NONLINEAR THEORY OF FERROMAGNETIC RESONANCE

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Translated from Fizika Tverdogo Tela, Vol. 13, No. 2,
pp. 523-532, February, 1971
Original article submitted October 23, 1970

A consistent steady-state theory of nonlinear ferromagnetic resonance is constructed. It takes into account not only the interaction of the uniform precession with the spin waves but also the interaction of parametrically excited spin waves with one another. The role of the spin wave-spin wave interaction in determination of the behavior of the uniform precession and the spin waves above the threshold is different in the cases when the instability of uniform precession is due to a three-wave interaction $\omega_0 = \omega_k + \omega_{-k}$ or four-wave interaction $2\omega_0 = \omega_k + \omega_{-k}$. For the three-wave instability in the case when the uniform precession is excited at resonance ($\omega_p = \omega_0$) the amplitude of uniform precession is "frozen" at the threshold level right up to $h/h_c \sim 10^3$ (where h_c is the critical value of the alternating magnetic field at the threshold). In the nonresonance case ($\omega_p - \omega_0 \sim \omega_0$) the spin wave-spin wave interaction becomes important immediately above the threshold and leads to virtually complete "defreezing" of the amplitude of uniform precession, which continues to increase in accordance with a linear law. For the four-wave instability ($\omega_p \simeq \omega_0 = 2\omega_k$) the spin wave-spin wave interaction also leads to a linear increase in the uniform precession for $(h - h_c)/h_c > 1$. A preliminary comparison of the results of the theory with experimental results obtained by the authors on yttrium iron garnet single crystals confirms that it is necessary to take into account the interaction of parametric spin waves with one another in the theory of nonlinear ferromagnetic resonance.

The experiments of Bloembergen, Damon, and Wang [1, 2] showed that the phenomenon of ferromagnetic resonance cannot be explained by a simple theory that treats ferromagnetic resonance as oscillations of a "rigid" magnetization vector that is uniform over the crystal. The physical reason for the anomalous effects in ferromagnetic resonance at high amplitudes of the alternating field was established by Anderson and Suhl [3]; they showed that uniform precession of the magnetization may become unstable against spatially inhomogeneous perturbations of the magnetization (spin waves). At a certain threshold amplitude spin waves are excited parametrically by the uniform precession. They grow linearly with time and expend the energy of uniform precession until nonlinear effects set in which restrict the amplitude of the waves at a certain level that is appreciably higher than the thermal level.

As is well known [4], one must distinguish two principal instability mechanisms of ferromagnetic resonance: 1) The excitation of spin waves with frequency $\omega_k = \omega_p/2$ (ω_p is the pumping frequency); 2) excitation of spin waves with the frequency $\omega_k = \omega_p$. We shall refer to these mechanisms as the nonlinear processes of first and second order, respectively.

The existing theory of the state above threshold does not take into account the interaction of the spin waves with one another (the spin wave-spin wave interaction), although the latter is very strong at large amplitudes and cannot be neglected in the majority of cases.[1] At the same time, it is widely believed that a restriction of parametric excitation occurs basically because of a reaction of the spin waves to the uniform precession (the spin wave-uniform precession interaction). This opinion is based on the phenomenon of "freezing" of the uniform precession above the threshold predicted by the spin wave-uniform precession theory which has been applied in ferrite power limiters.

It is well known that a virtually complete cessation of growth of the uniform precession amplitude is observed in ferromagnetic resonance for first-order processes [6]. On the other hand, there ~~are a number of~~ facts that cannot be explained by the spin wave-uniform precession interaction. In particular, the behavior of the nonlinear susceptibilities χ' and χ'' far from resonance differs from the theoretical prediction when the pumping power greatly exceeds the threshold [7]. A particularly important fact is that the real part of the susceptibility χ' remains practically constant after the threshold whereas the spin wave-uniform precession interaction theory predicts a decrease which varies as h^{-2} (h is the amplitide of the alternating magnetic field). We would like to also mention the experiments on the saturation of ferromagnetic resonance (in the case of second-order processes), which indicate an increase in the uniform precession amplitude above the threshold [8].[2]

The aim of the present paper is to construct a consistent nonlinear theory of ferromagnetic resonance which takes into account both the spin wave-uniform precession and spin wave-spin wave interaction. From the theoretical point of view it is clear that as the spin waves grow above the threshold the energy of the spin wave-spin wave interaction will first become comparable with and then exceed the energy of the spin wave-uniform precession interaction. The pumping power at which the spin wave-spin wave interaction becomes important depends on the properties of the spin system and the type of nonlinear processes.

