1 NOTATIONS

In the following notes, we will reserve the Greek letters $\alpha, \beta, \gamma, \ldots$ for tensor subscripts, while the Latin letters $i, j, k, \ldots$ will be used for particles and Eshelby inclusions.

2 NUMERICAL SIMULATIONS

To prepare quality data for the present discussion we have performed two-dimensional (2D) Molecular Dynamics simulations on a binary system which is an excellent glass former and is known to have a quasi-crystalline ground state [1, 2]. Each atom in the system is labeled as either “small” (S) or “large” (L) and all the particles interact via Lennard Jones (LJ) potential. All distances $|r_i - r_j|$ are normalized by $\lambda_{SL}$, the distance at which the LJ potential between the two species becomes zero and the energy is normalized by $\epsilon_{SL}$ which is the interaction energy between two species. Temperature was measured in units of $\epsilon_{SL}/k_B$ where $k_B$ is Boltzmann’s constant. For detailed information on the model potential and its properties, we refer the reader to Ref [1]. The number of particles in our simulations is 10000 at a number density $n = 0.985$ with a particle ratio $N_L/N_S = (1 + \sqrt{5})/4$. The mode coupling temperature $T_{MCT}$ for this system is known to reside close to 0.325. All particles have identical mass $m_0 = 1$ and time is normalized to $t_0 = \sqrt{\epsilon_{SL} \lambda_{SL}^2 / m_0}$. For the sake of computational efficiency, the interaction potential is smoothly truncated to zero along with its first two derivatives at a cut-off distance $r_c = 2.5$.

To prepare the glasses, we first start from a well equilibrated liquid at a high temperature of $T = 1.2$ which is supercooled to $T = 0.35$ at a quenching rate of $3.4 \times 10^{-3}$. Secondly, we then equilibrate these supercooled liquids for times greater than $20\tau_{rel}$, where $\tau_{rel}$ is the time taken for the self intermediate scattering function to become 1% of its initial value. Lastly, following this equilibration, we quench these supercooled liquids deep into the glassy regime at a temperature of $T = 0.01$ at a reduced quench rate of $3.2 \times 10^{-6}$. Following this quench we take the glass to mechanical equilibrium (nearest energy minima) by a conjugate gradient energy minimization. At this point we start shearing our glasses under athermal quasi-static conditions.

3 TWO DIMENSIONAL CIRCULAR INCLUSION

Consider an elastic solid having a volume $V$ and surface area $S$ [Fig. 1]. The material will be assumed to be homogeneous with an elastic stiffness tensor given by $C_{\alpha\beta\gamma\delta}$. Let a sub-volume $V_0$ with surface area $S_0$ undergo a uniform permanent (inelastic) deformation, such as a structural phase transformation. The material inside $V_0$ is called an inclusion and the material outside is called the matrix. If we could remove this inclusion from its surrounding material then it would attain a state of uniform strain and zero stress. Such a stress free strain is referred to as the eigen-strain $\epsilon^*_{\alpha\beta}$. The eigen stress is then given by $\sigma^*_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\epsilon^*_{\gamma\delta}$.

In reality, the inclusion is surrounded by the matrix. Therefore, it is not able to reach the state of eigen-strain and zero stress. Instead, both the inclusion and the matrix will deform and experience an elastic stress field. The Eshelby’s transformed inclusion problem [3] is to solve the stress, strain and displacement fields both in the inclusion and in the matrix.

We consider a 2D circular inclusion that has been strained into an ellipse using an eigen-strain $\epsilon^*_{\alpha\beta}$ and which allows for a volume change ($\epsilon^*_{\nu\nu} \neq 0$). A general expression for such a tensor
Figure 1: Cartoon showing an elastic medium of volume $V$ and surface area $S$. Inside the medium a small ellipsoidal region (volume $V_0$ and surface area $S_0$) undergoes an irreversible (plastic) deformation. The material inside $V_0$ is called as the inclusion and the material outside is referred to as the matrix.

can be written in terms of a unit eigen vectors ($\hat{n}, \hat{k}$) and corresponding eigen values ($\zeta_n, \zeta_k$) as follows:

$$\epsilon^*_\alpha\beta = \zeta_n n_\alpha n_\beta + \zeta_k k_\alpha k_\beta \quad (1)$$

Using orthogonality of the eigen directions: $n_\alpha n_\beta + k_\alpha k_\beta = \delta_{\alpha\beta}$, we get:

$$\epsilon^*_\alpha\beta = \frac{(\zeta_n - \zeta_k)}{2} (2n_\alpha n_\beta - \delta_{\alpha\beta}) + \frac{(\zeta_n + \zeta_k)}{2} \delta_{\alpha\beta}$$

$$= \epsilon^*_{\alpha\beta} + \epsilon^{*,T}_{\alpha\beta} \quad (2)$$

where the trace-less part $\epsilon^*_{\alpha\beta}$ and the trace part $\epsilon^{*,T}_{\alpha\beta}$ are given as

$$\epsilon^*_{\alpha\beta} = \frac{(\zeta_n - \zeta_k)}{2} (2n_\alpha n_\beta - \delta_{\alpha\beta})$$

$$\epsilon^{*,T}_{\alpha\beta} = \frac{(\zeta_n + \zeta_k)}{2} \delta_{\alpha\beta} \quad (3)$$

We also assume that a homogeneous strain $\epsilon^\infty_{\alpha\beta}$ that acts globally (which in our case also triggers the local transformation of the inclusion). This strained ellipsoidal inclusion feels a traction exerted by the surrounding elastic medium resulting in a constrained strain $\epsilon^c_{\alpha\beta}$ in the inclusion and also exerts a traction at the inclusion-matrix interface resulting in the originally unstrained surroundings developing a constrained strain field $\epsilon^c_{\alpha\beta}(X)$.

