Scaling of correlation functions of velocity gradients in hydrodynamic turbulence

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As is demonstrated in Refs. 2 and 3 in the limit of infinitely large Reynolds numbers, the correlation functions of the velocity predicted by Kolmogorov's 1941 theory (K41) are actually solutions of diagrammatic equations. Here we demonstrate that correlation functions of the velocity derivatives, $\nabla_\alpha v_\beta$, should possess scaling exponents which have no relation to the K41 dimensional estimates. This phenomenon is referred to as anomalous scaling. This result is proved in diagrammatic terms: We have extracted a series of logarithmically diverging diagrams, whose summation leads to the renormalization of the normal K41 dimensions. For a description of the scaling of various functions of $\nabla_\alpha v_\beta$, an infinite set of primary fields O_n with independent scaling exponents Δ_n can be introduced. Symmetry reasons enable us to predict relations between the scaling of different correlation functions. We also formulate restrictions imposed on the structure of the correlation functions due to the incompressibility condition. We also propose some tests which make it possible to check experimentally the conformal symmetry of the turbulent correlation functions. Further, we demonstrate that the anomalous scaling behavior should reveal itself in the asymptotic behavior of the correlation functions of the velocity differences. We propose a method to obtain the anomalous exponents from the experiment.

The theory of turbulence is the theory of strongly fluctuating hydrodynamic motion. Systems with strong fluctuations are examined both in quantum field theory and in condensed matter physics, e.g., in treating second-order phase transitions. It is known that adequate tools of theoretical investigation of strong fluctuating systems are based on functional integration methods, on different versions of the diagrammatic technique, and on related methods. Therefore, a consistent theory of turbulence should also be constructed in these terms.

The diagram technique for the problem of turbulence was developed by Wyld,[1] who started from the Navier–Stokes equation with a pumping force. The Wyld technique enables one to represent any correlation function characterizing the turbulent flow as a series over the nonlinear interaction. Unfortunately, infrared divergences appear in the technique. To avoid the divergences we will make use of the quasi-

Lagrangian (qL) variables. The perturbation theory of the Wyld type in qL variables was developed by Belinicher and L'vov[2] (see also the review[3]).

The Wyld diagrammatic expansion is formulated in terms of the propagators G and F and vertices determined by the nonlinear term of the Navier–Stokes equation. The G-function is the linear susceptibility determining the average value $\langle v_\alpha \rangle$ of the velocity $\mathbf{v}$ which arises as a response to the nonzero average $\langle \tilde{f}_\alpha \rangle$:

$$G_{\alpha\beta}(t,\mathbf{r}_1,\mathbf{r}_2) = -i\delta\langle v_\alpha(t,\mathbf{r}_1)\rangle / \delta\langle f_\beta(0,\mathbf{r}_2)\rangle, \tag{1}$$

where $\mathbf{f}$ is the pumping force. The F-function is the pair correlation function of $\mathbf{v}$:

$$F_{\alpha\beta}(t,\mathbf{r}_1,\mathbf{r}_2) = \langle v_\alpha(0,\mathbf{r}_1) v_\beta(t,\mathbf{r}_2)\rangle. \tag{2}$$

Note that the propagators G and F in qL variables depend separately on the coordinates of the points $\mathbf{r}_1$, $\mathbf{r}_2$. Besides, the simultaneous correlation function $F(t=0,\mathbf{r}_1,\mathbf{r}_2)$, which coincides with the simultaneous correlation function of the Eulerian velocities, depends only on the difference $\mathbf{r}_1 - \mathbf{r}_2$.

