Physica D 57 (1992) 85–95
North-Holland

PHYSICA D

Conservation laws and two-flux spectra of hydrodynamic convective turbulence

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Received 1 March 1991
Accepted 25 February 1992
Communicated by V.E. Zakharov

The stationary spectrum of hydrodynamic convective turbulence is shown to be defined by influxes of two independent motion integrals: entropy and mechanical energy. A careful analysis of the conservation laws is performed. It is shown that in the inertial range of scales kinetic energy converts into potential energy due to presence of temperature fluctuations independently of the type of long-scale stratification (stable or unstable one). Under a purely entropic excitation (for example, by horizontal temperature gradient) the spectrum with constant entropy flux, $F_{vv} \sim k^{-21/5}$, fills the whole of the inertial interval and crossover to the Kolmogorov–Obukhov spectrum with constant energy flux, $F_{vv} \sim k^{-11/3}$, is absent. An estimate for crossover scale is obtained for a mixed method of excitation with both nonzero energy pumping and nonzero entropy extraction caused by an environment. A simple but consistent differential model is suggested for the description of the fluxes of energy and entropy in $\boldsymbol{k}$-space. Two-flux universal spectra of the velocity and temperature fluctuations are obtained.

1. Introduction

The description of the state of a moving ideal fluid is effected by means of two functions which give the distribution of the fluid velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ and of some thermodynamic quantity, e.g., the temperature $T(\boldsymbol{r}, t)$. There exist two motion integrals connected with the fields of velocity and temperature, viz. energy and entropy. So there are two principal ways to excite hydrodynamic turbulence.

The first one corresponds to barotropic motion (with temperature being constant throughout the volume at all times). If the velocity is large enough and Reynolds number is high ($\mathrm{Re} = vL/\nu \gg 1$, L is the typical length scale and ν is the kinematic viscosity), then the cascade of instabilities produces the well known Kolmogorov–Obukhov (KO) spectrum of fully developed barotropic turbulence [1]:

$$F_{vv}(\boldsymbol{k}) = C_{\mathrm{KO}}(\epsilon/\rho)^{2/3}k^{-11/3}, \quad T(\boldsymbol{r}, t) = T_0 . \tag{1}$$

Here $F_{vv}(\boldsymbol{k})$ is the Fourier transform of the simultaneous pair correlator of velocity field, ρ is the fluid density. The spectrum (1) corresponds to the constant flux ϵ of kinetic energy in $\boldsymbol{k}$-space.

The very possibility of a second way of turbulence excitation is due to the presence of the gravity acceleration $\boldsymbol{g}$. The large gradient of the temperature $\Delta T/L$ leads to the convective instability due to the buoyant force. If the Rayleigh number Ra is sufficiently high ($\mathrm{Ra} = \beta g\,\Delta T L^3/\nu\chi \gg 1$, β is the volume expansion coefficient and χ is the thermometric conductivity), then the respective Reynolds number

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of L-eddies $\mathrm{Re} = L\sqrt{\beta g\,\Delta TL}/\nu$ is high as well and the cascade of instabilities arises. Supposing turbulence spectra to be local and defined by temperature gradient it is easy to obtain from dimensional consideration (see refs. [1–4])

$$F_{vv}(k) \propto (\beta g)^{4/5} N^{2/5} k^{-21/5} ,$$
$$F_{\mathrm{TT}}(k) \propto (\beta g)^{-2/5} N^{4/5} k^{-17/5} . \qquad (2)$$

Here $F_{\mathrm{TT}}(\boldsymbol{k})$ is the Fourier transform of the simultaneous pair correlator of the temperature field, N is the flux of thermal fluctuation intensity $\int (T - T_0)^2 \,\mathrm{d}\boldsymbol{r}$ which is motion integral recently shown to coincide with entropy (see ref. [5]).

What is the physical meaning of the solution (2)? Initially it was suggested for a mechanically excited turbulence in media with stable thermal stratification [2, 3]. According to refs. [1–3], in addition to energy transfer over scales, there exists a loss of kinetic energy through work done against buoyancy forces in a broad range of scales. So the spectrum (2) $F_{vv} \propto k^{-21/5}$ drops steeper than (1). Such a physical picture appears to be correct though taking place under external parameters v, L, ΔT from a very narrow region in the space of possible parameters, as will be shown in our paper.

As far as an unstable stratification is concerned, there exists a belief shared by all the authors of refs. [1–4] that "different scales will draw additional kinetic energy from the potential energy" [1]. If it was the case, then turbulence spectrum $F_{vv}(\boldsymbol{k})$ would drop with k slower than KO-spectrum (1). Nevertheless, from dimensional analysis one can obtain only the spectrum (2) which is expressed in terms of temperature gradient. Such a paradox will be resolved in our paper by a proof that, whatever the long-scale gradients, the interaction between the velocity and temperature fields leads to the conversion of kinetic energy into the potential one in the inertial range of scales. Such a statement was previously postulated in ref. [5]. So the spectra (2) should concern to the convective turbulence as it was stated (without explanation) in ref. [4].

Though, according to [4], those spectra are intermediate asymptotics only; with growth of k, the spectrum $F_{vv} \propto k^{-21/5}$ turns into the Kolmogorov–Obukhov spectrum (1). On the face of it, it seems rather plausible, since the temperature gradient $\Delta T/L$ should excite a velocity field $v_{\mathrm{T}} \simeq \sqrt{\beta g\,\Delta TL}$ which produces a kinetic energy flux $\epsilon_{\mathrm{m}} \simeq v_{\mathrm{T}}^3/L$ into the turbulent eddies. Thus, the temperature gradients should be a source of KO-turbulence $F_{vv} \propto k^{-11/3}$. Since the latter drops with k more slowly than $k^{-21/5}$, then a short-scale asymptotics should be a Kolmogorov one. That, however, is not the case.