The eigen-strain $\epsilon^*_{\alpha\beta}$ in the inclusion is related to the constrained strain $\epsilon^c_{\alpha\beta}$ via a fourth order Eshelby tensor $S_{\alpha\beta\gamma\delta}$:

$$\epsilon^c_{\alpha\beta} = S_{\alpha\beta\gamma\delta} \epsilon^*_{\gamma\delta} \quad (4)$$

Now for an inclusion of arbitrary shape the constrained strain $\epsilon^c_{\alpha\beta}$, stress $\sigma^c_{\alpha\beta}$, displacement field $u^c(X)$ inside the inclusion are in general functions of space. For ellipsoidal inclusions, however, it was shown by Eshelby [3, 4, 5] that the Eshelby tensor and the constrained stress and strain fields inside the inclusion become independent of space. We work here with a circular inclusion.
which is a special case of an ellipse and hence for such an inclusion, the Eshelby tensor $S_{\alpha \beta \gamma \delta}$ reads

$$S_{\alpha \beta \gamma \delta} = \frac{(\lambda - \mu)}{4(\lambda + 2\mu)} \delta_{\alpha \beta} \delta_{\gamma \delta} + \frac{(\lambda + 3\mu)}{4(\lambda + 2\mu)} (\delta_{\alpha \delta} \delta_{\beta \gamma} + \delta_{\alpha \gamma} \delta_{\beta \delta})$$  \hspace{1cm} (5)

where $\lambda, \mu$ are the two Lame coefficients. Plugging Eq (5) in Eq (4), we get

$$\epsilon_{c}^{\alpha \beta} = \frac{(\lambda + \mu)}{(\lambda + 2\mu)} \epsilon_{c}^{\alpha \beta} + \frac{(\lambda + 3\mu)}{2(\lambda + 2\mu)} \epsilon_{c}^{\alpha \beta}$$  \hspace{1cm} (6)

The total stress, strain and displacement field inside the circular inclusion is then given by

$$\epsilon_{I}^{\alpha \beta} = \epsilon_{c}^{\alpha \beta} + \epsilon_{\infty}^{\alpha \beta}$$
$$\sigma_{I}^{\alpha \beta} = \sigma_{c}^{\alpha \beta} - \sigma_{\infty}^{\alpha \beta} + \sigma_{\infty}^{\alpha \beta} \equiv C_{\alpha \beta \gamma \delta} (\epsilon_{c}^{\gamma \delta} - \epsilon_{c}^{\gamma \delta} + \epsilon_{\infty}^{\gamma \delta})$$
$$u_{I}^{\alpha} = u_{c}^{\alpha} + u_{\infty}^{\alpha} = (\epsilon_{c}^{\alpha \beta} + \epsilon_{\infty}^{\alpha \beta}) X_{\beta}$$  \hspace{1cm} (7)

where the superscript $I$ indicates the inclusion. The eigen stress $\sigma_{\alpha \beta}^{*}$ is related to the eigen strain $\epsilon_{\alpha \beta}^{*}$ as

$$\sigma_{\alpha \beta}^{*} = C_{\alpha \beta \gamma \delta} \epsilon_{\gamma \delta}^{*}$$
$$= 2\mu \epsilon_{\alpha \beta}^{*} + \lambda \epsilon_{\eta \eta}^{*} \delta_{\alpha \beta}$$  \hspace{1cm} (8)

where we have used the following definition of the fourth order elastic stiffness tensor $C_{\alpha \beta \gamma \delta}$ for an isotropic elastic medium

$$C_{\alpha \beta \gamma \delta} = \lambda \delta_{\alpha \beta} \delta_{\gamma \delta} + \mu (\delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma})$$  \hspace{1cm} (9)

The stress in the inclusion can now be written down in terms of independent variables using equations (8) as

$$\sigma_{I}^{\alpha \beta} = 2\mu \left( \epsilon_{c}^{\alpha \beta} - \epsilon_{c}^{\alpha \beta} + \epsilon_{\infty}^{\alpha \beta} \right) + \lambda \left( \epsilon_{\eta \eta}^{\alpha \beta} - \epsilon_{\eta \eta}^{\alpha \beta} + \epsilon_{\infty}^{\alpha \beta} \right) \delta_{\alpha \beta}$$  \hspace{1cm} (10)

4 CONSTRAINED FIELDS IN THE MATRIX

In the surrounding elastic matrix, the stress, strain and displacement fields are all explicit functions of space and can be written as

$$\epsilon_{m}^{\alpha \beta}(X) = \epsilon_{c}^{\alpha \beta}(X) + \epsilon_{\infty}^{\alpha \beta}$$
$$\sigma_{m}^{\alpha \beta}(X) = \sigma_{c}^{\alpha \beta}(X) + \sigma_{\infty}^{\alpha \beta}$$
$$u_{m}^{\alpha}(X) = u_{c}^{\alpha}(X) + u_{\infty}^{\alpha}$$  \hspace{1cm} (11)

Thus for a circular inclusion in order to compute the displacement field $u_{c}^{\alpha}(X)$ in the isotropic elastic medium. This displacement field will satisfy the Lame-Navier equation (without any body forces)

$$(\mu + \lambda) \frac{\partial^{2} u_{c}^{\alpha}}{\partial X_{\alpha} \partial X_{\gamma}} + \mu \frac{\partial^{2} u_{\infty}^{\alpha}}{\partial X_{\beta} \partial X_{\beta}} = 0$$  \hspace{1cm} (12)

The constrained fields in the inclusion will supply the boundary conditions for the displacement field in the matrix at the inclusion boundary. Also as $r \to \infty$, the constrained displacement field will vanish.

All solutions of Eq. (12) also obey the higher order bi-harmonic equation

$$\frac{\partial^{4} u_{c}^{\alpha}}{\partial X_{\beta} \partial X_{\beta} \partial X_{\lambda} \partial X_{\lambda}} = \nabla^{4} u_{c}^{\alpha} = 0$$  \hspace{1cm} (13)
Thus our objective is to construct from the radial solutions of the bi-Laplacian equation Eq. (13) derivatives which also satisfy Eq. (12). Note that the bi-Laplacian equation is only a necessary (but not a sufficient) condition for the solutions and Eq. (12) still needs to be satisfied.