We shall restrict our treatment to the interaction of parametrically excited waves with one another and we shall not touch on the various mechanisms of nonlinear damping due to the interaction of parametric waves with the thermal bath.[3]

The main factor determining the character of the spin wave-spin wave interaction above the threshold is the establishment of a phase correlation of waves with equal wave vectors but opposite directions. The sum of the phase of the waves forming a pair, similar to a Cooper pair in superconductors, is in a definite relationship with the pumping phase, i.e., the phase within every pair is completely correlated whereas the individual phases of the waves are statistically independent. Above the threshold one should not therefore speak of the interaction of individual waves but of the interaction of pairs.

The basic nonlinear equations describing the "pairing" and interaction of pairs have been obtained by Zakharov, L'vov, and Starobinets [11]; in what follows we shall refer to the latter papers as I. The paper I treats the interaction of parametric waves due to processes of the type

$$\omega_k + \omega_{-k} = \omega_{k'} + \omega_{-k'} = \omega_p, \tag{1}$$

which are the dominant spin wave-spin wave processes for not too large excesses above the threshold ($\zeta \equiv h/h_c < (\omega/\gamma\ )^{1/2}$. The investigation of the interaction (1) showed that it can be represented by an additional pumping proportional to the total amplitude of the pairs. It is important that the phase of this pumping differs from that of the external pumping. The occurrence of additional pumping with a different phase results in a self-consistent change in the phases of the pairs and, hence, to a weakening of their coupling and a limitation of their amplitude.

The importance of the spin wave-spin wave interaction for processes occurring above the instability threshold of ferromagnetic resonance is different for nonlinear processes of the first and second orders. A characteristic feature of the first-order processes is the strong reaction which the spin waves exert on the uniform precession as a result of three-wave processes of the type

[1]It is interesting to cite Schlömann's opinion [5]. "This approximation (the elimination of the spin wave-spin wave interaction) is made to simplify the mathematical side of the problem and cannot, in general, be justified. In applying this approximation we probably lose a large part of the important physical information."

[2]We should like to mention the interesting investigations [9, 10] in which it is shown that allowance must be made for the spin wave-spin wave interaction if one wishes to explain the phenomenon of frequency doubling at saturation of the ferromagnetic resonance.

[3]The nonlinear damping due to to third-order processes may be important; to take it into account one must use Eq. (42) instead of Eq. (30) of [11]. We have not done this for simplicity, an omission that is also justified by the fact that there is a range of values of the constant magnetic field in which the three-wave processes are forbidden by the form of the spectrum.

$$\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}=\omega_0. \tag{2}$$

At resonance ($\omega_0 = \omega_p$, $\omega_k = \omega_p/2$) the processes (2) are more important than those of (1) right up to very large excesses $\zeta > 10^3$ (this is proved in Sec. 2). In this region the uniform precession amplitude is virtually "frozen."

The investigation of the nonresonance case ($\omega_0 - \omega_p \sim \omega_0$, $\omega_k = \omega_p/2$) shows that at small excesses ($\zeta - 1 < 1$) the spin wave-uniform precession dominates, as before. However, if $\zeta - 1$ is not small compared with 1, the spin wave-spin wave interaction is as important as the spin wave-uniform precession interaction. It is interesting that for $\zeta \gg 1$ the spin wave-spin wave interaction may become predominant and the reaction can be neglected. The uniform precession amplitude is then completely "defrozen" and increases linearly.

For second-order processes ($\omega_0 \simeq \omega_p$, $\omega_k = \omega_p$) the spin wave-uniform precession interaction is due to four-wave processes of the type

$$\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}=2\omega_0, \tag{3}$$

which are the analog of the processes (1) with k' = 0. It follows that both the processes (1) and (3) must be taken into account except in the region of very small excesses ($\zeta - 1 \ll 1$), in which the processes (3) are predominant.

It should be noted that in crystals with magnetic inhomogeneities there is a more effective mechanism of spin wave-uniform precession interaction than the processes of the type (3). Inhomogeneities lead to a scattering of spin waves which occurs without alteration of frequency but with a change in the momentum, i.e., $\omega_k = \omega_0$ [5]. The theory of second-order processes considered in ~~this paper treats~~ a perfect crystal that does not contain inhomogeneities although even a few inhomogeneities exert an appreciable influence on the behavior of the uniform precession near threshold. It is to be expected that in a crystal with a small number of inhomogeneities there are three characteristic regions: 1) The region of small excesses ($\zeta - 1 \ll 1$), in which the inhomogeneity mechanism is predominant; 2) an intermediate region in which both mechanisms are important; 3) the region of large excesses, in which the processes (1) and (3) are predominant. In Sec. 4 we show that for $\zeta \gg 1$ the uniform precession amplitude increases linearly with the field although more slowly by a factor of approximately two than below the threshold.

