that $\boldsymbol{\varepsilon}$ is fundamentally different: To begin with, $\boldsymbol{E}$ and $\boldsymbol{e}$ are small everywhere (remember that $h^2-1\approx 1-h^{-2}\approx O(H/L)\ll 1$) and the diagonal elements of $\boldsymbol{\varepsilon}$ are not; $\boldsymbol{E}$ and $\boldsymbol{e}$ have a 0 eigenvalue, while $\boldsymbol{\varepsilon}$ does not (except for $\theta=0$ or 2π); and most importantly, $\boldsymbol{E}$ and $\boldsymbol{e}$ are θ-independent (in a proper sense) and $\boldsymbol{\varepsilon}$ is not.

This is very weird: for $X=R$ (that is, $h=1$) we have $\boldsymbol{E}=\boldsymbol{e}=0$, but $\boldsymbol{\varepsilon}\neq 0$!! Aren't they supposed to be equal to linear order? Or at least agree on whether they vanish or not? Which of the above is better?

The error lies in $\boldsymbol{\varepsilon}$! The system is indeed θ-invariant. The physical picture you should have in mind is that each "layer" in the rod (i.e. constant X), which was initially of length L, is stretched and attains the length $2\pi X$ in the deformed configuration. The ratio of the two lengths, $2\pi X/L=h$, is the stretch in the Y direction. In the X direction the material is not stretched. Note that the eigenvalues of $\boldsymbol{E}$, and $\boldsymbol{e}$ are exactly what you should expect, if you happen to remember Eqs. (5) and (6). Furthermore, the principal directions are also exactly what we should expect: they are indeed X and Y, and in the deformed coordinates they are $\hat{r}$ and $\hat{\theta}$. Since we work in Cartesian spatial coordinates, the principal directions are simply given by rotating the (x,y) directions by an angle θ.

So we got a good intuition as to why $\boldsymbol{E}$ and $\boldsymbol{e}$ do exactly what we expect them to, and all is well and nice. But what goes wrong in $\boldsymbol{\varepsilon}$? The answer is that the displacement gradient $\boldsymbol{F}$ is not small, due to the finite large rotations. Sadly, $\boldsymbol{\varepsilon}$ is not rotationally invariant. To see this, consider the following rigid body rotation

$$\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= Y\cos\theta + X\sin\theta \end{aligned} \tag{17}$$

We have

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{\varphi} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{18}$$

and

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{F}+\boldsymbol{F}^T\right) - \boldsymbol{I} = \begin{pmatrix} \cos\theta-1 & 0 \\ 0 & \cos\theta-1 \end{pmatrix}, \qquad \boldsymbol{E}=\boldsymbol{e}=\boldsymbol{0} \tag{19}$$

That means that $\boldsymbol{\varepsilon}\neq 0$ for finite rotations. For infinitesimal rotations, $\theta\ll 1$, we have

$$\varepsilon_{xx}\simeq\varepsilon_{yy}\simeq O(\theta^2) \qquad\qquad \varepsilon_{xy}=0\ . \tag{20}$$

So $\boldsymbol{\varepsilon}=0$ to linear order in θ, i.e. only for *infinitesimal rotations*. This is rather trivial as $\boldsymbol{\varepsilon}$ is a linearized version of the rotationally invariant strain measures $\boldsymbol{E}$ and $\boldsymbol{e}$ (they are the same to linear order). We see that the large values of $\boldsymbol{\varepsilon}$ at θ's far from zero do not stem from physical stretches in the material, but rather from rotation of the axes, which has no physical significance.

A word of caution: Note that $\boldsymbol{\varepsilon}$ is the "plain vanilla" strain measure. If you go to Rami Levy and ask for a strain measure, this is what they'll give you. It is the absolute standard in the majority of works in linear elasticity and is often presented as the "natural" one. Beware!

2 Symmetry of Cauchy's stress tensor

In this section, we'll see why the Cauchy stress tensor must be symmetric. We'll do this in two ways: the first is a bit technical and uses the machinery of continuum theories, and the second is intuitive and physically transparent. I hope that you'll learn to appreciate both, and remember that often the easy way can only be traveled after the hard way has been explored and the answer is known...

To this end, we will need to use Reynold's transport theorem. In class you have proven that

$$\frac{D}{Dt}\int_{\Omega}\psi(\boldsymbol{x},t)d\boldsymbol{x}^3=\int_{\Omega}\Big[\partial_t\psi(\boldsymbol{x},t)+\nabla_{\boldsymbol{x}}\cdot\Big(\psi(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\Big)\Big]d\boldsymbol{x}^3\ . \tag{21}$$

You'll be glad to know that there is a more useful version of this theorem. Since in the theorem ψ can by any field, one can replace it by $\rho\psi$. Then, using Leibniz's rule $\nabla(fg)=f\nabla g+g\nabla f$, we get (I omit all the arguments of the functions for readability. Remember that everything is a function of $(\boldsymbol{x},t)$):

$$\begin{aligned}\frac{D}{Dt}\int_{\Omega}\rho\psi d\boldsymbol{x}^3&=\int_{\Omega}\left[\partial_t(\rho\psi)+\psi\boldsymbol{v}\nabla_{\boldsymbol{x}}\cdot(\rho)\right]d\boldsymbol{x}^3\\&=\int_{\Omega}\Big[\rho\Big(\partial_t\psi+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\psi\Big)+\psi\Big(\partial_t\rho+\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})\Big)\Big]d\boldsymbol{x}^3\end{aligned} \tag{22}$$

Note that the expression in the first brackets is exactly $\frac{D}{Dt}\psi$. Also, the term in the second brackets vanishes identically due to mass conservation (Eq. (4.5) in Eran's lecture notes). Thus, we conclude that Reynold's theorem can be reformulated in a more pleasant way:

$$\frac{D}{Dt}\int_{\Omega}\rho(\boldsymbol{x},t)\psi(\boldsymbol{x},t)d\boldsymbol{x}^3=\int_{\Omega}\rho(\boldsymbol{x},t)\frac{D}{Dt}\psi(\boldsymbol{x},t)d\boldsymbol{x}^3 \tag{23}$$

Very loosely speaking, this means that in the material coordinates, the operator $\frac{D}{Dt}(\cdot)$ commutes with the operator $\int_{\Omega}\rho(\boldsymbol{x},t)(\cdot)d\boldsymbol{x}^3$. Remember that ψ can be a tensor of any rank.