4.1 SOLUTION OF THE NAVIER-LAME EQUATION

From the foregoing section, we note that the constrained displacement field due to the Eshelby solution is given as

\[
\mathbf{u}_c^\alpha = \epsilon_{\alpha\beta}^* X_\beta
\]

where we take the expression for \(\epsilon_{\alpha\beta}^*\) from Eq.(6). We will look for linear combinations of the derivatives of the radial solutions of the bi-harmonic equation (13) which are linear in the eigen-strain and go to zero at large distance. In addition the terms must transform as a vector field. Let \(\mathbf{u}_{c,T}\) and \(\mathbf{u}_{c,0}\) be the solutions to the Navier-Lame equations. We then have:

\[
(\mu + \lambda) \frac{\partial^2 u_{c,T}^\alpha}{\partial X_\alpha \partial X_\gamma} + \mu \frac{\partial^2 u_{c,0}^\alpha}{\partial X_\beta \partial X_\beta} = 0
\]

Using the radial solutions of Navier-Lame equation we can construct the following combinations which transform as a vector and also go to zero as \(r \to 0\).

\[
\mathbf{u}_{c,T}^\alpha = A \epsilon_{\alpha\beta}^{*T} \frac{\partial \ln r}{\partial X_\beta} + B \epsilon_{\beta\gamma}^{*T} \frac{\partial^3 \ln r}{\partial X_\alpha \partial X_\beta \partial X_\gamma} + C \epsilon_{\gamma\lambda}^{*T} \frac{\partial^3 (r^2 \ln r)}{\partial X_\alpha \partial X_\beta \partial X_\lambda}
\]

and

\[
\mathbf{u}_{c,0}^\alpha = A' \epsilon_{\alpha\beta}^{*0} \frac{\partial \ln r}{\partial X_\beta} + B' \epsilon_{\beta\gamma}^{*0} \frac{\partial^3 \ln r}{\partial X_\alpha \partial X_\beta \partial X_\gamma} + C' \epsilon_{\gamma\lambda}^{*0} \frac{\partial^3 (r^2 \ln r)}{\partial X_\alpha \partial X_\beta \partial X_\lambda}
\]

Using the identities

\[
\frac{\partial^2 \ln r}{\partial X_\beta \partial X_\beta} = 0, \quad \frac{\partial^2 (r^2 \ln r)}{\partial X_\beta \partial X_\beta} = 4 \ln r + 4
\]

we see from Eq (16)

\[
\frac{\partial^2 u_{c,T}^\alpha}{\partial X_\beta \partial X_\beta} = 4 C \epsilon_{\alpha\beta}^{*T} \frac{\partial^3 \ln r}{\partial X_\alpha \partial X_\gamma \partial X_\lambda}
\]

and similarly

\[
\frac{\partial^2 u_{c,0}^\alpha}{\partial X_\alpha \partial X_\gamma} = (A + 4 C) \epsilon_{\alpha\beta}^{*0} \frac{\partial^3 \ln r}{\partial X_\alpha \partial X_\gamma \partial X_\lambda}
\]

Plugging Eq (19), (20) into Eq (15), we get

\[
C = -\frac{A(\lambda + \mu)}{4(\lambda + 2\mu)}
\]

We can now rewrite Eq (16) as

\[
\mathbf{u}_{c,T}^\alpha = A \epsilon_{\alpha\beta}^{*T} \frac{\partial \ln r}{\partial X_\beta} + B \epsilon_{\beta\gamma}^{*T} \frac{\partial^3 \ln r}{\partial X_\alpha \partial X_\beta \partial X_\gamma} - A(\lambda + \mu) \epsilon_{\gamma\lambda}^{*T} \frac{\partial^3 (r^2 \ln r)}{4(\lambda + 2\mu) \partial X_\alpha \partial X_\beta \partial X_\lambda}
\]
We now compute the following identities

\[
\frac{\partial \ln \rho}{\partial X_{\beta}} = \frac{X_{\beta}}{r^2} \quad \frac{\partial^3 \ln \rho}{\partial X_{\alpha} \partial X_{\beta} \partial X_{\gamma}} = \frac{-2r^2(X_{\alpha} \delta_{\beta \gamma} + X_{\beta} \delta_{\alpha \gamma} + X_{\gamma} \delta_{\alpha \beta}) + 8X_{\alpha}X_{\beta}X_{\gamma}}{r^6} \]

\[
\frac{\partial^3 (r^2 \ln \rho)}{\partial X_{\alpha} \partial X_{\beta} \partial X_{\gamma}} = \frac{2r^2(X_{\alpha} \delta_{\beta \gamma} + X_{\beta} \delta_{\alpha \gamma} + X_{\gamma} \delta_{\alpha \beta}) - 4X_{\alpha}X_{\beta}X_{\gamma}}{r^4} \tag{23}
\]

Using these identities, we can now write down Eq (22) as

\[
u_{\alpha}^{c,T} = A_{\alpha \beta}^T \frac{X_{\beta}}{r^2} - \left[ \frac{2B}{r^4} + \frac{A(\lambda + \mu)}{2(\lambda + 2\mu)r^2} \right] \epsilon_{\beta \gamma}^T (X_{\alpha} \delta_{\beta \gamma} + X_{\beta} \delta_{\alpha \gamma} + X_{\gamma} \delta_{\alpha \beta}) + \left[ \frac{8B}{r^6} + \frac{A(\lambda + \mu)}{(\lambda + 2\mu)r^4} \right] \epsilon_{\beta \gamma} X_{\alpha} X_{\beta} X_{\gamma} \tag{24}
\]

and similarly

\[
u_{\alpha}^{c,0} = A'_{\alpha \beta} \frac{X_{\beta}}{r^2} - \left[ \frac{2B'}{r^4} + \frac{A'(\lambda + \mu)}{2(\lambda + 2\mu)r^2} \right] \epsilon_{\beta \gamma}^0 (X_{\alpha} \delta_{\beta \gamma} + X_{\beta} \delta_{\alpha \gamma} + X_{\gamma} \delta_{\alpha \beta}) + \left[ \frac{8B'}{r^6} + \frac{A'(\lambda + \mu)}{(\lambda + 2\mu)r^4} \right] \epsilon_{\beta \gamma} X_{\alpha} X_{\beta} X_{\gamma} \tag{25}
\]

Now using Eq. (3), we can rewrite equations (24) and (25) as

\[
u_{\alpha}^{c,T} = \left( \frac{\nu_{\alpha} T + \nu_{\alpha}^{c,0}}{2(\lambda + 2\mu)r^2} \right) \epsilon_{\beta \gamma}^T X_{\alpha} X_{\beta} X_{\gamma} \]

\[
u_{\alpha}^{c,0} = \left( \frac{A'_{\alpha \beta}}{(\lambda + 2\mu)r^2} - \frac{4B'}{r^4} - \frac{A' \epsilon_{\alpha \beta}^0}{(\lambda + 2\mu)r^2} \right) \epsilon_{\beta \gamma}^0 X_{\alpha} X_{\beta} X_{\gamma} \tag{26}
\]