As a matter of fact, the spectra (2) are as fundamental as KO-spectra (1). As it was recently stated [5], those spectra correspond to a constancy of the flux of another motion integral, viz. the entropy. Two motion integrals (energy and entropy) exactly correspond to the two ways of turbulence excitation. These integrals are mutually independent, so one can suggest plenty of purely entropic methods of excitation. If in the rest fluid one creates temperature gradients, which do not vary the mean mechanical energy, then the spectra (2) with constant entropy flux will be shown below to fill the whole of the inertial interval. That statement seems to be almost obvious. As a matter of fact, that means that the whole kinetic energy influx into the inertial range completely converts there into the potential energy, so the energy attenuation rate $\bar{\epsilon}$ in the dissipative range turns into zero as $\nu \to 0$. Such an exact annihilation of two fluxes (of kinetic and potential energy) certainly connected with the law of energy conservation. In particular, the exact flux compensation takes place for turbulence generated by random temperature force or by horizontal temperature gradient as will be demonstrated below.

We shall also consider a mixed case of turbulence excitation, when external forces produce both velocity and temperature gradients. There are two influxes of motion integrals in this case. According to our understanding of turbulence universality hypothesis [6], a stationary spectrum in the inertial range of scales should be defined by both influxes (of mechanical energy, ϵ_{ext}, and of entropy N_{ext}). In particular, the short-scale asymptotics of such spectra is a Kolmogorov one at $k \gg k_b \simeq (\beta g)^{1/2} N_{ext}^{3/4}/\epsilon_{ext}^{5/4} L$. If $k_b L \simeq N_{ext}^{3/4} (\beta g)^{1/2}/\epsilon_{ext}^{5/4} \gg 1$ (where L is a typical scale of external forces), then a crossover can be observed: at $L^{-1} \ll k \ll k_b$ there exist the spectra (2), while at $k \gg k_b$ the spectrum (1) exists. It is seen, however, that the very existence of crossover scale k_b is connected with the nonzero external influx of mechanical energy ϵ_{ext}. For purely entropic excitation, $\epsilon_{ext} = 0$ and spectra (2) take place at the entire inertial interval, as was stated above.

In conclusion we shall propose a simple but consistent differential approximation of the equation system for energy and entropy fluxes in $\boldsymbol{k}$-space. We obtain a two-flux stationary solution evidently expressed in terms of the rates of energy dissipation and entropy production. This solution clearly illustrates the aforementioned properties of hydrodynamic convective turbulence.

2. Basic equations and conservation laws

To discuss our problem in the simplest situation we shall assume that the temperature inhomogeneities are small in comparison with the mean temperature of the medium $\langle T \rangle = T_0$ and the motion equations of an ideal fluid can therefore be written in the Boussinesq approximation (BA) (see, for example, section 56 in ref. [7]):

$$\partial \boldsymbol{v}/\partial t + (\boldsymbol{v} \cdot \nabla) v + \nabla p'/\rho_0 + \beta g T' = f_V , \quad \operatorname{div} \boldsymbol{v} = 0 , \tag{3}$$

$$\partial T'/\partial t + (\boldsymbol{v}\nabla) T' = f_T . \tag{4}$$

Here $T' = T - T_0$, $p' = p - \rho_0 \boldsymbol{g} \cdot \boldsymbol{r}$. Following Kraichnan [8] and Wyld [9] we simulate an excitation of velocity field with the help of a space-distributed variable force $f_V(\boldsymbol{r}, t)$. In the same way, following Procaccia and Zeitak [4] we assume that temperature force $f_T(\boldsymbol{r}, t)$ represents the effect of boundary conditions on temperature field. According to the Kolmogorov–Obukhov locality hypothesis [1] one can believe that in the limit of a large Reynolds and Rayleigh numbers, the properties of the small-scale part of the turbulence (in the inertial range) should be independent of the fine structure of the boundary conditions. So the latter could be simulated by the exciting forces $f_V(\boldsymbol{r}, t)$ and $f_T(\boldsymbol{r}, t)$ with simple statistical properties.

We suppose f_V and f_T to be random forces which do not excite the mean flow and the mean temperature:

$$\int \langle f_V(\boldsymbol{r}, t) \rangle \, d\boldsymbol{r} = 0 , \quad \int \langle f_T(\boldsymbol{r}, t) \rangle \, d\boldsymbol{r} = 0 .$$

Their pair correlators depend only on the coordinate and time differences:

$$\langle f_{V,i}(\boldsymbol{r}_1, t)\, f_{V,j}(\boldsymbol{r}_2, t) \rangle = D_{VV,ij}(\boldsymbol{r}_1 - \boldsymbol{r}_2, t_1 - t_2) ,$$
$$\langle f_T(\boldsymbol{r}_1, t)\, f_T(\boldsymbol{r}_2, t) \rangle = D_{TT}(\boldsymbol{r}_1 - \boldsymbol{r}_2, t_1 - t_2) . \tag{5}$$

These formulas are the conditions for the turbulence excitation to be stationary and homogeneous. Since there is no forced excitation of the turbulence in the inertial interval, then the correlators $D_{VV}(R)$ and $D_{TT}(R)$ must be concentrated in the energy-contained interval $R \leq L$. In the inertial interval it is necessary to put $f_V = f_T = 0$. Then eqs. (3), (4) conserve the total mechanical energy

$$\frac{dE}{dt} = 0, \quad E = \rho_0 \int \left(\tfrac{1}{2} v^2 + \beta \boldsymbol{g} \boldsymbol{r} T \right) d\boldsymbol{r} , \tag{6}$$

and the total square of temperature

$$\frac{d\theta^2}{dt} = 0, \quad \theta^2 = \int T^2(\boldsymbol{r}, t)\, d\boldsymbol{r} . \tag{7}$$