To establish the behavior of (1) and (2), we can utilize the dimensional estimates by Kolmogorov and Obukhov.[4,5] For the pair correlation function (2) we obtain

$$F(t,\mathbf{r}_1,\mathbf{r}_2) \sim (\bar{\varepsilon} R)^{2/3}, \tag{3}$$

where R is the characteristic scale, and $\bar{\varepsilon}$ is the average value of the energy dissipation rate per unit mass

$$\varepsilon = (\nu/2)(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha)^2. \tag{4}$$

The Green's function also possess the scaling behavior with

$$G(t,\mathbf{r}_1,\mathbf{r}_2) \sim R^{-3}. \tag{5}$$

Can such scaling behavior be obtained as a solution of the diagrammatic equations? To answer this question, one should first reformulate the diagram technique in terms of the bare vertices and the dressed propagators F and G. Then one can easily check that the scaling behavior of F and G, determined by the estimates (3) and (5), is reproduced in any order of the perturbation theory. But this is not sufficient to justify the assertion that F and G actually possess such scaling behavior. The reason for this was recognized long ago in the theory of second-order phase transitions. Reformulating the diagrammatic series for the correlation functions of the order parameter in terms of the bare interaction vertex with the dressed correlation function with its suitable scaling exponent, one can check that this exponent is reproduced in each order of the perturbation theory. Furthermore, one immediately encounters logarithmic ultraviolet divergences which arise in each order of the perturbation expansion. The logarithmic corrections are summed up to generate power corrections which strongly renormalize the naive exponents.

Fortunately, this phenomenon does not occur in the theory of turbulence. As was demonstrated by Belinicher and L'vov[2] in the qL variables, there are neither infrared nor ultraviolet divergences in the diagrammatic expansion for G and F, if (3) and (5) are used. Such a theorem can be proved for high-order correlation functions of $\mathbf{u}$. This property is the basis for the assertion that in the consistent theory the simultaneous

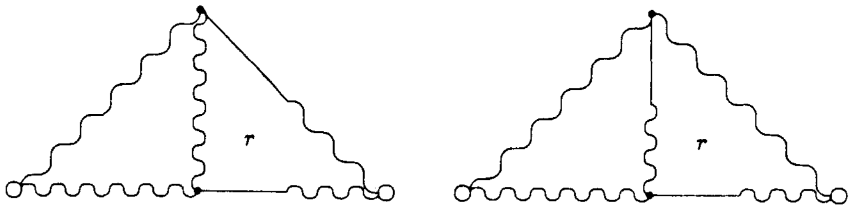

FIG. 1. The first diagrams for $K_{\varepsilon\varepsilon}$ producing ultraviolet logarithms.

correlation functions actually have naive K41 exponents (see also Ref. 6). Nevertheless, ultraviolet logarithms immediately arise in the diagrams for the correlation functions of the powers of the velocity gradient $\nabla_\alpha v_\beta$. The simplest example of such a correlation function is the following irreducible correlation function:

$$K_{\varepsilon\varepsilon}(R) \equiv \langle\langle \varepsilon(t,\mathbf{r})\varepsilon(t,\mathbf{r}+\mathbf{R})\rangle\rangle. \tag{6}$$

Let us analyze the diagrammatic series for $K_{\varepsilon\varepsilon}(\mathbf{r}_1,\mathbf{r}_2)$. The first one-loop diagram for $K_{\varepsilon\varepsilon}(\mathbf{r}_1,\mathbf{r}_2)$ gives the expression which has a normal K41 behavior $\propto R^{-8/3}$. The diagrams of the next order are shown in Fig. 1, where circles denote the points $\mathbf{r}_1$ and $\mathbf{r}_2$, the vertex is determined by the nonlinear term in the Navier–Stokes equation, the wavy line corresponds to the pair correlation function (3), and the combined wavy–straight line represents the Green's function (1). Using the estimates (3) and (5), we find that these diagrams give us the expressions for $K_{\varepsilon\varepsilon}$, which are $\propto R^{-8/3}$. However, there are also logarithmic divergences in these diagrams which are related to the loops marked by the letter "r." Therefore, the final result behaves as $R^{-8/3}\ln(R/\eta)$. Generalizing the above analysis, we conclude that diagrams of the nth order will produce the normal K41 factor $R^{-8/3}$ with prefactors which are different powers of the logarithm up to the nth power. Thus, we encounter a series over the large logarithm $\ln(R/\eta)$, which could be an arbitrary function. Below we argue that this function is an exponential function which is a power of R/η. Such a function in the prefactor produces an anomalous scaling.