The complete solution for the displacement field in the matrix is then given as

\[
u_{\alpha}^c = \nu_{\alpha}^{c,T} + \nu_{\alpha}^{c,0} = \frac{(\nu_{\alpha} T + \nu_{\alpha}^{c,0})}{2(\lambda + 2\mu)} \epsilon_{\beta \gamma}^T X_{\alpha} X_{\beta} X_{\gamma} \]

Now at \( r = a \) (inclusion boundary), the form of solution (28) must match with the Eshelby solution (14). This implies,

\[
A = \frac{a^2(\lambda + \mu)}{\mu} \quad A' = a^2 \quad B' = -\frac{a^4(\lambda + \mu)}{8(\lambda + 2\mu)} \tag{29}
\]

Plugging Eq. (29) into Eq. (28), we get:

\[
u_{\alpha}^c = \frac{(\lambda + \mu)}{2(\lambda + 2\mu)} \left( \frac{\nu_{\alpha} T + \nu_{\alpha}^{c,0}}{r^2} \right) \left[ \epsilon_{\alpha \beta}^0 X_{\beta} + \left( \frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2} \right) \epsilon_{\alpha \beta}^0 X_{\beta} + 2 \left( \frac{a^2}{r^2} \right) \epsilon_{\beta \gamma}^0 X_{\alpha} X_{\beta} X_{\gamma} \right] \tag{30}
\]

Noting that

\[
\epsilon_{\alpha \beta}^0 X_{\beta} = \frac{(\nu_{\alpha} T - \nu_{\alpha})}{2} (\hat{n} \cdot X) X_{\alpha} \tag{31}
\]

and

\[
\epsilon_{\beta \gamma}^0 X_{\alpha} X_{\beta} X_{\gamma} = \frac{(\nu_{\alpha} T - \nu_{\alpha})}{2} (2(\hat{n} \cdot X)^2 - r^2) X_{\alpha} \tag{32}
\]
we find that Eq. (30) finally becomes

\[
\mathbf{u}^c(x) = \frac{\left(\zeta_n - \zeta_k\right)\left(\lambda + \mu\right)}{4\left(\lambda + 2\mu\right)} \left(\frac{a^2}{r^2}\right) \left[ 2\left(\zeta_n + \zeta_k\right) \mathbf{X} + \left(\frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \mathbf{X}\right)\hat{n} - \mathbf{X} \right] \\
+ 2 \left(1 - \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \hat{r}\right)^2 - 1 \mathbf{X} \]
\]

(33)

where \( \hat{r} = \frac{\mathbf{X}}{r} \). For the purpose of visualizing the displacement field in space, it proves useful to have expressions for the Cartesian components of the field given by Eq. (33). If we consider the unit vector \( \hat{n} \) making an angle \( \phi \) with the x-axis, then the Cartesian components of \( \mathbf{u}^c(x) \) are:

\[
u_x^c(x) = \frac{\left(\zeta_n - \zeta_k\right)\left(\lambda + \mu\right)}{4\left(\lambda + 2\mu\right)} \left(\frac{a^2}{r^2}\right) \left[ 2\left(\zeta_n + \zeta_k\right) \frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right] (\cos^2 \phi + \sin^2 \phi) \\
+ 2 \left(1 - \frac{a^2}{r^2}\right) \left(\frac{x^2 - y^2}{r^2}\cos^2 \phi + 2x y \sin^2 \phi\right) x
\]

\[
u_y^c(x) = \frac{\left(\zeta_n - \zeta_k\right)\left(\lambda + \mu\right)}{4\left(\lambda + 2\mu\right)} \left(\frac{a^2}{r^2}\right) \left[ 2\left(\zeta_n + \zeta_k\right) \frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right] (\cos^2 \phi - \sin^2 \phi) \\
+ 2 \left(1 - \frac{a^2}{r^2}\right) \left(\frac{x^2 - y^2}{r^2}\cos^2 \phi + 2x y \sin^2 \phi\right) y
\]

(34)

Taking derivatives of displacement field

\[
\frac{\partial \mathbf{u}^c}{\partial \mathbf{X}_\beta} = \frac{\left(\zeta_n - \zeta_k\right)\left(\lambda + \mu\right)}{4\left(\lambda + 2\mu\right)} \left(\frac{a^2}{r^2}\right) \left[ 2\left(\zeta_n + \zeta_k\right) \frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right] \left(\delta_{\alpha\beta} - 2\frac{X_\alpha X_\beta}{r^2}\right) \\
- 4 \left(\frac{\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \hat{r}\right) \hat{n}_\alpha - \frac{X_\alpha}{r} \frac{X_\beta}{r} \\
+ \left(\frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right) \left(2\hat{n}_\alpha \hat{n}_\beta - \delta_{\alpha\beta}\right) - 4 \left(1 - \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \hat{r}\right)^2 - 1 \frac{X_\alpha X_\beta}{r^2} \\
+ 8 \left(1 - \frac{a^2}{r^2}\right) \left(\hat{n} \cdot \hat{r}\right) \hat{n}_\alpha - \left(\hat{n} \cdot \hat{r}\right)^2 \frac{2X_\alpha}{r} \frac{X_\beta}{r} + 2 \left(1 - \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \hat{r}\right)^2 - 1 \delta_{\alpha\beta}\]
\]

(35)

and

\[
\frac{\partial \mathbf{u}^c}{\partial \mathbf{X}_\alpha} = \frac{\left(\zeta_n - \zeta_k\right)\left(\lambda + \mu\right)}{4\left(\lambda + 2\mu\right)} \left(\frac{a^2}{r^2}\right) \left[ 2\left(\zeta_n + \zeta_k\right) \frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right] \left(\delta_{\alpha\beta} - 2\frac{X_\alpha X_\beta}{r^2}\right) \\
- 4 \left(\frac{\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \hat{r}\right) \hat{n}_\beta - \frac{X_\alpha}{r} \frac{X_\beta}{r} \\
+ \left(\frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2}\right) \left(2\hat{n}_\alpha \hat{n}_\beta - \delta_{\alpha\beta}\right) - 4 \left(1 - \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \hat{r}\right)^2 - 1 \frac{X_\alpha X_\beta}{r^2} \\
+ 8 \left(1 - \frac{a^2}{r^2}\right) \left(\hat{n} \cdot \hat{r}\right) \hat{n}_\beta - \left(\hat{n} \cdot \hat{r}\right)^2 \frac{2X_\alpha}{r} \frac{X_\beta}{r} + 2 \left(1 - \frac{a^2}{r^2}\right) \left(2\hat{n} \cdot \hat{r}\right)^2 - 1 \delta_{\alpha\beta}\]
\]