We should like to note here that eq. (4) with

$f_V = f_T = 0$ conserves not only θ^2, but also $\int \Phi(T(\boldsymbol{r}, t)\, d\boldsymbol{r}$, where Φ is an arbitrary function of T. Though only one of such functions – total entropy of the system S – has the physical meaning and is remained to be a motion integral when one refuses the Boussinesq approximation. Within the framework of the BA the entropy just corresponds to the square of temperature fluctuations [5]:

$$S = \tfrac{1}{2}\theta^2 ,$$
$$b = \left(\frac{\partial^2(\rho s)}{\partial T^2}\right) \simeq \rho\left(\frac{\partial(C_p/T)}{\partial T}\right)_p \simeq \frac{\rho C_p}{T_0^2} . \quad (8)$$

The external forces f_V and f_T serve as sources of motion integrals E and S:

$$\frac{dE}{dt} = \rho_0 \int \langle v_j(\boldsymbol{r}, t)\, f_{V,j}(\boldsymbol{r}, t)\rangle\, d\boldsymbol{r} - \beta\rho_0 g \times \int z\langle f_T(\boldsymbol{r}, t)\rangle\, d\boldsymbol{r} , \quad (9)$$

$$dS/dt = b\langle T(\boldsymbol{r}, t)\, f_T(\boldsymbol{r}, t)\rangle . \quad (10)$$

The velocity force increases the kinetic energy, so $\langle v_j(\boldsymbol{r}, t)\, f_{V,j}(\boldsymbol{r}, t)\rangle > 0$. The second term in (9) describes the variation of potential energy due to unhomogeneous heating. It can be either positive or negative for different types of vertical temperature stratification. We shall suppose z-axis to be directed upward. In the model of turbulence excitation by forces f_V and f_T random in every point $\boldsymbol{r}$, the second term in (9) disappears due to the condition $\langle f_T\rangle = 0$. However, if one considers the physical system with the mean vertical external gradient of temperature, then such a gradient should serve as a source of E. The mean horizontal gradient produces entropy flux without energy flux.

In the limit of high Re and Ra the energy is dissipated and entropy is produced in the region of small sizes (dissipative region with $l \leq L_{\text{int}}$). The question is how to find the spectra of turbulence in the inertial interval of sizes ($L_{\text{ext}} \geq l \geq L_{\text{int}}$) where there are no sources and sinks of energy and entropy? Answering to this question it is convenient to use the $\boldsymbol{k}$-representation in the basic equation (3), (4):

$$\frac{\partial v_i(t, \boldsymbol{k})}{\partial t} = \tfrac{1}{2} i \int \Gamma_{ijl}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \times v_j(t, \boldsymbol{k}_1)\, v_l(t, \boldsymbol{k}_2)\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2 + \beta g \delta_{iz} T(t, \boldsymbol{k}) + f_{V,i}(t, \boldsymbol{k}) , \quad (11a)$$

$$\frac{\partial T(t, \boldsymbol{k})}{\partial t} = \mathrm{i} k_j \int \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \times v_j^*(t, \boldsymbol{k}_1)\, T(t, \boldsymbol{k}_2)\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2 + f_T(t, \boldsymbol{k}) . \quad (11b)$$

Here Γ is the Euler vertex for barotropic turbulence and P is the transverse projector:

$$\Gamma_{ijl}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = P_{im}(k)\, \gamma_{mjl}(\boldsymbol{k})\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) ,$$
$$\gamma_{mjl}(\boldsymbol{k}) = (k_m \delta_{ml} + k_l \delta_{mj}) ,$$
$$P_{im}(k) = \delta_{im} - k_i k_m / k^2 . \quad (12)$$

To obtain the statistical description of turbulence let us determine in the usual manner the two-point simultaneous pair correlators of velocity and temperature fields in $\boldsymbol{k}$-representations:

$$F_{VV,ij}(\boldsymbol{k})\, \delta(\boldsymbol{k} - \boldsymbol{k}') = \langle v_i(t, k)\, v_j^*(t, k')\rangle ,$$
$$F_{TT}(\boldsymbol{k})\, \delta(\boldsymbol{k} - \boldsymbol{k}') = \langle T'(t, k)\, T'^*(t, k')\rangle ,$$
$$F_{TV,j}(\boldsymbol{k})\, \delta(\boldsymbol{k} - \boldsymbol{k}') = \langle T'(t, k)\, v_j^*(t, k')\rangle . \quad (13)$$

It is obvious that

$$F_{VV,ij}(\boldsymbol{k}) = F^*_{VV,ij}(-\boldsymbol{k}) = F^*_{VV,ji}(\boldsymbol{k}) ,$$
$$F_{TT}(-\boldsymbol{k}) = F^*_{TT}(\boldsymbol{k}),\ F_{TV,j}(-\boldsymbol{k}) = F^*_{TV,j}(\boldsymbol{k}) . \quad (14)$$

The condition of incompressibility leads to the following relations:

$$k_i F_{VV,ij}(\boldsymbol{k}) = k_j F_{VV,ij}(\boldsymbol{k}) = 0, \quad k_j F_{TV,j}(\boldsymbol{k}) = 0 . \quad (15)$$

Convective turbulence is supposed to be spatially homogeneous though not isotropic even in the inertial range, since the buoyant force $\beta g T'$ acts only in the z-direction ($g \| z$). The axial symmetry with respect to the z-axis still remains. Together with relations (14), (15) it allows to seek the correlators (13) in the following form:

$$F_{\mathrm{VV},ij}(\boldsymbol{k}) = P_{ij}(\boldsymbol{k})\, F_{\mathrm{VV},1}(\boldsymbol{k}) + [P_{iz}\delta_{jz} + P_{jz}\delta_{iz} - \delta_{ij}(\delta_{iz}\delta_{jz} - k_z^2/k^2)] \times F_{\mathrm{VV},2}(\boldsymbol{k})\,,$$

$$F_{\mathrm{TV},j}(\boldsymbol{k}) = P_{jl}(\boldsymbol{k})\, n_l F_{\mathrm{TV},1}(\boldsymbol{k}) + \mathrm{i}\varepsilon_{jlm} k_l n_m F_{\mathrm{TV},2}(\boldsymbol{k})\,. \tag{16}$$