On the basis of the Wyld technique a formally exact diagram representation for $K_{\varepsilon\varepsilon}$ can be formulated by working from the fact that in each diagram for $K_{\varepsilon\varepsilon}$ there exists only one cut going along all F-functions.[3] This enables us to formulate the representation shown in Fig. 2. There we have classified diagrams for $K_{\varepsilon\varepsilon}$. In accor-

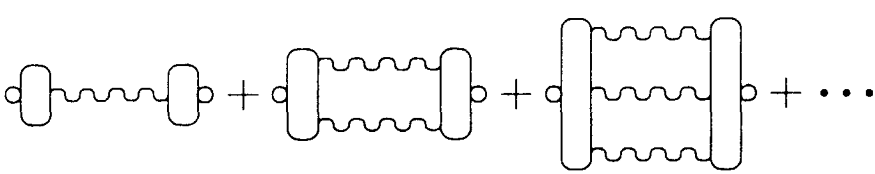

FIG. 2. The formally exact diagrammatic representation for $K_{\varepsilon\varepsilon}$, the first terms of an infinite series.

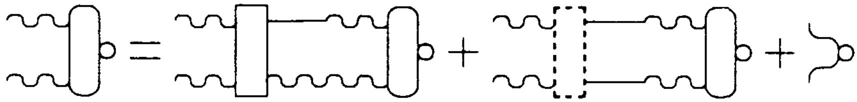

FIG. 3. The diagrammatic equation for the three-leg object Υ contained in the diagrammatic expression for $K_{\varepsilon\varepsilon}$.

dance with the number of F-functions in our marked cut, the ovals designate objects which are sums of the blocks at the left and the right sides of the marked cut.

The first "one-bridge" term of the diagrammatic series, shown in Fig. 2, is actually reduced to the objects arising in the second "two-bridge" term. Therefore, we begin our analysis with this three-leg object Υ, which corresponds to an oval in this "two-bridge" term. Designating by boxes the sums of the four-leg parts of the diagrams which cannot be cut along two lines, we come after summation to the diagrammatic relation, shown in Fig. 3, where the last term designates the bare contribution. The diagrammatic relation can be rewritten in analytical form, which gives the integral equation for Υ. The kernel B of this equation corresponds to the sum of the boxes in Fig. 3 with the attached lines. Following the analysis given in Refs. 2 and 3, we can demonstrate that there are neither ultraviolet nor infrared divergences in the higher-order diagrams for B. This means that the first contributions give the correct scaling for B. Using (3) and (5), we conclude that the integration in the equation is dimensionless. It follows that this equation admits scaling solutions for Υ. Actually, an infinite number of terms with different exponents are present in Υ, since the equation for Υ is an integral equation. Analogously, the higher-order terms of the series shown in Fig. 3 can be analyzed. Thus we conclude that $K_{\varepsilon\varepsilon}$ has a complicated scaling behavior. The same is true for all correlation functions of the local fields $\varphi_j(\mathbf{r})$ which are constructed as different single-point products of the velocity gradients, since the gradients produce logarithms and consequently anomalous dimensions.

To proceed with the analysis of the scaling, it is worthwhile to extract a set of local fields A_n with a "clean" scaling behavior; specifically, each local field A_n is characterized by its scaling dimension Δ_n which means that

$$\langle A_n(\mathbf{R})A_m(0)\rangle \propto R^{-\Delta_n-\Delta_m}. \tag{7}$$

From the set A_n we can extract the subset of the so-called primary fields O_n, which give rise to all other fields A_n by differentiation. These "field-descendants" A_n are usually called *secondary fields.* The dimension Δ of any secondary field A differs from the dimension Δ_n of the corresponding primary field O_n by an integer number l: $\Delta=\Delta_n+l$, where the number l is the number of the differentiations needed to obtain A from O_n. An example of a primary turbulent field is the velocity $\mathbf{v}$, which has the normal K41 scaling dimension $\Delta_v=-1/3$.