1. BASIC EQUATIONS

We shall describe nonlinear phenomena in ferromagnetic resonance in the framework of the classical Hamiltonian formalism. The Hamiltonian of a system that includes uniform precession and spin waves has the form

$$\mathcal{H}=\omega_0 a_0 a_0^*+\sum_{\mathbf{k}}\omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^*+\mathcal{H}_p+\mathcal{H}_{00}+\sum_{\mathbf{k}}\mathcal{H}_{0\mathbf{k}}+\sum_{\mathbf{k}\mathbf{k}'}\mathcal{H}_{\mathbf{k}\mathbf{k}'}. \tag{4}$$

Here

$$\mathcal{H}_p = h e^{i\omega_p t} U a_0^* + \text{c.c.} \tag{4a}$$

describes the interaction of uniform precession a_0 with the uniform pumping magnetic field with frequency ω_p (for simplicity we assume that this field is circularly polarized in a plane perpendicular to the constant magnetization);

$$\mathcal{H}_{00} = T_{00}|a_0|^4 \tag{4b}$$

describes the intrinsic nonlinear shift of the uniform precession frequency;

$$\mathcal{H}_{0\mathbf{k}}=\tfrac{1}{2}(V_{0\mathbf{k}}^* a_0^* a_{\mathbf{k}} a_{-\mathbf{k}}+\text{c.c.})+(S_{0\mathbf{k}}^* a_0^* a_0^* a_{\mathbf{k}} a_{-\mathbf{k}}+\text{c.c.}) + 2T_{0\mathbf{k}}|a_0|^2|a_{\mathbf{k}}|^2, \tag{4c}$$

describes the uniform precession-spin wave interaction. The first two terms determine the parametric interaction of the first and second order, respectively, and the last term the nonlinear frequency shift; finally,

$$\mathcal{H}_{\mathbf{k}\mathbf{k}'}=S_{\mathbf{k}\mathbf{k}'}^* a_{\mathbf{k}}^* a_{-\mathbf{k}}^* a_{\mathbf{k}'} a_{-\mathbf{k}'}(1+2\Delta_{\mathbf{k}\mathbf{k}'}) + T_{\mathbf{k}\mathbf{k}'}|a_{\mathbf{k}}|^2|a_{\mathbf{k}'}|^2(2-\Delta_{\mathbf{k}\mathbf{k}'})$$
$$(\Delta_{\mathbf{k}\mathbf{k}}=1,\quad \Delta_{\mathbf{k}\mathbf{k}'}=0 \ \text{ for } \mathbf{k}\neq\mathbf{k}') \tag{4d}$$

describes the spin wave-spin wave interaction. The first term is the parametric interaction of pairs of spin waves and the second is the mutual nonlinear frequency shift.

For a system consisting of one or two pairs the Hamiltonian (4d) is exact. For a large number of pairs the terms that are nondiagonal in the pairs and are omitted in (4d) are averaged to zero because of the random distribution of the individual phases. The physical justification for the choice of $\mathcal{H}_{\mathbf{k}\mathbf{k}'}$ in the form (4d) and the limits of applicability of this approximation are discussed in detail in I.

The equations of motion of the medium have the form

$$\left(\frac{\partial}{\partial t}+\gamma_0\right)a_0=i\frac{\delta\mathscr{H}}{\delta a_0^*},\quad \left(\frac{\partial}{\partial t}+\gamma_{\mathbf{k}}\right)a_{\mathbf{k}}=i\frac{\delta\mathscr{H}}{\delta a_{\mathbf{k}}^*}. \tag{5}$$

In these equations we have introduced phenomenologically the terms $\gamma_0 a_0$ and $\gamma_{\mathbf{k}}a_{\mathbf{k}}$, which describe the damping of the uniform precession and the spin waves. Rather than the canonical variables $a_{\mathbf{k}}$ and a_0 that characterize individual waves it is convenient to introduce new canonical variables

$$n_{\mathbf{k}}=a_{\mathbf{k}}a_{\mathbf{k}}^*,\quad \Psi_{\mathbf{k}}=\varphi_{\mathbf{k}}+\varphi_{-\mathbf{k}},$$
$$n_0=a_0a_0^*,\quad \Psi_0=2\varphi_0,$$

which characterize the "amplitudes" and "phases" of the pairs. The equations of motion for these variables obtained from (4) and (5) have the form

$$\left.\begin{aligned}\frac{dn_0}{2dt}&=n_0[-\gamma_0+\mathrm{Im}\,(P_0^*e^{i\Psi_0})],\\ \frac{d\Psi_0}{2dt}&=\tilde{\omega}_0+\mathrm{Re}\,(P_0^*e^{i\Psi_0}),\end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned}\frac{dn_{\mathbf{k}}}{2dt}&=n_{\mathbf{k}}[-\gamma_{\mathbf{k}}+\mathrm{Im}\,(P_{\mathbf{k}}^*e^{i\Psi_{\mathbf{k}}})],\\ \frac{d\Psi_{\mathbf{k}}}{2dt}&=\tilde{\omega}_{\mathbf{k}}+\mathrm{Re}\,(P_{\mathbf{k}}^*e^{i\Psi_{\mathbf{k}}}).\end{aligned}\right\} \tag{7}$$