It is worthwhile to mention here that the function $F_{\mathrm{VV},1}$ arises already in an isotropic turbulence, while $F_{\mathrm{VV},2}$ arises due to the presence of anisotropy. As far as the vector structure of F_{TV} is concerned, its z-component is evidently nonzero due to the second term in the right part of (11a). In addition, the rest of the components, $F_{\mathrm{TV},x}$ and $F_{\mathrm{TV},y}$ are nonzero as well, since the incompressibility condition for $\boldsymbol{v}$ (div $\boldsymbol{v} = 0$) leads to the condition (15) for F_{TV}. Thus, if $F_{\mathrm{TV},z} \neq 0$, then $F_{\mathrm{TV},x} = F_{\mathrm{TV},y} \neq 0$ (compare with [4]).

3. Continuity equations for energy and entropy in the k-space

It is important for us to study the conservation laws (9), (10) in the $\boldsymbol{k}$-space. Using eqs. (6), (7) and (12), (13) one can obtain

$$\rho_0 \frac{\partial}{\partial t}\left(\tfrac{1}{2}F_{\mathrm{VV},jj}(\boldsymbol{k},t) - \mathrm{i}\beta g\delta(\boldsymbol{k})\frac{\partial}{\partial k_z}\langle T(\boldsymbol{k},t)\rangle\right) + \frac{\partial}{\partial \boldsymbol{k}}\mathscr{E}(\boldsymbol{k},t) = \beta g\rho_0\,\mathrm{Re}\left(F_{\mathrm{TV},z}(\boldsymbol{k},t) - \delta(\boldsymbol{k})\int F_{\mathrm{TV},z}(\boldsymbol{k}',t)\,\mathrm{d}\boldsymbol{k}'\right) + \rho_0[P_{\mathrm{VV}}(\boldsymbol{k},t) + P_{\mathrm{VT}}(\boldsymbol{k},t)]\,,$$

$$\frac{\partial}{\partial t}[\tfrac{1}{2}F_{\mathrm{TT}}(\boldsymbol{k},t)] + \frac{\partial}{\partial \boldsymbol{k}}\mathscr{N}(\boldsymbol{k},t) = P_{\mathrm{TT}}(\boldsymbol{k},t)\,. \tag{17}$$

First of all it should be pointed out that potential energy is contained in the zero harmonic only (such a harmonic can be referred to as condensate). So for $k \neq 0$ there exist only kinetic energy expressed in terms of $F_{\mathrm{VV}}(\boldsymbol{k})$. The values P_{VV}, P_{VT} and P_{TT} represent the sources of integrals of motion:

$$P_{\mathrm{VV}}(\boldsymbol{k},t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1) = \langle v_j(\boldsymbol{k},t)\, f_{\mathrm{V},j}(\boldsymbol{k}_1,t)\rangle\,,$$

$$P_{\mathrm{VT}}(\boldsymbol{k},t) = -\beta g\delta(\boldsymbol{k})\int z\langle f_{\mathrm{T}}(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r}\,,$$

$$P_{\mathrm{TT}}(\boldsymbol{k},t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1) = \langle T(\boldsymbol{k},t)\, f_T(\boldsymbol{k}_1,t)\rangle\,. \tag{18}$$

Functions $\mathscr{E}(\boldsymbol{k})$ and $b\mathscr{N}(\boldsymbol{k})/2$ are the fluxes of kinetic energy and entropy in three-dimensional $\boldsymbol{k}$-space:

$$\frac{\partial\mathscr{E}(\boldsymbol{k})}{\partial\boldsymbol{k}} = \rho_0\,\mathrm{Im}\left(\int \Gamma_{ijl}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) \times F_{\mathrm{VVV},ijl}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\right), \tag{19a}$$

$$\frac{\partial}{\partial\boldsymbol{k}}\mathscr{N}(\boldsymbol{k}) = \mathrm{Im}\left(k_j\int \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) \times F_{\mathrm{TVT},j}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\right). \tag{19b}$$

Here F_{VVV} and F_{TVT} are the third-order correlators of velocity and temperature fields. Integrals over $\boldsymbol{k}$ of the right-hand parts of eqs. (19) equal zero due to symmetry relations of third order correlators and vertex functions. Due to that very reason, the left-hand parts of eqs. (19) may be written in the divergent form (as a continuity equation) and fluxes $\mathscr{E}(\boldsymbol{k})$ and $\mathscr{N}(\boldsymbol{k})$ may be introduced in the theory.

And finally let us explain the physical meaning of the first two terms in the right-hand part of the first equation (17). The first term is equal to the rate of kinetic energy variation and the second one to that of potential energy. So for $k \neq 0$ only the first term defines the behavior of energy. We shall define the sign of that term in the next subsection.