Any local field φ_j can be expanded in a series over the fields A_n with certain coefficients:

$$\varphi_j(\mathbf{r}) = \sum_n \varphi_{j(n)} A_n(\mathbf{r}). \tag{8}$$

This expansion enables one to reduce the correlation functions of φ_j to the correlation functions of the fields A_n. Unfortunately, it is impossible to find the values of Δ_n, but one can express the scaling behavior of the observable quantities in terms of Δ_n. It is convenient to order the fields A_n over the values of their scaling dimensions: $\Delta_1 \leqslant \Delta_2 \leqslant \Delta_3 \ldots$. It is clear that the principal scaling behavior of the correlation functions of φ_j is determined by the first nonzero term of its expansion in a series over A_n. If the first terms of the expansions of φ_1 and φ_2 are not equal to zero, for example, the scaling behavior of the principal term in the correlation function $\langle\langle \varphi_1(\mathbf{R})\varphi_2(0)\rangle\rangle$ is $\propto R^{-2\Delta_1}$. We expect that this behavior is inherent to the correlation function of the two scalar fields. Since (6) is such a function, we conclude that $2\Delta_1 = \mu$, where by definition $K_{\varepsilon\varepsilon} \propto R^{-\mu}$.

Now we can formulate for hydrodynamic turbulence the fusion rules for fluctuating fields, as introduced by Polyakov.[7] It is obvious that the product of the fields $A_n(\mathbf{r}_1)A_m(\mathbf{r}_2)$, taken at the nearby points, behaves like a single-point object, which can be expanded in a series over $A_n(\mathbf{r})$. Thus we obtain the relations

$$A_n(\mathbf{r}_1)A_m(\mathbf{r}_2) = \sum_l C_{mn,l}(\mathbf{r}_1 - \mathbf{r}_2) A_l[(\mathbf{r}_1 + \mathbf{r}_2)/2], \tag{9}$$

which are known as the operator algebra.[8,9] The relations (8) and (9) can be used to investigate any correlation function of the fields φ_j with two neighboring points.

Special consideration is needed for the correlation functions of the first power of the velocity $\mathbf{v}$ and of its derivatives because of the incompressibility condition. The cross-correlation function of the velocity with any scalar field φ_j, for example, is equal to zero. To prove this point, note that the correlation function $\langle \mathbf{v}(\mathbf{r})\varphi_j(0)\rangle$ is a vector which can be directed only along $\mathbf{r}$. We know, however, that the divergence of this vector should be equal to zero because of incompressibility, $\nabla \cdot \mathbf{v} = 0$.

Note that if the system has conformal symmetry, then there exists a set of strong selection rules for the coefficients in the r.h.s. of (7), which were established by Polyakov.[10] These coefficients are not equal to zero for different values of Δ_n and Δ_m only if these fields are secondary fields of the same primary field. This is a consequence of the "orthogonality rule": The correlation functions of different primary fields O_n are equal to zero if the system has conformal symmetry. This question arises in connection with the recent work of Polyakov,[11] who treated the 2D turbulence in the framework of the conformal approach. As we know,[12] for 2D systems the conformal symmetry permits one to establish many properties of the correlation functions, particularly the possible sets of dimensions Δ_n. The conformal symmetry imposes also some restrictions on the r-dependence of the correlation functions in 3D (Ref. 13).