(36)

The constrained strain in the matrix can be written as

\[
\epsilon^c_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial \mathbf{u}^c}{\partial \mathbf{X}_\beta} + \frac{\partial \mathbf{u}^c}{\partial \mathbf{X}_\alpha}\right)
\]

(37)
Using equations (35) and (36), Eq. (37) becomes

\[
\epsilon^c_{\alpha\beta}(X) = \frac{(\zeta_n - \zeta_k)(\lambda + \mu)}{4(\lambda + 2\mu)} \left( \frac{a^2}{r^2} \right) \left[ \frac{2(\zeta_n + \zeta_k)}{(\zeta_n - \zeta_k)} \delta_{\alpha\beta} - 2 \frac{X_a X_\beta}{r^2} \right] \\
- 4 \left( \frac{\mu}{\lambda + \mu} + \frac{a^2}{r^2} \right) \left\{ \left( \hat{n} \cdot \hat{r} \right) \left( \frac{\hat{n} \alpha X_\beta}{r} + \frac{\hat{n} \beta X_\alpha}{r} \right) - \frac{X_a X_\beta}{r^2} \right\} \\
+ \left( \frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2} \right) \left( 2\hat{n} \alpha \hat{n} \beta - \delta_{\alpha\beta} \right) - 4 \left( 1 - 2 \frac{a^2}{r^2} \right) \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \frac{X_a X_\beta}{r^2} \\
+ 4 \left( 1 - \frac{a^2}{r^2} \right) \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \delta_{\alpha\beta} \left] \right. \\
+ 2 \left( 1 - \frac{a^2}{r^2} \right) \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \delta_{\alpha\beta}\right. \\
(38)
\]

It is easy to see that the trace of \( \epsilon^c_{\alpha\beta}(X) \) is given as

\[
\epsilon^c_{\eta\eta}(X) = -\frac{(\zeta_n - \zeta_k)\mu}{(\lambda + 2\mu)} \left( \frac{a^2}{r^2} \right) \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \\
(39)
\]

We can now calculate the constrained stress in the elastic matrix due to the deformed Eshelby inclusion. It follows from Hooke’s law:

\[
\sigma^c_{\alpha\beta}(X) = 2\mu \epsilon^c_{\alpha\beta}(X) + \lambda \epsilon^c_{\eta\eta}(X) \delta_{\alpha\beta} \\
(40)
\]

Plugging Eq. (38) in Eq. (40), we get the final expression for the constrained stress

\[
\sigma^c_{\alpha\beta}(X) = \frac{(\zeta_n - \zeta_k)\mu(\lambda + \mu)}{2(\lambda + 2\mu)} \left( \frac{a^2}{r^2} \right) \left[ \frac{2(\zeta_n + \zeta_k)}{(\zeta_n - \zeta_k)} \delta_{\alpha\beta} - 2 \frac{X_a X_\beta}{r^2} \right] \\
- 4 \left( \frac{\mu}{\lambda + \mu} + \frac{a^2}{r^2} \right) \left\{ \left( \hat{n} \cdot \hat{r} \right) \left( \frac{\hat{n} \alpha X_\beta}{r} + \frac{\hat{n} \beta X_\alpha}{r} \right) - \frac{X_a X_\beta}{r^2} \right\} \\
+ \left( \frac{2\mu}{\lambda + \mu} + \frac{a^2}{r^2} \right) \left( 2\hat{n} \alpha \hat{n} \beta - \delta_{\alpha\beta} \right) - 4 \left( 1 - 2 \frac{a^2}{r^2} \right) \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \frac{X_a X_\beta}{r^2} \\
+ 4 \left( 1 - \frac{a^2}{r^2} \right) \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \delta_{\alpha\beta} - \frac{2\lambda}{\lambda + \mu} \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \delta_{\alpha\beta} \right. \\
+ 2 \left( 1 - \frac{a^2}{r^2} \right) \left( 2(\hat{n} \cdot \hat{r})^2 - 1 \right) \delta_{\alpha\beta} \left] \right. \\
(41)
\]

5 ENERGY OF \( N \) ESHELBY INCLUSIONS EMBEDDED IN THE MATRIX

The energy of the \( N \) Eshelby inclusions embedded in a linear elastic matrix \( m \) is given by the following expression

\[
E = \frac{1}{2} \sum_{i=1}^{N} \int_{V_0}^{V_i} \sigma^i_{\alpha\beta} \epsilon^i_{\alpha\beta} dV + \frac{1}{2} \int_{V - \sum_{i=1}^{N} V_0}^{V} \sigma^m_{\alpha\beta} \epsilon^m_{\alpha\beta} dV \\
(42)
\]

where the superscript \( i \) denotes the index of the inclusion and \( m \) denotes the matrix. Eq (42) can be re-written using the definition of strain \( \epsilon_{\alpha\beta} = (1/2)(u_{\alpha\beta} + u_{\beta\alpha}) \), where \( u_{\alpha\beta} = \frac{\partial u}{\partial x_\beta} \):

\[
E = \frac{1}{2} \sum_{i=1}^{N} \int_{V_0}^{V_i} \sigma^i_{\alpha\beta} \left( u^i_{\alpha\beta} + u^i_{\beta\alpha} \right) dV + \frac{1}{2} \int_{V - \sum_{i=1}^{N} V_0}^{V} \sigma^m_{\alpha\beta} \left( u^m_{\alpha\beta} + u^m_{\beta\alpha} \right) dV \\
(43)
\]

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Using the symmetry of the stress tensor, we obtain

\[ E = \frac{1}{2} \sum_{i=1}^{N} \int_{V_0^i} \sigma_{\alpha\beta}^i u_{\beta,\alpha}^i dV + \frac{1}{2} \int_{V - \sum_{i=1}^{N} V_0^i} \sigma_{\alpha\beta}^m u_{\beta,\alpha}^m dV \]  

(44)