4. Energy and entropy types of turbulence excitation and crossover scale

In the stationary case, when $\partial[\cdots]/\partial t = 0$ let us integrate eqs. (17) over the sphere $k \le k_0$ with $1/L_{\rm ext} \ll k_0 \ll 1/L_{\rm int}$. The result is

$$\frac{\epsilon(k_0)}{\rho_0} = -\beta g \int \mathrm{d}\Omega \int_{k_0}^{\infty} k^2 F_{\mathrm{TV},z}(k, \Omega)\, \mathrm{d}k + (P_{\mathrm{VV}} + P_{\mathrm{VT}})\,, \tag{20a}$$

$$\mu(k_0) = bP_{\mathrm{TT}}/2\,. \tag{20b}$$

Here Ω is a solid angle and

$$\epsilon(k_0) = k_0^2 \int \mathscr{E}(k_0, \Omega)\, \mathrm{d}\Omega,$$

$$\mu(k_0) = \tfrac{1}{2} b k_0^2 \int \mathscr{N}(k_0, \Omega)\, \mathrm{d}\Omega \tag{21}$$

are kinetic-energy flux and entropy flux through the surface of the sphere in k-space with radius k_0; the constants

$$P_{\mathrm{VV}} = \int P_{\mathrm{VV}}(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}\,,$$

$$P_{\mathrm{VT}} = \int P_{\mathrm{VT}}(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}\,,$$

$$P_{\mathrm{TT}} = \int P_{\mathrm{TT}}(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k} \tag{22}$$

represent the total intensity of sources (18). Since the main contribution in the integrals (22) arises in the region $k \le 1/L_{\rm ext} \ll k_0$, then it is possible to put in (22) $k_0 = \infty$ and thus to neglect the dependences of the integrals (22) on k_0.

According to definition (18) the sources P_{VV} and P_{TT} are naturally supposed to be positive. As far as the influx of potential energy due to temperature gradient, i.e. P_{VT}, is concerned, it may be either positive or negative. The sign of P_{VT} depends on the sign of mean long-scale temperature gradient created by the force f_{T}. Moreover, while considering (20) and (22) one can forget about the artificial force f_{T} and regard influxes themselves. So if the hotter layers correspond to negative z and colder ones to positive, so the temperature gradient increases the potential energy and $P_{\mathrm{VT}} > 0$. It corresponds to an unstable stratification. In the case of stable long-scale stratification $P_{\mathrm{VT}} < 0$.

And now let us consider the energy exchange in the inertial range of scales. The only k-dependent term in the right-hand part of (20a) is the first. What is the sign of this term describing the variation of the kinetic energy of k-harmonic? As it will be proved below (see (27c)) it is positive whatever sign of βg is. And such a result can be understood with the help of simple physical speculations. Let βg be positive which corresponds to usual situation. A nonzero value of $F_{\mathrm{TV},z}$ arises due to those eddies for which upward and downward parts of flow correspond to different temperatures (yet that speculation is enough to understand that sign of such term independent of long-scale vertical stratification since only horizontal temperature gradients with the same k as for $v(k)$ contribute the value of $F_{\mathrm{TV},z}(k)$). One can meet hot pieces moving upward and cold ones downward with the same probability as otherwise (which corresponds to equal probabilities of two opposite orientations of horizontal temperature gradient with respect to eddy rotation). In the first case, $F_{\mathrm{TV},z}$ is positive while in the second one is negative. It is easy to understand that buoyance force accelerate hotter pieces moving upward and decelerate those moving downward so the absolute value of v_z should be smaller in the first case to provide a stationary distribution. Therefore, resulting value of $F_{\mathrm{T}v,z}$ (and of $\beta g F_{\mathrm{TV},z}$) should be negative. One can repeat such speculations for $\beta < 0$ and find $F_{\mathrm{TV},z}$ to be positive in this case. Indeed, for $\beta g = 0$, buoyant force is absent and turbulence should be isotropic in the inertial interval. The vector $\boldsymbol{F}_{\mathrm{TV}}$ is thus equal to zero. Therefore, $F_{\mathrm{TV},z} \propto \beta g$ (see (27c) below), so the variation of kinetic energy (which is proportional to $\beta g F_{\mathrm{TV},z}$) should not depend on the mutual orientation of large-scale temperature gradient and βg.

So the sign of the long-scale influx P_{VT} of the potential energy (but not a small-scale energy exchange) depends on the type of stratification.

Relation (20a) shows that at pure entropy method of turbulence excitation (when $P_{VV} = P_{VT} = 0$) the flux $\epsilon(k_0)$ tends to zero if $k_0 \to \infty$. That means that in this case there is no crossover to KO-spectrum (with constant kinetic-energy flux) even in the region of large k. Indeed, in this case the kinetic energy of fluid arises in the scale L_{ext} only from the potential one. This process is described by the term $-\beta g \delta(k) \int F_{TV}(k')\,dk'$ in eq. (17). But the term $\beta g F_{TV,z}(k)$ describes the reverse process of conversion of the kinetic energy into potential one in the small scale. It is important that the integral over k in the entire k-space from this two terms equals zero. This is a consequence of the conservation law of total mechanical energy which is the sum of the kinetic and potential energy of the turbulent fluid. In such a way here we prove that only external sources of kinetic energy (represented by the terms P_{VV} and P_{VT} in eq. (20a)) produce the KO-asymptotics of the power spectrum.

It is easy now to estimate the crossover scale k_b between the convective turbulence spectra (2) and barotropic KO-spectra (1). To this end let us use the kinetic energy flux $\epsilon_0(k)$ for purely entropic type of excitation (with $P_{VT} = P_{VV} = 0$) which was obtained in Ref. [5] from dimensional consideration (see also (33) below):

$$\epsilon_S(k) \simeq \rho[(\beta g)^2 P_{TT}]^{3/5} k^{-4/5} . \tag{23}$$

According to (20a) in the presence of P_{VT} and P_{VV} it is necessary to add the sum $\rho_0(P_{VV} + P_{VT})$ to the expression (23) for $\epsilon_S(k)$:

$$\epsilon(k) \simeq \epsilon_S(k) + \epsilon_0, \quad \epsilon_0 = \rho_0(P_{VV} + P_{TT}) . \tag{24}$$

At $k < k_b$ it should be $\epsilon(k) \simeq \epsilon_0(k)$. At $k > k_b \epsilon(k) \simeq (P_{VV} + P_{VT})$. Therefore $\epsilon_0(k_b) \simeq (p_{VV} + P_{VT})\rho_0$ and finally we get

$$k_b^4 \simeq [(\beta g)^2 P_{TT}]^3 / (P_{VV} + P_{VT})^5 . \tag{25a}$$

This result coincides with the estimate

$$k_b^4 \simeq [(\beta g)^2 N]^3 / \bar{\epsilon}^5 , \tag{25b}$$

which was given in [2] and [3] for the case of stable stratification, if one keeps in mind that $P_{TT} = N$ and replaces the rate of energy dissipation $\bar{\epsilon}$ on the rate of energy pumping $(P_{VV} + P_{VT})$.