Here

$$\tilde{\omega}_0=\omega_0-\omega_p+2T_{00}n_0+2\sum_{\mathbf{k}}T_{0\mathbf{k}}n_{\mathbf{k}} \tag{6a}$$

and

$$\tilde{\omega}_{\mathbf{k}}=\omega_{\mathbf{k}}-l\frac{\omega_p}{2}+2T_{0\mathbf{k}}n_0+2\sum_{\mathbf{k}'}T_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}+2S_{\mathbf{k}\mathbf{k}}n_{\mathbf{k}}-T_{\mathbf{k}\mathbf{k}}n_{\mathbf{k}} \tag{7a}$$

describe the nonlinear detuning of the frequency of uniform precession and the spin waves (l = 1, 2, respectively, for the processes of first and second order). The expressions for P_0 and $P_{\mathbf{k}}$, which represent "complex" energy fluxes into the uniform precession and the k-th pair, depend on the process considered. For first-order processes we have

$$P_0=\frac{e^{i\frac{\Psi_0}{2}}}{\sqrt{n_0}}\left[hU+\frac{1}{2}\sum_{\mathbf{k}}V_{0\mathbf{k}}n_{\mathbf{k}}e^{i\Psi_{\mathbf{k}}}\right], \tag{6b}$$

$$P_{\mathbf{k}}=\overline{V_{0\mathbf{k}}}\sqrt{n_0}\,e^{i\frac{\Psi_0}{2}}+\sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}e^{i\Psi_{\mathbf{k}'}}, \tag{7b}$$

and for second-order processes,

$$P_0=\frac{e^{i\frac{\Psi_0}{2}}}{\sqrt{n_0}}hU+\sum_{\mathbf{k}}S_{0\mathbf{k}}n_{\mathbf{k}}e^{i\Psi_{\mathbf{k}}}, \tag{6c}$$

$$P_{\mathbf{k}}=S_{0\mathbf{k}}n_0e^{i\Psi_0}+\sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}e^{i\Psi_{\mathbf{k}'}}. \tag{7c}$$

Our problem reduces to an investigation of the solutions of the coupled equations (6) and (7) as a function of the pumping amplitude h. It is convenient to begin with a simpler problem in which the uniform precession is assumed given. It is readily seen that this problem is identical with that of the parallel pumping of spin waves in a given alternating magnetic field.

Equations of type (7) are investigated in detail in I. It is shown that all the stationary solutions of these equations for which $n_{\mathbf{k}} \neq 0$ are found on the following surface in k space:

$$\tilde{\omega}_{\mathbf{k}}=0. \tag{8}$$

The distribution of $n_{\mathbf{k}}$ on this surface depends on the specific form of the coefficients $S_{\mathbf{k}\mathbf{k}'}$, $S_{0\mathbf{k}}$, and $V_{0\mathbf{k}}$. In the case when $|P_{\mathbf{k}}|$ has a maximum at the point $\mathbf{k} = \mathbf{k}_1$ dynamical conditions in the form of a single pair with $\mathbf{k} = \mathbf{k}_1$ are realized at not too large excesses above the threshold; for this pair we have

$$n_1=\frac{\sqrt{V_{01}^2n_0^2-\gamma_1^2}}{|S_{11}|},\quad \sin\left(\Psi_1-\frac{\Psi_0}{2}\right)=\frac{\gamma_1}{V_{01}\sqrt{n_0}} \tag{9}$$

for first-order processes and

$$n_1=\frac{\sqrt{S_{01}^2n_0^2-\gamma_1^2}}{|S_{11}|},\quad \sin\left(\Psi_1-\Psi_0\right)=\frac{\gamma_1}{S_{01}n_0} \tag{10}$$

for second-order processes.

At sufficiently high values of ζ stochastic conditions are established and there is a uniform distribution of $n_{\mathbf{k}}$ with the direction of k. The total amplitude and phases of the pairs for $\zeta \gg 1$ are then

$$N=2\pi\int n_\theta\sin\theta d\theta=\left|\frac{\overline{W}}{\overline{S}}\right|, \tag{11}$$

$$n_\theta\left(\Psi_\theta-l\frac{\Psi_0}{2}\right)=-\frac{\gamma_\theta}{4\pi\overline{S}}, \tag{12}$$

where $\overline{S}$ is the mean value of $S_{\mathbf{k}\mathbf{k}'}$, and $\overline{W}$ is the mean value of the pumping and has the values $h_{\parallel}V_{\mathbf{k}}$, $\sqrt{n_0}V_{0\mathbf{k}}$, or $n_0S_{0\mathbf{k}}$ (depending on the process considered).

