Quasiequilibrium solution of the $1+d$ KPZ model

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The properties of the correlation functions of solutions of the $1+d$ KPZ equation in the region of the strong interaction of fluctuations are considered. It is proved that analytical continuation of the solution realized at $d=1$ for the dimensions $1<d<2$ gives a solution with the dynamic index $z=(d+2)/2$. The possibility of alternative solutions is discussed.

Kardar-Parisi-Zhang (KPZ) equation is written as

$$\partial h/\partial t=\nu_0\nabla^2h+\lambda(\nabla h)^2+\xi. \tag{1}$$

Here $h(t,\mathbf{r})$ is a scalar field, λ is a coupling constant, ν_0 is a diffusion coefficient, and $\xi(t,\mathbf{r})$ is a random Gaussian "force" with the correlation function

$$\langle\xi(t_1,\mathbf{r}_1)\xi(t_2,\mathbf{r}_2)\rangle=2T\nu_0\delta(t_1-t_2)\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{2}$$

where T is the effective temperature. We stress that a system described by (1) is far from equilibrium.

The KPZ equation describes roughening of an interface in different cases, like growth of solids,[1] two-fluid flows,[2,3] motion of domain walls[4] or boundaries of clusters,[5] etc. This equation is equivalent to the Burgers equation.[6–8] It is also equivalent to the equation for the partition function of directed polymers[9] and of dislocations[10] or vortices[11] in a random potential (in these cases we should take the third coordinate instead of the time t). This variety of physical contexts is associated with the universal character of the KPZ equation representing the long-wavelength dynamics of any field h if it is invariant under $h\to h+\text{const}$ but not invariant under $h\to -h$.

Taking into consideration the interface in the $3d$ space or the vortex in the $3d$ lattice, the quantity h should be considered as a function of the $2d$ radius-vector $\mathbf{r}$. The fluctuations of the field h will then be relevant. It appears that the case of "asymptotic freedom" is realized; i.e., the dimensionless coupling constant grows with increasing scale.[12] In this situation one cannot say anything definite about the long-wavelength properties of the correlation functions of h on the basis of perturbative methods like renormalization-group equations. Numerical experiments[13–15] indicate a scaling long-wavelength behavior. From a theoretical point of view, this behavior is surprising, since in the known, exactly solvable models, where the "asymptotic freedom" is simulated, the long-wavelength behavior of the correlation functions is not of the scaling type.[16,17] The possibility of the scaling behavior of the correlation functions of h is related to famous cancellations of ultraviolet divergences in the KPZ model.[18,19]

Therefore, the problem appears to determine theoretically the dynamic index z which characterizes the correlation functions of h. As we know from the theory of second-order phase transitions, it is useful to investigate the system with strong fluctuations not only in the physical dimensionality but also near it. Equation (1) may be considered in the space of any dimension d of $\mathbf{r}$ (the physical value is $d=2$). We will investigate the behavior of the solutions of (1) for dimensions d between 1 and 2. Some conclusions concerning $d=2$ can be derived further by extrapolation of obtained results. For $d=1$ the exponent z can be derived exactly.[12,4,20,21] Its value is $z=1.5$, which has been confirmed numerically.[13-15] We will demonstrate that analytical continuation of the solution with $z=1.5$ to the dimensions $1<d<2$ gives a solution with the exponent

$$z=\frac{d+2}{2}. \tag{3}$$

This solution may be called "quasiequilibrium" since it has the same index z as the equilibrium solution.

To examine statistical properties of solutions of the KPZ equations, we will utilize a diagram technique of the type initially developed by Wyld[22] for the problem of hydrodynamic turbulence and extended to a wide class of physical systems by Martin, Siggia, and Rose.[23] A textbook description of the diagram technique can be found in the book by Ma.[24] Note that this diagram technique is a classical limit of the Keldysh diagram technique[25] which is applicable to any physical system. As was demonstrated by de Dominicis[26] and Janssen[27] (see also Refs. 28 and 29), Wyld's diagrammatic technique is generated by a conventional quantum field theory, starting from an effective action I. The corresponding methods of investigation can be found in the monograph by Popov.[30]

The explicit expression for the effective action I can be constructed on the basis of nonlinear dynamic equations of a system. For the KPZ equation the effective action is

$$I=\int dt d\mathbf{r}\ [\hat{h}\partial h/\partial t-\lambda\hat{h}(\nabla h)^2+\nu_0\nabla\hat{h}\nabla h+iT\nu_0\hat{h}^2]. \tag{4}$$