It should be pointed out that in the entropic region (at $k < k_b$) energy flux $\epsilon_S(k)$ depends on k according to (23). Consequently, unlike a barotropic turbulence with $\epsilon(k) = \epsilon_0 = \text{const}$, the value $\bar{\epsilon}$ differs from the kinetic energy flux ϵ_{max} at an entrance to the inertial interval. It is necessary to know k_b for obtaining a relation between ϵ_{max} and $\bar{\epsilon}$. As a result, the estimate (25b) is proved to be inconvenient since the crossover scale is not expressed in terms of external parameters which characterized a turbulence excitation. Unlike (25b), our estimate (25a) relates the crossover scale k_b with external influxes of entropy $bP_{TT}/2$ and mechanical energy $\rho(P_{VV} + P_{VT})$. The next step is to connect influxes with more simple characteristics of turbulence excitation such as external temperature gradient, heat flux through system, long-scale velocity etc. To do it one has to describe dynamic processes of turbulence excitation. This is beyond our aims in this paper. However, we adduce some qualitative consideration of convective turbulence in the next section.

5. Convective turbulence produced by external temperature gradient

First of all let us discuss a turbulence excited by the large horizontal gradient of mean temperature $T_0(r)$. As it was already mentioned, the influx of the mechanical energy is absent in this case. Indeed, horizontal gradient do not change the potential energy. Thus, for $\nabla T \perp g$ we have $P_{VT} = 0$ and $k_b \to \infty$ so the entropic spectrum (2) fills the whole inertial interval.

Let us estimate P_{VT} under vertical temperature gradient starting with the case of unstable stratification when $\beta g \cdot \nabla T_0(r) < 0$. Since β is positive, the last inequality means that the bottom heat source permanently warms and expands subsequent portions of fluid which come with the velocity v_T and with some temperature averaged over the volume. Consequently, a bottom heater lifts the centre of inertia and thus increases the potential energy doing work against gravity force. So $P_{VT} > 0$ and it can be estimated as follows:

$$P_{VT} \simeq \beta g |\nabla T_0| L v_T \simeq \beta g \, \Delta T_0 v_T ,$$

where ΔT_0 is the variation of mean temperature on the typical length L. Note that the scalar P_{VT} can be obtained only as a scalar product of the two vectors, g and ∇T_0 so our estimate is easily generalized to the case of arbitrary angle between those vectors:

$$P_{VT} \simeq -\beta(g \cdot \nabla T_0) L v_T .$$

It agrees with the fact that $P_{VT} = 0$ for $g \perp \nabla T_0$.

The entropy influx should depend not only on the regular gradient but also on the intensity of temperature fluctuations. It can be evidently estimated as follows:

$$P_{TT} \simeq [(\Delta T)^2 + (\Delta T_0)^2] v_T / L ,$$

where ΔT is the rms fluctuation of temperature. A more strict estimate of the turbulent velocity is the following one:

$$v_T^2 \simeq \beta g L \sqrt{(\Delta T)^2 + (\Delta T_0)^2} ,$$

which takes into account that velocity produced both by regular and by chaotic temperature gradients. Substituting those estimations into (25a) one obtains, under $P_{VV} = 0$,

$$k_b L \simeq [1 + (\Delta T / \Delta T_0)^2]^{5/8} . \tag{26}$$

Without mean vertical gradient – for example, for turbulence excitation by random gradients – $\Delta T_0 = 0$ and $k_b \to \infty$. Simple supposition $\Delta T \simeq \Delta T_0$ would yield $k_b L \simeq 1$ i.e. in fact, it would mean that interval of spectrum (2) is absent. Nevertheless, there exist experimental evidence for both the inequality $\Delta T > \Delta T_0$ and the presence of spectrum (2) in the wide interval of scales [10, 11]. Rather satisfactory physical picture of the high-Rayleigh excitation of turbulent convection was described in [10, 11]. It is based on the notions of mixing zone which separates central region from the boundary layer [10] and of coherent flow around the central region [11]. Without going into details, it is worth mentioning here that typical value of temperature fluctuations substantially greater than mean vertical gradient ΔT_0 in the central region [10] which only our universal theory may concern with. Moreover, the frequency spectrum $F_{TT}(\omega)$ of temperature fluctuations [11, 12] is in the good agreement with the formula (2) if one takes into account that [5]

$$F_{TT}(\omega) \simeq k^2 F_{TT}(k) / v_T \simeq (\omega^2 / v_T^3) F_{TT}(\omega / v_T) \propto \omega^{-7/5} .$$

Such a spectrum had been observed in the frequency range $\omega_{max}/\omega_{min} \simeq 30$–$50$ under Rayleigh numbers from the interval 10^8–10^{11}. It is worthwhile to recall that the flux spectra could be realized in the inertial interval, i.e. for scales greater than dissipation length L_{int}. Such a length can be easily estimated for spectra (2) comparing the term $v \nabla v$ with dissipative term $\nu \nabla v$ omitted in (3). The result was given in [5]:

$$L_{int} \simeq L(\mathrm{Ra}_{cr}/\mathrm{Ra})^{5/16} .$$

Here Ra_{cr} corresponds to the transition of the "soft" turbulence to the "hard" one when the inertial interval arises [10]. The authors of [13] have introduced a so-called "inner scale" λ_* below which isothermal surfaces are smooth. Of course, λ_* is nothing else but L_{int} as can be seen from eq. (5.19) of [13]. And the following interesting circumstance was marked in [13]: the dissipation length decreases with Ra faster than the height l_m of the mixing zone. So at $Ra > 10^{11}$ according to [13] $l_m < l_{int}$. One can conclude that a separation scale l_m arises in the inertial interval. The spectra (2) which correspond to the constant fluxes should take place in the region $L > l > l_m$ which expands with the growth of Ra. At $l \simeq l_m$ there may exist an additional excitation mechanism, possibly connected with the thermal plumes which are formed in the mixing zone [10]. Whether some universal spectrum takes place at $l_m > l > L_{int}$ is not clear so far.