Note that in the limit of high pumping values Eqs. (9) and (10) give amplitudes and phases of the same order of magnitude as Eqs. (11) and (12). This means that in a qualitative analysis of the state above threshold one can use Eqs. (9) and (10) in the whole pumping range. In addition, Eqs. (9) and (10) are exact when W and γ do not depend on k if S_{11} is replaced by $\bar{S}$. In this case, the distribution $n(\theta)$ is isotropic.

Equations (9-12) cannot be applied at very high pumping amplitudes $W \sim \sqrt{\gamma\omega}$ when the fourth-order terms (which describe the interaction of individual parametric waves with one another) that are omitted in the Hamiltonian (4d) become important. The region of applicability of the theory is roughly $n_0/n_c \leqslant \omega/\gamma \sim 10^3$ for first-order processes and $n_0/n_c \sim 10^{3/2}$ for second-order processes; here, n_c is the threshold amplitude of uniform precession.

2. FIRST-ORDER PROCESSES

The stationary solution of Eqs. (6) for uniform precession obtained by means of (9) has the form

$$\sqrt{n_0}\, e^{i\frac{\Psi_0}{2}} = \frac{hU}{\tilde{\omega} + i\tilde{\gamma}}, \tag{13}$$

where the "effective" frequency detuning for the uniform precession is

$$\tilde{\omega} = \tilde{\omega}_0 - \frac{S_{11} n_1^2}{2n_0}, \tag{14}$$

and its "effective" damping is

$$\tilde{\gamma} = \gamma_0 + \frac{\gamma_1 n_1}{2n_0}. \tag{15}$$

We now have the complete system of equations (9), (13-15), and (6a) needed to calculate the steady-state conditions for the first-order processes. For what follows it is convenient to put these equations into a dimensionless form, which makes it easier to estimate the relative importance of the different terms. To this end, we introduce the dimensionless quantities

$$\left.\begin{aligned} x_0 &= \frac{n_0}{n_c}; \quad x_1 = \frac{n_1}{n_c}; \quad \zeta = \frac{h}{h_c}; \quad \delta = \frac{\omega_0 - \omega_p}{\gamma_0}, \\ \varepsilon &= \frac{2\gamma_0 S_{11}}{V_{01}^2}; \quad a = \frac{T_{00}}{S_{11}}\left(\frac{\gamma_1}{\gamma_0}\right)^2; \quad b = \frac{2T_{01}}{S_{11}}\frac{\gamma_1}{\gamma_0}, \end{aligned}\right\} \tag{16}$$

where

$$n_c = \left(\frac{\gamma_1}{V_{01}}\right)^2; \quad h_c = \frac{\gamma_0\gamma_1}{UV_{01}}, \tag{17}$$

and we reduce Eqs. (9) and (13-15) to the form

$$x_1 = \frac{2\gamma_0}{\gamma_1}\frac{\sqrt{x_0 - 1}}{|\varepsilon|}, \tag{18}$$

$$\left[\left(\frac{\tilde{\omega}}{\gamma_0}\right)^2 + \left(\frac{\tilde{\gamma}}{\gamma_0}\right)^2\right] x_0 = \zeta^2, \tag{19}$$

$$\frac{\tilde{\gamma}}{\gamma_0} = 1 + \frac{\sqrt{x_0 - 1}}{|\varepsilon|\, x_0}, \tag{19a}$$

$$\frac{\tilde{\omega}}{\gamma_0} = \delta - \frac{x_0 - 1}{\varepsilon x_0} + b\sqrt{x_0 - 1} + \varepsilon a x_0. \tag{19b}$$

These equations contain the small parameter $\varepsilon = 2\gamma_0 S_{11}/V_{01}^2 \simeq \gamma/\omega_M \sim 10^{-3}$-$10^{-4}$, which is assumed to vanish in a theory which does not take into account the spin wave-spin wave interaction.

1. Behavior of uniform precession at resonance. As follows from (19) the amplitude x_0 is maximal for $\tilde{\omega} = 0$ and is given by the equation

$$\sqrt{x_0 - 1} + |\varepsilon|\, x_0 = \varepsilon\zeta\sqrt{x_0}. \tag{20}$$

Since ε is small, the solution of this equation as a function of ζ has two characteristic regions: the region of "small excesses" ($|\varepsilon|\zeta \ll 1$), in which

$$x_0 = 1 + \varepsilon^2(\zeta - 1)^2; \quad x_1 = \frac{2\gamma_0}{\gamma_1}(\zeta - 1), \tag{21}$$

and the region of "large excesses" ($|\varepsilon|\zeta \gg 1$), in which

$$x_0 = \frac{1}{1 - (\varepsilon\zeta)^2}; \quad x_1 = \frac{2\gamma_0}{\gamma_1}\frac{\zeta - 1}{1 - (\varepsilon\zeta)^2}. \tag{22}$$

These relations are valid as long as $x_0 < |\varepsilon|^{-2/3}$. At large excesses Eq. (20) shows that the uniform precession must increase in accordance with the linear law $\sqrt{x_0} = \zeta$, although we then enter the region $x_0 \sim |\varepsilon|^{-1}$, in which the basic equations (9) cease to hold.