Here $\hat{h}$ is an auxiliary field which is "conjugated" with the field h. Introduction of I enables us to express the correlation functions of h in terms of the functional integrals. For example, the pair correlation function is

$$F(t_1-t_2,\ \mathbf{r}_1-\mathbf{r}_2)=-\langle h(t_1,\mathbf{r}_1)h(t_2,\mathbf{r}_2)\rangle\equiv\int\mathscr{D}h\mathscr{D}\hat{h}\exp(iI)h(t_1,\mathbf{r}_1)h(t_2,\mathbf{r}_2), \tag{5}$$

where the functional integration over the fields h and $\hat{h}$ is implied. It is also useful to define the Green's function

$$G(t_1-t_2,\ \mathbf{r}_1-\mathbf{r}_2)=-\langle h(t_1,\mathbf{r}_1)\hat{h}(t_2,\mathbf{r}_2)\rangle. \tag{6}$$

The susceptibility which determines the linear response of the system to the external "force" should be added to the right-hand side of Eq. (1). Here G is an integral kernel in the linear relationship between the external "force" and the average value of h. Therefore, the value of $G(t,r)$ is equal to zero at $t<0$ as a result of the causality principle.

We introduce instead of h, $\hat{h}$ the new variables $\mathbf{p}$, $\tilde{\mathbf{v}}$ in accordance with the definition

$$\hat{h}=\nabla_i p_i, \quad \nabla_i h=-\tilde{v}_i+iTp_i, \tag{7}$$

where p_i and $\tilde{v}_i$ are potential fields, i.e.,

$$\nabla_i p_k=\nabla_k p_i, \quad \nabla_i\tilde{v}_k=\nabla_k\tilde{v}_i. \tag{8}$$

These relations enable us to reduce the fields p_i and $\tilde{v}_i$ to the gradients of the scalar fields. The correlation functions $\langle\tilde{v}p\rangle$ and $\langle\widetilde{vv}\rangle$ can be reduced to the functions G and F. The introduction of new fields in accordance with (7) is analogous to the one performed in Ref. 18 for the case $d=1$. For $d=1$ this new representation enabled us to prove some famous properties of the perturbation series generated by the KPZ equation,[18] which appears to be useful for any dimension $1<d<2$.

In the new terms the effective action (4) is rewritten as the sum $I_0+I_1+I_2$, where

$$I_0=\int dt d\mathbf{r}(p_i\partial\tilde{v}_i/\partial t+\nu_0\nabla_i p_i\nabla_k\tilde{v}_k), \tag{9}$$

$$I_1=-\lambda\int dt d\mathbf{r}[\nabla_i p_i\tilde{v}_k^2+iT(2p_i p_k\nabla_i\tilde{v}_k-p_i^2\nabla_k\tilde{v}_k], \tag{10}$$

$$I_2=\int dt d\mathbf{r}\lambda T^2 p_i^2\nabla_k p_k. \tag{11}$$

The correlation functions $\langle\tilde{v}p\rangle$ and $\langle\widetilde{vv}\rangle$ can now be calculated in the framework of the perturbation series. The bare values of G and D are determined by the second-order part I_0 of the effective action (9). Interaction vertices are determined by the third-order terms I_1 and I_2 in the effective action. The renormalized value of the Green's function $G=-\langle\tilde{v}_i p_i\rangle$ in the Fourier representation can be written as

$$G(\omega,\mathbf{k})=[\omega+i\nu_0 k^2-\Sigma(\omega,\mathbf{k})]^{-1}, \tag{12}$$

where Σ is the self-energy function which is represented by an infinite sum of one-particle irreducible diagrams.

The expressions for Σ and G in the region of the scaling behavior are

$$\Sigma(\omega,\mathbf{k})=\mu_k\sigma(\omega/\mu_k), \tag{13}$$

$$G(\omega,\mathbf{k})=[\omega-\mu_k\sigma(\omega/\mu_k)]^{-1}, \tag{14}$$

where σ is a dimensionless function, and

$$\mu_k\propto k^z. \tag{15}$$

Here z is the dynamic scaling exponent. The term $\nu_0 k^2$ in the region can be ignored.

First, we will prove that the "truncated" Green's function G_1, which is related to an average

$$\int \mathscr{D}\tilde{v}\mathscr{D}p \exp(iI_0+I_1)v_i p_k, \tag{16}$$

has the exponent (3). We will therefore include the term I_2 (11) of the effective action and show that this inclusion will not change z. The proof is close to the one proposed in Ref. 18, where the case $d=1$ was examined.

The "truncated" Green's function G_1 can be written in the same form (14) as the complete function with the self-energy function Σ_1 which is represented by an infinite sum of one-particle irreducible diagrams, where only the vertices determined by (10) are present. We would like to stress that the "truncated" dressed correlation functions $\langle \tilde{v}_i\tilde{v}_k\rangle_1$ and $\langle p_i p_k\rangle_1$ are equal to zero. After a partial summation of the diagrams we obtain a diagrammatic series for Σ_1, where the bare vertices and the dressed pair correlation functions are taken into account. Since the correlation functions $\langle \tilde{v}_i\tilde{v}_k\rangle_1$ and $\langle pp\rangle_1$ are zero, this series contains the functions G_1 only.