If the long-scale temperature stratification is stable, then $P_{VT} < 0$. The sufficiently powerful source of kinetic energy $P_{VV} > -P_{VT}$ is necessary for the very existence of turbulence in this case since the energy flux should be positive. If the external influx of kinetic energy is close to the outflux of the potential one due to stable stratification ($P_{VV} + P_{VT} \to +0$) then $k_b \to \infty$ and there exist large interval of scales filled by spectrum (2) with entropy flux. As was mentioned in the introduction, such a spectrum was suggested in refs. [2, 3].

6. Differential model for energy and entropy fluxes in *k*-space

In conclusion we would like to propose a simple model for studying the two-flux universal spectrum depending on P_{TT} and $P_{VV} + P_{VT}$. To this end let us estimate the correlators $F_{VVV}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ and $F_{TVT}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ at $k \simeq k_1 \simeq k_2$ in the simplest way:

$$F_{VVV}(k) \simeq [F_{VV}(k)/k]^{3/2}\,, \tag{27a}$$

$$F_{TVT}(k) \simeq \sqrt{F_{VV}(k)}\, F_{TT}(k)/k^{3/2}\,. \tag{27b}$$

The simplest dimensional estimate for F_{VT} given in ref. [4]

$$F_{VT} \simeq \sqrt{F_{VV}F_{TT}}$$

is, generally speaking, wrong. Indeed, it does not account that F_{VT} should be proportional to βg as it was explained above. More careful analysis of motion equations (3) with the use of estimates (27a,b) gives

$$F_{VT}(k) \simeq -\beta g\, \frac{F_{TT}(k)k^{-5/2}}{\sqrt{F_{VV}(k)}}\,. \tag{27c}$$

Substituting these relations in the eqs. (19) we have, under the assumption of the interaction locality,

$$\mathscr{E}(k) \simeq \rho_0 k^{7/2} F_{VV}^{3/2}(k)\,,$$

$$\mathscr{N}(k) \simeq k^{7/2} F_{TT}(k)\sqrt{F_{VV}(k)}\,.$$

Then eqs. (21) give

$$\epsilon(k) \simeq \rho_0 k^{11/2} F_{VV}^{3/2}(k)\,,$$

$$\mu(k) \simeq b k^{11/2} F_{TT}(k)\sqrt{F_{VV}(k)}\,. \tag{28}$$

In particular, in the case of barotropic turbulence with $\epsilon(k) = \bar{\epsilon}$ the relation (28) gives the KO-spectrum (1). In the mixed (two-flux) case the relations (20), (27c) and (28) give

$$k^{11/2}F_{VV}^{3/2}(k) = \left(\beta g \int_k^\infty \frac{F_{TT}(k)\,dk}{\sqrt{kF_{VV}(k)}} + (P_{VV} + P_{VT}) \right), \tag{29a}$$

$$k^{11/2}F_{TT}(k)\sqrt{F_{VV}(k)} = P_{TT}\,. \tag{29b}$$

These integral equations are equivalent to the following differential equation:

$$11k^{21/2}F_{VV}^{5/2} + 3k^{23/2}F_{VV}^{3/2}\frac{dF_{VV}}{dk} = -2(\beta g)^2 P_{TT} \tag{30}$$

with the boundary condition

$$\lim F_{VV}(k) = (P_{VV} + P_{VT})^{2/3} k^{-11/3} \quad \text{at } k \to \infty\,. \tag{31}$$

System (30), (31) has the solution

$$F_{VV}(k) = k^{-11/3}\Big[15(\beta g)^2 P_{TT}/4k^{4/3} + (P_{VV} + P_{VT})^{5/3}\Big]^{2/5} . \quad (32)$$

This equation together with (28a) yields

$$\epsilon(k) = \rho_0[(P_{VV} + P_{VT})^{5/3} + 15(\beta g)^2 P_{TT}/4k^{4/3}]^{3/5} . \quad (33)$$

With the help of eqs. (27c), (29b), (32) and (33) one can now express the solutions obtained in terms of entropy and kinetic energy fluxes:

$$F_{VV} \simeq [\epsilon(k)/\rho_0]^{2/3} k^{-11/3} , \quad (34a)$$

$$F_{TT} \simeq P_{TT}[\epsilon(k)/\rho_0]^{-1/3} k^{-11/3} , \quad (34b)$$

$$F_{TV} \simeq \beta g P_{TT}[\epsilon(k)/\rho_0]^{-2/3} k^{-13/3} . \quad (34c)$$

It should be noted that two-flux stationary spectrum was obtained for the first time for acoustic turbulence [6].