In Fig. 1a we have plotted the dependences of x_0 and x_1 on ζ in a double logarithmic scale.

2. Profile of the resonance curve. When exact resonance is absent or impossible, the threshold condition has the form

$$\zeta = \zeta_c \equiv \sqrt{1 + \delta^2}.$$

For $\zeta - \zeta_c \gg \varepsilon$ Eq. (19) yields

$$x_0 = 1 + \varepsilon^2\frac{\zeta^2 - \delta^2}{(1 - \delta\varepsilon)^2}, \tag{23}$$

which holds as long as $x_0 < |\varepsilon|^{-2/3}$. Equation (23) indicates that in the limit of large ζ the uniform precession amplitude increases in accordance with the linear law

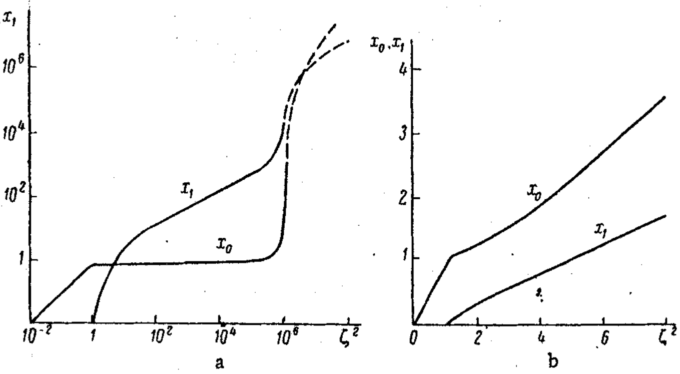

Fig. 1. Dependence of the amplitudes of the uniform precession $x_0 = n_0/n_c$ and the spin waves $x_1 = n_1/n_c$ on the excess above the threshold $\xi = h/h_c$. a) First-order processes; b) second-order processes. The dashed lines indicate the region in which the theory ceases to hold.

$$\sqrt{x_0} = \frac{|\varepsilon|\sqrt{1+\delta^2}}{|1-\delta\varepsilon|}\,\frac{\zeta}{\zeta_c},$$

the coefficient of proportionality being ~ 1 far from resonance, when $(1-\delta\varepsilon) \sim 1$, i.e., it is of the same order of magnitude as under linear conditions.

The profile of the resonance curve near ferromagnetic resonance can be understood after the identity transformation of (23) to the form

$$x_0 = \frac{1}{1-(\varepsilon\zeta)^2}\left[1-\left(\varepsilon\,\frac{\varepsilon\zeta^2-\delta}{1-\delta\varepsilon}\right)^2\right]. \tag{24}$$

We see that the resonance value of x_0 is given by Eq. (2) and is attained at

$$\delta_{\text{res}} = \varepsilon\zeta^2, \tag{25}$$

which corresponds to the condition of ferromagnetic resonance above the threshold. The resonance curve (24) depicted in Fig. 2 differs strongly from the normal curve of ordinary ferromagnetic resonance. The half-width of the curve at half height (to the right and left of resonance) is

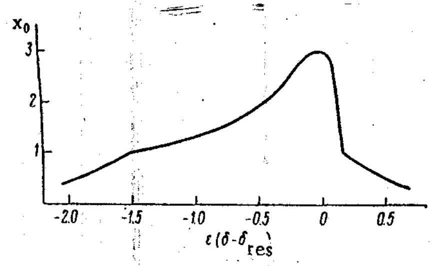

Fig. 2. Profile of the resonance curve at a high power level for first-order processes $(\varepsilon\xi)^2 = {}^2/_3$.

$$\delta_\pm = \delta_{\text{res}} \pm \delta_{1/2} = \pm\frac{1-(\varepsilon\zeta)^2}{\varepsilon(\sqrt{2}\mp 1)},$$

i.e., $\delta_+/\delta_- \simeq 6$.