If there are no body forces, we also have the identity

\[ \sigma_{\alpha\beta} u_{\beta,\alpha} = (\sigma_{\alpha\beta} u_{\beta})_{,\alpha} - \sigma_{\alpha\beta,\alpha} u_{\beta} = (\sigma_{\alpha\beta} u_{\beta})_{,\alpha} \]  

(45)

Thus we can write Eq. (44) as

\[ E = \frac{1}{2} \sum_{i=1}^{N} \int_{V_0^i} \left( \sigma_{\alpha\beta}^i u_{\beta,\alpha}^i \right)_{,\alpha} dV + \frac{1}{2} \int_{V - \sum_{i=1}^{N} V_0^i} \left( \sigma_{\alpha\beta}^m u_{\beta,\alpha}^m \right)_{,\alpha} dV \]  

(46)

Using Gauss’s theorem, we convert these volume integrals into area integrals to obtain

\[ E = \frac{1}{2} \sum_{i=1}^{N} \int_{S_0^i} \sigma_{\alpha\beta}^i u_{\beta,\alpha}^i \hat{n}_a dS - \frac{1}{2} \sum_{i=1}^{N} \int_{S_0^i} \sigma_{\alpha\beta}^m u_{\beta,\alpha}^m \hat{n}_a dS + \frac{1}{2} \int_{S_\infty} \sigma_{\alpha\beta}^m u_{\beta,\alpha}^m \hat{n}_a dS \]

(47)

where \( \hat{n}^i \) and \( \hat{n}_\infty \) are unit vectors both pointing outwards respectively from the inclusion volume \( V_0^i \) and the matrix boundary. Eq. (47) can be re-written as follows

\[ E = \frac{1}{2} \int_{S_\infty} \sigma_{\alpha\beta}^m u_{\beta,\alpha}^m \hat{n}_a dS + \frac{1}{2} \sum_{i=1}^{N} \int_{S_0^i} \left( \sigma_{\alpha\beta}^i u_{\beta,\alpha}^i - \sigma_{\alpha\beta}^m u_{\beta,\alpha}^m \right) \hat{n}_a dS \]

\[ = \frac{1}{2} \sigma_{\alpha\beta}^\infty \epsilon_{\alpha\beta}^\infty \int_{S_\infty} \nabla \cdot \vec{u} \hat{n}_a dS + \frac{1}{2} \sum_{i=1}^{N} \int_{S_0^i} \left( \sigma_{\alpha\beta}^i u_{\beta,\alpha}^i - \sigma_{\alpha\beta}^m u_{\beta,\alpha}^m \right) \hat{n}_a dS \]

(48)

Thus we can re-write the Eq. (11) as

\[ \epsilon_{\alpha\beta}^m (X) = \sum_{i=1}^{N} \epsilon_{\alpha\beta}^{c,i} (X) + \epsilon_{\alpha\beta}^\infty \]

\[ \sigma_{\alpha\beta}^m (X) = \sum_{i=1}^{N} \sigma_{\alpha\beta}^{c,i} (X) + \sigma_{\alpha\beta}^\infty \]

\[ u_{\alpha\beta}^m (X) = \sum_{i=1}^{N} u_{\alpha\beta}^{c,i} (X) + u_{\alpha\beta}^\infty \]

(49)

where \( \epsilon_{\alpha\beta}^{c,i} (X) \) denotes the constrained strain at \( X \) in the matrix due to the \( i^{th} \) Eshelby inclusion. We also have for locations \( X \) inside the inclusions

\[ \epsilon_{\alpha\beta}^i (X) = \sum_{j \neq i} \epsilon_{\alpha\beta}^{c,j} (X) + \epsilon_{\alpha\beta}^{c,i} (X) - \epsilon_{\alpha\beta}^{s,i} + \epsilon_{\alpha\beta}^\infty \]

\[ \sigma_{\alpha\beta}^i (X) = \sum_{j \neq i} \sigma_{\alpha\beta}^{c,j} (X) + \sigma_{\alpha\beta}^{c,i} (X) - \sigma_{\alpha\beta}^{s,i} + \sigma_{\alpha\beta}^\infty \]

\[ u_{\alpha}^i (X) = \sum_{j \neq i} u_{\alpha}^{c,j} (X) + u_{\alpha}^{c,i} (X) - \epsilon_{\alpha\beta}^{s,i} X_{\beta} + u_{\alpha}^\infty \]

(50)
where $\epsilon_{a\beta}^{*i}$ is the eigen-strain of the $i^{th}$ Eshelby inclusion and so on. Note that in the expression for the strain in the inclusion given by Eq. (50), we have subtracted the contribution of eigenstrain from the constrained strain leaving only the elastic contribution to calculate correctly the elastic contribution to the energy. Using these expressions, the elastic energy of the system can be written from Eq. (48)

$$E = \frac{1}{2} \sigma_{a\beta}^{c} \epsilon_{a\beta}^{c} V + \frac{1}{2} \sum_{i=1}^{N} \int_{S_{0}} \left( \sigma_{a\beta}^{i} u_{\beta} - \sigma_{a\beta}^{m} u_{\beta} \right) \hat{n}_{a} dS$$

(51)

Since the traction force has to be continuous at the inclusion boundary (Newton’s IIIrd law), we have

$$\sigma_{a\beta}^{i} \hat{n}_{a} = \sigma_{a\beta}^{m} \hat{n}_{a}$$

(52)

which gives us from Eq. (51),

$$E = \frac{1}{2} \sigma_{a\beta}^{c} \epsilon_{a\beta}^{c} V + \frac{1}{2} \sum_{i=1}^{N} \int_{S_{0}} \sigma_{a\beta}^{i} \hat{n}_{a} \left( u_{\beta} - u_{\beta}^{m} \right) \hat{n}_{a} dS$$

(53)

We also have from equations (49) and (50)

$$u_{\beta}^{i} - u_{\beta}^{m} = -\epsilon_{\beta\xi}^{*i} X_{\xi}$$

(54)