Now we should take into consideration the omitted interaction terms I_2 (11) in the effective action. Consider the "complete" self-energy function Σ which we will treat as a series over I_2. Each term in this series is actually determined by an infinite sequence of diagrams generated by the interaction terms in I_1 (10). We assume that the partial summation of the diagrams dressing the bare Green's function G_0 into G_1 is the same. Then Σ will be represented as a sum of diagrams, where the "truncated" Green's functions G_1 and the bare vertices determined by the terms I_1 and I_2 in the effective action are taken into account. Note that the "complete" correlation function $\langle v_i v_k\rangle$ is not equal to zero, whereas the "complete" correlation function $\langle pp\rangle$ is zero.

Let us use G_1 in the form (14) with $z=(d+2)/2$ (3). Simple dimension estimations show that any contribution to Σ_1 and Σ will then have the form (13) with the same index z. We can thus conclude that the above assumption is self consistent and consequently (3) is the true exponent for the KPZ model. But such a conclusion is potentially very dangerous. The problem is that in the diagrammatic series for Σ_1 or Σ ultraviolet logarithmic divergences might appear. These divergences can change the exponent of the solution[31] or even destroy the scaling behavior. The famous peculiarity of KPZ model is that these divergences do not appear in the diagrammatic series for the "truncated" and the "complete" problems which are demonstrated below (the absence of divergences in the "complete" KPZ problem in another language was proved in Ref. 19).

As is well known, a loop of a diagram gives rise to an integration over a wave vector $\mathbf{q}$ in the corresponding analytical expression. Therefore, we should prove the absence of the ultraviolet divergences in the integrals corresponding to all loops of the diagrams which represent contributions to Σ. The diagrams are constructed from lines representing the G_1 functions which are marked by arrows directed from p to $\tilde{v}$. Then a loop constructed from lines directed clockwise or anticlockwise give zero contribution because of causality properties of the G functions.[18] Therefore, we may consider only the loops with some "inputs" and "outputs," where the two G lines begin or end

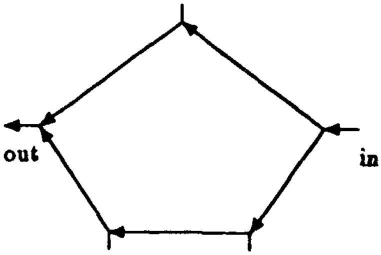

FIG. 1.

(a loop with one "input" and one "output" is shown in Fig. 1. Input is designated by "in" and output is denoted by "out").

The "outputs" are produced only by the term I_1 (10) and the "inputs" are produced by the term I_1 (10) and the term I_2 (11). We see that in the "input" and "output" vertices produced by I_1 the derivative ∇ acts on a field which is external to the loop. The same assertion is valid for the "input" vertices produced by I_2 since this term can be rewritten in the form

$$I_2 \to \int dt d\mathbf{r} \lambda T^2 \left(\frac{3}{2} p_i^2 \nabla_k p_k' - p_i p_k \nabla_k p_i' \right), \tag{17}$$

where the prime designates the field external to the loop. This property enables us to prove that at large q the integral corresponding to the loop behaves as

$$q^{d+z} \chi q^{n-2m} \chi q^{-nz}, \tag{18}$$

where n is the number of vertices of the loop, and m is the number of "inputs" or "outputs". It is not difficult to verify that the exponent here is negative for $n \geqslant 2$, $m \geqslant 1$ and z determined by (3) if $d < 2$. Therefore, the integral converges at large q.

Thus, we proved that at $1 < d < 2$ there is no ultraviolet divergence in the diagrammatic series for Σ_1 or Σ, where the bare vertices and the dressed functions G_1 are taken into account. It implies that the scaling solution with the value (3) of the dynamic index does actually exist.

Let us now return to the original variable h. Using relations (7), we find

$$\int dt d\mathbf{r} \exp(i\omega t - i\mathbf{k}\mathbf{r}) F(t,\mathbf{r}) = k^{-2-z} f(\omega/\mu_k), \tag{19}$$

which is a consequence of Refs. 12 and 14.

For $d=2$ the numerical experiments show that the value of z is between 1.6 and 1.7 (Refs. 13–15), whereas the solution (3) gives $z=2$ at $d=2$, which does not coincide with the above value. The reason for this contradiction is that the system of diagrammatic equations is very complicated, and a solution with another exponent or even with a nonscaling behavior may be realized. Our assertion concerns only the existence of a quasiequilibrium solution. Therefore, the observed value of the index z

cannot be found in the framework of $1+\epsilon$ expansion. The quasiequilibrium solution might be realized as a metastable solution under some conditions, for example, at different short wavelength terms in the dynamic equation for h.

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