Physically, the symmetry of Cauchy's stress tensor is the local version of the conservation of angular momentum. To see this, we first define the total angular momentum $\boldsymbol{J}$ (do not confuse with the Jacobian J):

$$\boldsymbol{J}=\int_{\Omega}\boldsymbol{r}\times\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)d\boldsymbol{x}\ . \tag{24}$$

Then, we apply Reynold's theorem to this definition, getting

$$\frac{D\boldsymbol{J}}{Dt}=\int_{\Omega}\rho\ (\dot{\boldsymbol{r}}\times\boldsymbol{v}+\boldsymbol{r}\times\dot{\boldsymbol{v}})\,d\boldsymbol{x}\ =\int_{\Omega}\rho\,\boldsymbol{r}\times\dot{\boldsymbol{v}}\,d\boldsymbol{x}\ , \tag{25}$$

where we used the fact that $\dot{\boldsymbol{r}}\times\boldsymbol{v}=\boldsymbol{v}\times\boldsymbol{v}=0$. Newton's second law says that

$$\frac{D\boldsymbol{J}}{Dt}=\int_{\Omega}(\boldsymbol{r}\times\boldsymbol{b})d\boldsymbol{x}+\int_{\partial\Omega}(\boldsymbol{r}\times\boldsymbol{t})ds \tag{26}$$

Where $\boldsymbol{b},\boldsymbol{t}$ are the body force and traction fields, respectively. We use the relation $\boldsymbol{t}=\boldsymbol{\sigma}\boldsymbol{n}$, and equate both expressions:

$$\int_{\Omega}\boldsymbol{r}\times[\rho\partial_t\boldsymbol{v}-\boldsymbol{b}]-\int_{\partial\Omega}(\boldsymbol{r}\times\boldsymbol{\sigma}\boldsymbol{n})ds=0\ . \tag{27}$$

Here we use a little lemma:

Lemma: Let $\boldsymbol{u}, \boldsymbol{A}$ be vector and tensor fields defined in the region Ω. Then

$$\int_{\partial\Omega} \boldsymbol{u} \times \boldsymbol{A}\boldsymbol{n}\, ds = \int_{\Omega} \left[\boldsymbol{\mathcal{E}} : (\operatorname{grad} \boldsymbol{u})\boldsymbol{A}^T + \boldsymbol{u} \times \operatorname{div} \boldsymbol{A}\right] dv \tag{28}$$

where $\boldsymbol{\mathcal{E}}$ is the Levi-Civita tensor. To see this, write in index notation

$$\int_{\partial\Omega} \boldsymbol{u} \times \boldsymbol{A}\boldsymbol{n}\, ds = \int_{\partial\Omega} \mathcal{E}_{ijk} u_j (\boldsymbol{A}\boldsymbol{n})_k\, ds = \int_{\partial\Omega} \mathcal{E}_{ijk} u_j A_{kl} n_l\, ds \tag{29}$$

$$= \int_{\partial\Omega} (\boldsymbol{\mathcal{E}} : \boldsymbol{u}\boldsymbol{A})_{il}\, n_l\, ds = \int_{\Omega} \operatorname{div} (\boldsymbol{\mathcal{E}} : \boldsymbol{u}\boldsymbol{A})\, dv \tag{30}$$

$$= \int_{\Omega} \partial_l \left(\mathcal{E}_{ijk} u_j A_{kl}\right) dv = \int_{\Omega} \Big[\underbrace{\mathcal{E}_{ijk} (\partial_l u_j) A_{kl}}_{\boldsymbol{\mathcal{E}}:(\operatorname{grad} \boldsymbol{u})\boldsymbol{A}^T} + \underbrace{\mathcal{E}_{ijk} u_j (\partial_l A_{kl})}_{\boldsymbol{u}\times\operatorname{div} \boldsymbol{A}} \Big] dv \tag{31}$$

We now plug that into (27) to get

$$\int_{\Omega} \boldsymbol{r} \times [\rho \partial_t \boldsymbol{v} - \boldsymbol{b} - \operatorname{div} \boldsymbol{\sigma}]\, dv - \int_{\Omega} \boldsymbol{\mathcal{E}} : \boldsymbol{\sigma}^T\, dv = 0 \tag{32}$$

The first integrand vanishes identically, as this is an equation of motion. The second one can be integrated on an arbitrary volume and so we see that $\boldsymbol{\mathcal{E}} : \boldsymbol{\sigma}^T = 0$, or in other words, $\boldsymbol{\sigma}$ is symmetric.

The second, "easier", way is as follows. Examine the torque applied on a small cube of linear size L. The torque scales as $(\sigma_{xy} - \sigma_{yx})\, L^3$ while the moment of inertia goes like $ML^2 \simeq \rho L^5$. Therefore, as one takes smaller and smaller cubes the angular acceleration diverges unless $\boldsymbol{\sigma}$ is symmetric.