3. Nonlinear susceptibility. We shall now calculate a quantity that is convenient for experimental determination, the susceptibility χ'', which is the ratio of the power absorbed by the system to the pumping power $\omega_p h^2/2$. Using (6b) and (13), we obtain

$$\chi'' = \frac{2hU\sqrt{n_0}\sin\frac{\Psi_0}{2}}{h^2} = \frac{2U^2\tilde{\gamma}}{\tilde{\omega}^2+\tilde{\gamma}^2}. \tag{26}$$

Similarly, we define the susceptibility

$$\chi' = \frac{2hU\sqrt{n_0}\cos\frac{\Psi_0}{2}}{h^2} = \frac{2U^2\tilde{\omega}}{\tilde{\omega}^2+\tilde{\gamma}^2}, \tag{27}$$

which characterizes the frequency "detuning." We write χ' and χ'' in the dimensionless variables

$$\chi'' = \chi_0\frac{\tilde{\gamma}}{\gamma_0}\frac{x_0}{\zeta^2}; \quad \chi' = \chi_0\frac{\tilde{\omega}}{\gamma_0}\frac{x_0}{\zeta^2}, \tag{28}$$

where $\chi_0 = 2U^2/\gamma_0$ is the resonance susceptibility under linear conditions.

Let us now consider the behavior of the nonlinear susceptibilities far from resonance ($\delta \gg 1$) in the region of subsidiary absorption. Equations (28) and (23) yield

$$\chi'' = \frac{\chi_0}{\zeta^2}\,\frac{\sqrt{\zeta^2-\delta^2}}{|1-\delta\varepsilon|}, \quad \zeta-\zeta_c \gg \varepsilon. \tag{29}$$

It follows, in particular, that χ'' increases above the threshold in accordance with a linear law and attains the maximum

$$\chi''_{max} = \frac{\chi_0}{2\delta\,|1 - \delta\varepsilon|} \qquad (30)$$

at $\zeta = \sqrt{2\delta} \simeq \sqrt{2}\zeta_c$. Equation (30) shows that χ'' decreases at high excesses in accordance with a ζ^{-1} law.

The real part χ' of the nonlinear susceptibility varies in a much smaller range. Equations (28) and (23) yield

$$\chi' = \frac{\chi_0}{\delta}\left[1 + \frac{1}{\varepsilon\delta - 1}\left(1 - \frac{\delta^2}{\zeta^2}\right)\right] \quad \text{for } \zeta - \zeta_c \gg 1. \qquad (31)$$

In a theory which does not take into account the spin wave-spin wave interaction, $\varepsilon = 0$ and, in accordance with Eq. (31), χ' decreases to zero as ζ^{-2}. Allowance for the spin wave-spin wave interaction radically alters this result: At large excesses $\chi' \to \chi_0\varepsilon/(\delta\varepsilon - 1) \sim \chi_0/\delta$, i.e., the susceptibility is of the same order of magnitude as below the threshold. This explains why the phase constants of microwave ferrite instruments are practically independent of the power level [12].

3. SECOND-ORDER PROCESSES

Proceeding as in the foregoing section, we use Eqs. (6) and (10) to derive a system of equations for the determination of the steady-state conditions for the second-order processes:

$$x_0^2 = 1 + C^2\left(\frac{\gamma_1}{\gamma_0}\right)^2 x_1^2, \qquad (32)$$

$$\left[\left(\frac{\tilde{\omega}}{\gamma_0}\right)^2 + \left(\frac{\tilde{\gamma}}{\gamma_0}\right)^2\right]x_0 = \zeta^2, \qquad (33)$$

$$\frac{\tilde{\gamma}}{\gamma_0} = 1 + \frac{\sqrt{x_0^2 - 1}}{Cx_0}, \qquad (33a)$$

$$\frac{\tilde{\omega}}{\gamma_0} = \delta + \frac{\gamma_1}{\gamma_0}\left[\frac{2T_{00}}{S_{01}}x_0 + \frac{2T_{01}}{S_{11}}\sqrt{x_0^2 - 1} - \frac{S_{01}}{S_{11}}\frac{x_0^2 - 1}{x_0}\right]. \qquad (33b)$$

Here, x_0 and x_1 denote, as before, the dimensionless amplitudes n_0/n_c and n_1/n_c though the threshold amplitude is now

$$n_c = \frac{\gamma_1}{|S_{01}|},$$

and instead of the small parameter ε we have the coefficient

$$C = \frac{\gamma_0 S_{11}}{\gamma_1 S_{01}} \simeq 1.$$

This last circumstance is responsible for the marked difference between the processes of first and second orders.