On plugging this expression into Eq. (53) we finally get

$$E = \frac{1}{2} \sigma_{a\beta}^{c} \epsilon_{a\beta}^{c} V - \frac{1}{2} \sum_{i=1}^{N} \int_{S_{0}} \sigma_{a\beta}^{i} \hat{n}_{a} \epsilon_{\beta\xi}^{*i} X_{\xi} \hat{n}_{a} dS$$

$$= \frac{1}{2} \sigma_{a\beta}^{c} \epsilon_{a\beta}^{c} V - \frac{1}{2} \sum_{i=1}^{N} \epsilon_{\beta\xi}^{*i} \int_{V_{0}} \left( \sigma_{a\beta}^{i} X_{\xi} \right) \hat{n}_{a} dV$$

$$= \frac{1}{2} \sigma_{a\beta}^{c} \epsilon_{a\beta}^{c} V - \frac{1}{2} \sum_{i=1}^{N} \epsilon_{\beta\xi}^{*i} \int_{V_{0}} \sigma_{a\beta}^{i} \delta_{\xi a} dV$$

$$= \frac{1}{2} \sigma_{a\beta}^{c} \epsilon_{a\beta}^{c} V - \frac{1}{2} \sum_{i=1}^{N} V_{0} \epsilon_{\beta\xi}^{*i} \sigma_{a\beta}^{i}$$

(55)

where $\overline{\sigma_{a\beta}^{i}} \equiv \left( 1/V_{0} \right) \int_{V_{0}} \sigma_{a\beta}^{i} dV$. Using the expression for $\sigma_{a\beta}^{i}$ from equation Eq.(50), we obtain

$$\overline{\sigma_{a\beta}^{i}}(X) = \sigma_{a\beta}^{c} + \sum_{j \neq i} \sigma_{a\beta}^{c\beta}(r^{ij}) + \sigma_{a\beta}^{c\beta}(r^{ij}) - \sigma_{a\beta}^{*i}$$

(56)

Eq. (56) is a far field approximation which assumes that $r^{ij} \gg a$. As $r^{ij} \rightarrow a$, clearly the spatial integrals contributing to $\overline{\sigma_{a\beta}^{i}}$ must be computed explicitly and cannot be replace by the single distance $r^{ij}$ between the centers of the Eshelby inclusions $i$ and $j$.

Using equations (55) and (56), we can write down the final form of the elastic energy expression.

$$E = \frac{1}{2} \sigma_{a\beta}^{c} \epsilon_{a\beta}^{c} V - \frac{1}{2} \sum_{i=1}^{N} \epsilon_{\beta\xi}^{*i} \sigma_{a\beta}^{i} + \frac{1}{2} \sum_{i=1}^{N} \epsilon_{\beta\xi}^{*i} \sigma_{a\beta}^{i}$$

$$- \frac{1}{2} \sum_{i=1}^{N} \epsilon_{\beta\xi}^{*i} \sigma_{a\beta}^{i}$$

$$= E_{mat} + E_{\infty} + E_{esh} + E_{inc}$$

(57)
where each component of energy defined as

\[ E_{\text{mat}} = \frac{1}{2} \sigma_{\alpha\beta}^\infty \epsilon_{\beta\alpha}^\infty V \]

\[ E_\infty = -\frac{1}{2} \sigma_{\alpha\beta}^\infty \sum_{i=1}^{N} \epsilon_{\beta\alpha}^{x,i} V_0^i \]

\[ E_{\text{esh}} = \frac{1}{2} \sum_{i=1}^{N} (\sigma_{\alpha\beta}^{x,i} - \sigma_{\alpha\beta}^{c,i}) \epsilon_{\beta\alpha}^{x,i} V_0^i \]

\[ E_{\text{inc}} = -\frac{1}{2} \sum_{i=1}^{N} \epsilon_{\beta\alpha}^{x,i} V_0^i \sum_{j \neq i} \sigma_{\alpha\beta}^{c,j} (r_{ij}) \] (58)

Here the eigen-strain \( \epsilon_{\alpha\beta}^{x,i} \) and volume \( V_0^i \) associated with any \( i^{\text{th}} \) Eshelby inclusion are given as

\[ V_0^i = \frac{\pi a^2}{2} \]

\[ \epsilon_{\alpha\beta}^{x,i} = \left(\frac{\zeta_n - \zeta_k}{2}\right)(2n_{\alpha}^i n_{\beta}^i - \delta_{\alpha\beta}) + \left(\frac{\zeta_n + \zeta_k}{2}\right) \delta_{\alpha\beta} \] (59)

Also for a 2D material being loaded under uni-axial strain with free boundaries along \( \hat{y} \), we can write the form of the global stress tensor as

\[ \sigma^\infty = \begin{pmatrix} \sigma_{xx}^\infty & 0 \\ 0 & 0 \end{pmatrix} \] (60)

By Hooke’s law, we get the expression for applied global stress tensor

\[ \sigma_{\alpha\beta}^\infty = 2\mu \epsilon_{\alpha\beta}^\infty + \lambda \epsilon_{\eta\xi}^\infty \delta_{\alpha\beta} \] (61)

Taking trace of Eq. (61), we find

\[ \epsilon_{\eta\eta}^\infty = \frac{1}{2(\lambda + \mu)} \sigma_{\eta\eta}^\infty = \frac{1}{2(\lambda + \mu)} \sigma_{xx}^\infty \] (62)

Plugging Eq. (62) in Eq. (61), we find

\[ \sigma_{xx}^\infty = \frac{4\mu(\lambda + \mu) \gamma}{\lambda + 2\mu} \] (63)

where \( \gamma \) is the external strain. In the following, we discuss these components of energy in detail:

**\( E_{\text{mat}} \):** It is the elastic energy that would be present in the strained matrix in the absence of inclusions. Plugging equation (60) and (63) into Eq. (58), we get the following expression

\[ E_{\text{mat}} = \frac{2\mu(\lambda + \mu) \gamma^2 V}{\lambda + 2\mu} \] (64)