It is interesting to find an asymptotic of flux $\epsilon(k)$ (33) in the *entropy-inertial* subrange ($1/L_{ext} < k < k_b$) and in the *barotropic-inertial* subgrange ($k_b < k < 1/L_{int}$):

$$\epsilon(k) = \epsilon_S(k) + C_1\rho_0 \frac{(P_{VV} + P_{VT})^{5/3}}{(\beta g)^{4/5} P_{TT}^{2/5}} k^{8/5} \quad \text{at } k < k_b , \quad (35a)$$

$$\epsilon(k) = \epsilon_0 + C_2(\beta g)^2 \rho_0 \frac{P_{TT} k^{-4/3}}{(P_{VV} + P_{VT})^{2/3}} ,$$

$$\epsilon_0 = \rho_0(P_{VV} + P_{TT}) \quad \text{at } k > k_b . \quad (35b)$$

Here we replace numerical factors (which arise in the expansion of the eq. (33) but have no physical meaning) on some dimensionless (universal!) coefficients C_1 and C_2. One can see that only the main term in our previous rough estimation (24) for $\epsilon(k)$ coincides with the expansions (35a) and (35b). The additional (small) term in the expression for $\epsilon(k)$ is predicted correctly only in eqs. (35). It is possible to obtain these small corrections with the help of a perturbation theory in a more accurate approach.

In the same way, one can find a main term and a small correction in the expression for pair correlators. After expansion of eqs. (34) we have in the entropy-inertial subrange (at $k < k_b$):

$$F_{VV}(k) \simeq [(\beta g)^2 P_{TT}]^{2/5} k^{-21/5} \times \left(1 + C_3\left(\frac{\epsilon_0}{\rho}\right)^{5/3} \frac{k^{4/3}}{(\beta g)^2 P_{TT}}\right) , \quad (36a)$$

$$F_{TT}(k) \simeq P_{TT}[(\beta g)^2 P_{TT}]^{-1/5} k^{-17/5} \times \left(1 - C_4\left(\frac{\epsilon_0}{\rho}\right)^{5/3} \frac{k^{4/3}}{(\beta g)^2 P_{TT}}\right) , \quad (36b)$$

$$F_{VT}(k) \simeq P_{TT}^{1/2}[(\beta g)^2 P_{TT}]^{1/6} k^{-19/5} \times \left(1 - C_5\left(\frac{\epsilon_0}{\rho}\right)^{5/3} \frac{k^{4/3}}{(\beta g)^2 P_{TT}}\right) . \quad (36c)$$

Here C_3, C_4 and C_5 are universal positive dimensionless factors. The main terms in (36) correspond to the spectra of velocity and temperature fluctuations under pure entropy method of convective turbulence excitation. This term was given earlier (see eq. (2)). The corrections in eq. (36) describe the influence of small kinetic energy pumping $P_{VV} + P_{VT}$, which leads to increase in the velocity fluctuations and to depressing the temperature fluctuations. At the same time, the cross-correlator F_{TV} has to decrease under the influence of energy pumping. Note, that the sign of the corrections is determined by the sign of the power of $\epsilon(k)$ in the eqs. (34).

It should be noted that the knowledge of the correlators indices allows us to obtain other scaling properties, e.g. fractal dimensions. As it is known (see e.g. [14]), if the pair correlator of some function $Y(r)$ obeys a power law $F_{YY} \propto k^{-s}$ in k-space, then the graph of the function $Y(r)$ in real three-dimensional space as a fractal dimension D related to s as

$$D = 2 + (5 - s)/2\,.$$

For example, isothermal surfaces should possess $D = 2.8$ since the index of temperature correlator is equal to $\frac{17}{5}$. In ref. [13] the following connection between D and the index ζ of velocity scaling $v(k) \propto k^{-\zeta}$ was obtained: $D = (5 + \zeta)/2$. If one derives from (13), (36a) that in this case $\zeta = \frac{3}{5}$, then one should again obtain $D = 2.8$.

In the barotropic inertial subregion ($k_b < k < 1/L_{int}$) the expansion of eqs. (34) yields the well-known expressions for the KO-spectra of velocity and passive scalar fields and small stationary corrections to it:

$$F_{VV}(k) \simeq \left(\frac{\epsilon_0}{\rho_0}\right)^{2/3} k^{-11/3} \times \left(1 + C_6(\beta g)^2 \frac{P_{TT}k^{-4/3}}{(\epsilon_0/\rho)^{5/3}}\right), \tag{37a}$$

$$F_{TT}(k) \simeq P_{TT}\left(\frac{\epsilon_0}{\rho_0}\right)^{-1/3} k^{-11/3} \times \left(1 - C_7(\beta g)^2 \frac{P_{TT}k^{-4/3}}{(\epsilon_0/\rho)^{5/3}}\right), \tag{37b}$$

$$F_{TV}(k) \simeq \beta g P_{TT}\left(\frac{\epsilon_0}{\rho_0}\right)^{-2/3} k^{-13/3} \times \left(1 - C_8(\beta g)^2 \frac{P_{TT}k^{-4/3}}{(\epsilon_0/\rho)^{5/3}}\right). \tag{37c}$$

The first term in eq. (37a) corresponds to the KO-spectrum of barotropic turbulence (1), the second small term in (37a) (which is proportional to $(\beta g)^2$) describes the influence of small temperature fluctuation on the almost barotropic turbulence. This term may be found also with the help of first-order perturbation theory with respect to buoyancy term $-\beta g T'$ in the equation of motion (3). The main term in (37b), which is proportional to the intensity of source of the temperature fluctuation P_{TT}, describes the passive temperature fluctuations in the external turbulent velocity field with the KO-spectra (1). These fluctuations remain even in the limit $\beta g = 0$. Its origin is obviously determined by the sweeping term $(\boldsymbol{v}\cdot\nabla)T$ in (4). The term with C_7 arises due to small inverse influence of the temperature fluctuation on the velocity field. It should be pointed out that cross-correlator F_{TV} decreases faster than F_{VV} and F_{TT} at large k. Such a correlator arises due to buoyancy force.

This work was partially supported by the Meyerhoff Foundation of the Weizmann Institute of Science, Rehovot, Israel. The authors are grateful to B. Laikhtman, V. Steinberg and M. Stiassnie for useful discussions.

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