Using (8) and (9), we can examine the asymptotic behavior of the correlation functions of the velocity differences. Consider the case in which there are two sets of neighboring points—$\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, $\mathbf{r}_4$—which are separated by a large distance R. Take, for example, the second power of the velocity difference $(\mathbf{v}_1-\mathbf{v}_2)^2$. Using (9), we can "fuse" this object into a single point. It is natural to expect that the principal term in this expansion is determined by O_1:

$$(\mathbf{v}_1-\mathbf{v}_2)^2 \rightarrow f(r_{12})O_1[(\mathbf{r}_1+\mathbf{r}_2)/2], \tag{10}$$

where $r_{12}=|\mathbf{r}_1-\mathbf{r}_2|$ This means that

$$\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^2(\mathbf{v}_3-\mathbf{v}_4)^2\rangle\rangle \propto f(r_{12})f(r_{34})R^{-2\Delta_1}. \tag{11}$$

Recall that $2\Delta_1=\mu$. The r-dependence of the function $f(r)$ can also be determined if one remembers that the general scaling behavior of the correlation function of the velocity differences in the l.h.s. of (11) is determined by the conventional K41 index $-4/3$. Comparing this index with the scaling behavior (11), we conclude that $f(r) \propto r^{\Delta_1+2/3}$. We should take into consideration the terms of the expansion of $(\mathbf{v}_1-\mathbf{v}_2)^2$ over the vector and tensor fields. These terms describe the dependence of $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^2(\mathbf{v}_3-\mathbf{v}_4)^2\rangle\rangle$ on the angles between $\mathbf{R}$ and $\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{r}_3-\mathbf{r}_4$. The main term is determined by the smallest value $\Delta_{;1}$ of the principal scaling exponents of the tensor fields.

The proposed scheme can easily be generalized to all even powers $(\mathbf{v}_1-\mathbf{v}_2)^{2n}$. The main contributions to the correlation function $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^m(\mathbf{v}_3-\mathbf{v}_4)^n\rangle\rangle$ are as follows: The first contribution $\propto R^{-2\Delta_1}$ does not depend on the angles, the second contribution $\propto R^{-\Delta_1-\Delta_{;1}}$ is the sum of two terms which depend on the angle between $\mathbf{R}$ and $\mathbf{r}_1-\mathbf{r}_2$ or on the angle between $\mathbf{R}$ and $\mathbf{r}_3-\mathbf{r}_4$ only, and the third term $\propto R^{-2\Delta_{;1}}$ depends on both angles. This is also the point at which the conformal symmetry is revealed, since it cancels the second contribution $\propto R^{-\Delta_1-\Delta_{;1}}$.

Let us now analyze the correlation functions of odd powers. First, we consider the special case of the first power, since the difference $\mathbf{v}_1-\mathbf{v}_2$ has the normal K41 dimension. The main term of the expansion of this difference in a series over local fields is $\nabla_\alpha v_\beta$. This means, e.g., that $\langle(v_{1\alpha}-v_{2\alpha})(v_{3\beta}-v_{4\beta})\rangle \propto R^{-4/3}$. Consider now the correlation function $\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle$. As we have seen, the correlation function $\langle\mathbf{v}O_n\rangle$ is zero for any scalar field O_n. Therefore, only the vector and tensor fields A_n should be taken into account in the expansion of $(\mathbf{v}_3-\mathbf{v}_4)^{2n}$ which gives

$$\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle \propto R^{-2/3-\Delta_{;1}}, \tag{12}$$

where $\Delta_{;1}$, as above, is the smallest exponent of the tensor fields. Of course, among the fields A_n in the expansion (8) for $(\mathbf{v}_3-\mathbf{v}_4)^{2n}$ there is a term with $\nabla_\alpha v_\beta$. This means that in any case there is a term $\propto R^{-4/3}$ in the correlation function $\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle$. This is again the point at which the conformal symmetry can be checked: it admits only the behavior $\propto R^{-4/3}$.

Now consider a general odd power of the velocity difference $(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_1-\mathbf{v}_2)^{2n}$. It can be expanded in a series over the same fields A_n as even powers but with more complicated angular dependence of the coefficients. Therefore, the scaling behavior of the mutual correlation functions of the odd–odd and the odd–even correlation functions at large separations will be the same as the behavior of the even–even correlation function. The terms with different scaling exponents can in principle be separated on the basis of their angular dependence.

The conclusions made in this article concern the principal scaling behavior of the correlation functions of the velocity gradients and velocity differences. Therefore, we hope that our predictions permit direct experimental verification.

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