**\( E_\infty \):** It is the contribution to the elastic energy by the Eshelby inclusions themselves. Note that this term makes a negative contribution in comparison to \( E_{\text{mat}} \) itself. Again plugging equations (60), (63) and (59) into Eq. (58), we find

\[ E_\infty = \frac{-2\pi a^2 \mu(\lambda + \mu) \gamma}{(\lambda + 2\mu)} \sum_{i=1}^{N} \left\{ \frac{\left(\zeta_n - \zeta_k\right)}{2} (2n_{x}^i)^2 - 1 \right\} \] (65)
Eshelby inclusion can be written [using Eq. (8)] as approximation”. We note from Eq. (58)
In deriving Eq. (71), we have used the following identities
In deriving Eq. (67), we have used equations (9), (4) and (5). The eigen stress for the
Hence each Eshelby inclusion must be oriented along the principal stress direction (or \( \theta = 0 \)).
\textbf{E}_{\text{esh}}$: It is the self energy required to create the Eshelby inclusion and is positive. To calculate
this term, we need the following quantities:
\[
\sigma_{\alpha\beta}^{\alpha\beta} = C_{\alpha\beta\delta\varepsilon} \varepsilon_{\gamma\delta}^{\alpha\beta} = C_{\alpha\beta\gamma\delta} S_{\delta\varepsilon\beta} \varepsilon_{k\varepsilon}^{\beta\gamma}
\]
\[
= \frac{(\lambda + \mu)}{2(\lambda + 2\mu)} \left[ 2(\zeta_n + \zeta_k)(\zeta_n - \zeta_k) + \frac{\mu(\lambda + 3\mu)(\zeta_n - \zeta_k)}{\lambda + \mu}(2\hat{n}_n \hat{n}_\beta - \delta_{\alpha\beta}) \right]
\]
In deriving Eq. (67), we have used equations (9), (4) and (5). The eigen stress for the \( i \)th
Eshelby inclusion can be written [using Eq. (8)] as
\[
\sigma_{\alpha\beta}^{\alpha\beta} = (\lambda + \mu)(\zeta_n + \zeta_k)\delta_{\alpha\beta} + \mu(\zeta_n - \zeta_k)(2\hat{n}_n \hat{n}_\beta - \delta_{\alpha\beta})
\]
Combining this and the expression for eigen-strain from Eq. (59), Eq. (58) becomes
\[
E_{\text{esh}} = \frac{\pi a^2}{2} \sum_{i=1}^N (\sigma_{\alpha\beta}^{\alpha\beta} - \sigma_{\alpha\beta}^{\alpha\beta}) \varepsilon_{\beta\alpha}^{\alpha\beta}
\]
\[
= \frac{\pi a^2 N \mu(\lambda + \mu)}{4(\lambda + 2\mu)} \{2(\zeta_n + \zeta_k)^2 + (\zeta_n - \zeta_k)^2\}
\]
\textbf{E}_{\text{inc}}$: This term arises due to the interaction between Eshelby inclusions under the “far field
approximation”. We note from Eq. (58)
\[
E_{\text{inc}} = -\frac{1}{2} \sum_{i=1}^N \varepsilon_{\alpha\beta}^{\alpha\beta} \varepsilon_{\beta\alpha}^{\alpha\beta} \sigma_{\alpha\beta}^{\alpha\beta}(r_{ij})
\]
\[
= -\frac{\pi a^2}{2} \sum_{<ij>} \left\{ \varepsilon_{\alpha\beta}^{\alpha\beta} \sigma_{\alpha\beta}^{\alpha\beta}(r_{ij}) + \varepsilon_{\alpha\beta}^{\alpha\beta} \sigma_{\alpha\beta}^{\alpha\beta}(r_{ij}) \right\}
\]
Recalling the value of eigen-strain (Eq. (2)) and constrained stress (Eq. (41)) due to an \( i \)th
Eshelby inclusion, we write down Eq. (70) (after simplifying):
\[
E_{\text{inc}} = -2\pi a^2 \sum_{<ij>} \frac{\mu(\lambda + \mu)}{(\lambda + 2\mu)} \left[ \frac{a^2}{r_{ij}} \right] \left[ (\zeta_n - \zeta_k)^2 - \left( \frac{a^2}{r_{ij}} \right)^2 \right] + \frac{(\zeta_n - \zeta_k)^2}{8} \left\{ -4 \left(\frac{\mu}{\lambda + 2\mu} + \frac{a^2}{r_{ij}} \right) \left( 4(\hat{n}_i \cdot \hat{n}_j)(\hat{n}_i \cdot \hat{r}_{ij})(\hat{n}_j \cdot \hat{r}_{ij}) - 2(\hat{n}_i \cdot \hat{r}_{ij})^2 - 2(\hat{n}_j \cdot \hat{r}_{ij})^2 + 1 \right) + 2 \left(\frac{2\mu}{\lambda + 2\mu} + \frac{a^2}{r_{ij}} \right) \left( 2(\hat{n}_i \cdot \hat{n}_j)^2 - 1 \right) - 4 \left(\frac{a^2}{r_{ij}} \right)^2 \left( 2(\hat{n}_j \cdot \hat{r}_{ij})^2 - 1 \right) \left( 2(\hat{n}_j \cdot \hat{r}_{ij})^2 - 1 \right) \right\}
\]
In deriving Eq. (71), we have used the following identities

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\[ \varepsilon_{i}^{x} = \left( \alpha \beta - \frac{X_{ij}^{x} X_{ij}^{x}}{r_{ij}^2} \right) = (\zeta_{n} - \zeta_{k}) \left( 1 - 2(n_{i} \cdot r_{ij})^2 \right) \]

\[ \varepsilon_{i}^{x} = \left( \frac{n_{j} \cdot r_{ij}}{r_{ij}^2} \frac{n_{j}^{*} \cdot X_{ij}^{x} X_{ij}^{x}}{r_{ij}^2} \right) = (\zeta_{n} + \zeta_{k}) \left( (n_{j}^{*} \cdot r_{ij})^2 - \frac{1}{2} \right) + \\
(\zeta_{n} - \zeta_{k}) \left( (n_{j}^{*} \cdot n_{j})(n_{j}^{*} \cdot r_{ij})(n_{j} \cdot r_{ij}) - (n_{j}^{*} \cdot r_{ij}) - (n_{j}^{*} \cdot r_{ij})^2 + \frac{1}{2} \right) \]

To calculate the shear band angle with respect to the principal direction of strain, we must minimize \( E_{\text{inc}} \) with respect to \( \theta \). Assuming all eigen directions to be the same, i.e., taking \( n_{i} = n_{j} = \hat{n} \) and \( (n_{i} \cdot r_{ij})^2 = \cos^2 \theta = \chi \), we find, by putting \( \frac{d}{d\chi} E_{\text{inc}} = 0 \):

\[ \chi = \frac{1}{2} - \frac{1}{4} \frac{(\zeta_{n} + \zeta_{k})}{(\zeta_{n} - \zeta_{k})} \]

or

\[ \theta = \cos^{-1} \sqrt{\frac{1}{2} - \frac{1}{4} \frac{(\zeta_{n} + \zeta_{k})}{(\zeta_{n} - \zeta_{k})}} \]  

References