The dependence of the resonance amplitude x_0 on ζ is given by the equation

$$Cx_0 + \sqrt{x_0^2 - 1} = C\sqrt{x_0}\,\zeta. \qquad (34)$$

If the spin wave-spin wave interaction is ignored, then $C \to 0$ and $x_0^2 = 1$. In reality, "freezing" of the uniform precession does not occur and even at small $(\zeta - 1)$ the amplitude x_0 is appreciably different from unity:

$$x_0 = 1 + \frac{C^2}{2}(\zeta - 1)^2 \quad \text{for } \zeta - 1 \leqslant 1. \qquad (35)$$

Hence, it is clear that the spin wave-spin wave interaction can be ignored only near the threshold itself, when $(\zeta - 1) \ll C^{-1} \sim 1$. For first-order processes the corresponding condition is satisfied in a much wider range of pumpings: $(\zeta - 1) \ll \varepsilon^{-1} \sim 10^3$–$10^4$. At large excesses $(\zeta \gg 1)$ we have

$$x_0 = \left(\frac{C\zeta}{C + 1}\right)^2 + \frac{C + 1}{C^2\zeta^2}. \qquad (36)$$

The dependence (35-36) is shown in Fig. 1b.

Experimentally [8] one observes a strong growth of the uniform precession amplitude above the threshold; this can be regarded as a qualitative confirmation of the mechanism of the spin wave-spin wave interaction. The profile of the resonance curve observed at large ζ usually has a much more complicated form than follows from (32) and (33) and it contains oscillations and "dips" [13-15]. We leave these questions open for the time being. They can only be solved within the framework of a nonstationary nonlinear theory of ferromagnetic resonance. We should only like to mention that the *stationary* solutions obtained in the foregoing sections will be realized only if they are stable against small changes in the amplitudes and the phases of the uniform precession and the spin waves (internal stability). We have considered the conditions under which the system loses this stability in [16]. The question of the nonlinear stage of the development of internal instability has not hitherto been considered. Steady-state conditions of amplitude oscillations are apparently established.

4. COMPARISON WITH EXPERIMENTAL RESULTS

Let us compare the results obtained with measurements of the nonlinear susceptibilities χ' and χ'' as a function of the pumping amplitude ζ. In Fig. 3 we show these dependences obtained in an

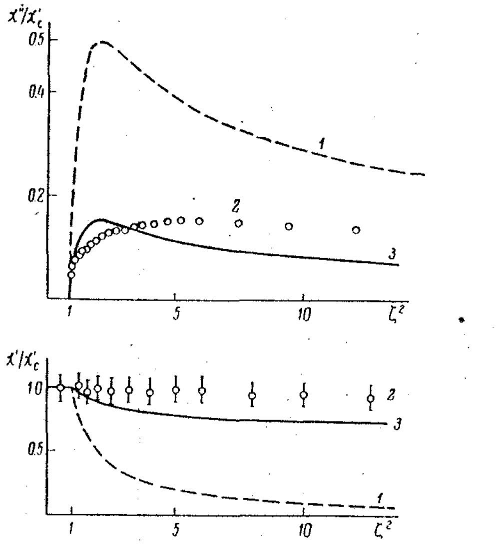

Fig. 3. Comparison of theoretical and experimental results. 1) Suhl's theory ($\varepsilon = 0$); 2) experimental points; 3) theoretical dependence for $\varepsilon = 3 \cdot 10^{-4}$.

experiment we made on a small single-crystal sphere of yttrium iron garnet at the frequency $\omega_p = 2\pi \cdot 9450$ MHz in the region of subsidiary absorption. In the same figure we give the theoretical dependences calculated from Eqs. (29) and (31). The parameter ε that enters the theory was determined from the condition of equality of the experimental and theoretical values of χ''_{max}. It was found that $\varepsilon \simeq 3 \cdot 10^{-4}$, a value that agrees well with the order of magnitude of the theoretical estimate [see (16)]. Despite its small value, allowance for ε (i.e., allowance for the spin wave-spin wave interaction) has an appreciable influence on the nature of the dependences $\chi'(\zeta)$ and $\chi''(\zeta)$. For comparison we have included in Fig. 3 the theoretical dependences for the case $\varepsilon = 0$ corresponding to a theory that takes into account only the spin wave-uniform precession interaction.

A detailed quantitative comparison of the theory considered in this paper with the experimental results requires a calculation of the coefficients of the Hamiltonian $S_{kk'}$ for specific ferromagnets. Then, using these coefficients, one would determine the real distribution function n_k as a function of ζ and substitute it into the equation of motion for uniform precession. It may then happen that weak interactions ~~(for example,~~ magnetic anisotropy), which do not exert an appreciable influence on the threshold for parametric excitation, play an important role in the spin wave-spin wave interaction and, consequently, determine the nature of the behavior above the threshold. This program can only be implemented with a computer and is clearly worthwhile only for some important cases (for example, for an yttrium iron garnet single crystal).

In our view it would be more interesting and promising to study simple systems (for example, with W_k = const) in which the main features of the behavior of the waves above the threshold of parametric excitation appear most clearly. In this connection it would be interesting to perform experiments on antiferromagnets [17-19], on ferromagnetic films, and also on other nonlinear media.

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