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Isotropic and Anisotropic Turbulence in Clebsch Variables

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Abstract—Three-dimensional turbulence of incompressible fluid is described by using Clebsch canonical variables. This reveals the families of new local integrals of motion so that there are additional cascade spectra besides the energy cascade. A weakly anisotropic spectrum of developed turbulence is shown to be as universal as isotropic Kolmogorov spectrum. The correlation functions of three-dimensional incompressible turbulence approach their isotropic values in the inertial interval so that the share taken by the anisotropic parts of velocity correlators decrease with the wavenumber as $k^{-2/3}$, which satisfactorily fits the experimental data. The complementarity of the turbulence description in Clebsch and velocity variables is demonstrated.

1. INTRODUCTION

The peculiarity of incompressible fluid turbulence is the existence of different motion invariants. Energy conservation gives the Kolmogorov spectrum with constant energy flux [1–5] and the respective symmetry of the equation makes it possible to establish some exact (Kolmogorov) relation for the triple velocity correlation function (flux constancy—see Section 4 below). In addition, helicity conservation [6] allows one to find the spectrum of gyrotropic turbulence [7] (Section 4).

Here we discuss additional integrals of motion and respective cascade spectra: action cascade of isotropic turbulence and momentum cascade in weakly anisotropic turbulence. The main subject of this paper is turbulence approach to isotropy that is the behaviour of nonisotropic perturbations of the Kolmogorov spectrum. This has been the subject of much theoretical and experimental work. Besides the phenomenological [8] and numerical approaches [9], the only analytical result is that by Leslie [10] for the particular case of mean flow with constant shear. Meanwhile, the approach of three-dimensional turbulence to isotropy is governed by a law as universal as Kolmogorov spectrum itself. This law satisfactorily fits the experimental data summarized in [11]. An exact relation similar to the Kolmogorov one can be proved for an anisotropic part of the correlation function (Section 2.3). Previous analytical approaches developed in terms of velocity variables could not reach this solution, since the respective integral of motion that governs approach to isotropy cannot be expressed via the double velocity correlator. This integral has a simple form and physical meaning in the Clebsch variables. Going over to the canonical Clebsch variables (Section 2.1) makes the problem of hydrodynamic turbulence similar to a large variety of cases of wave turbulence, where the problem of approach to (or departure from) isotropy was satisfactorily solved during the last two decades [12, 13]. The key notion in solving this problem is the conservation of momentum. An anisotropic perturbation carries

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some wave momentum conservation of which governs the behaviour of the perturbation. Incompressible fluid flow has no momentum except that of the mean flow. Unlike waves, eddies cannot transport momentum with respect to the fluid. However, if we represent velocity through the Clebsch variables and introduce normal canonical amplitudes, then the auxiliary wave field does have a momentum, and the theory can be constructed by analogy with the general theory of wave turbulence [13] by using the method of conformal transformations (Section 2.3).

We write down the equation that expresses the time derivative of the simultaneous double correlation functions via the simultaneous fourth-order correlation function in the Clebsch variables. That allows us to avoid the problem of infrared divergencies of integrals in the perturbation theory caused by the sweeping interaction, which contributes only in the ω-dependence of different-time propagators. The first angular harmonic of the simultaneous fourth-order correlation function (that is the momentum flux in the Clebsch variables) could be thus found as an exact steady solution of the equation linearized with respect to small anisotropy (Section 2.3). Behaviour of the first angular harmonics of other correlation functions and of the Green's function can be found by using the knowledge of the fourth correlator and by analysing diagrammatic perturbation series which is possible at any order due to the weakness of the anisotropy (Sections 2.4, 2.5). To find the behaviour of subsequent angular harmonics, a linear approach is insufficient, though the smallness of the anisotropic parts of the correlators makes it possible to develop a regular calculation procedure (Section 2.5). The solution thus obtained can be then expressed in the velocity variables (Section 2.6). It is worth emphasizing that we cannot rigorously prove that the set of correlation functions thus found is an exact solution of the Euler equation (this is possible only for the first angular harmonic of the fourth correlation function). Neither can one prove the same for the isotropic Kolmogorov spectrum. Our aim here is to develop the theory of anisotropic turbulence to the same extent as that of isotropic turbulence: finding an exact relation for the flux and analysing convergency of the perturbation expansion (locality check).

Clebsch variables reveal also an additional isotropic cascade of three-dimensional turbulence $E(k) \propto k^{-1}$. We discuss this spectrum and its structural stability from different viewpoints in Section 3. In addition to obtaining a new universal spectrum in turbulence theory, this paper aims at demonstrating the complementarity of turbulence descriptions in the velocity and Clebsch variables. It is shown in Sections 3, 4 that some of the solutions can be obtained after a reduction of the perturbation series in both representations, while some can be obtained only in one of them. The reason is that reduction of the series in terms of the Eulerian velocity may break symmetries which are preserved by a similar reduction in terms of Clebsch variables and vice versa.

2. WEAKLY ANISOTROPIC STEADY SPECTRUM

2.1. *Clebsch canonical variables*

In the present paper we consider spectra of turbulence that are certainly nonequilibrium and have fluxes of different quantities. It is a standard approach (that can be substantiated by showing that there are no ultraviolet divergences for values in question) to consider the spectra in the inertial interval of scales in the framework of Euler rather than Navier–Stokes equation, using viscous as a sink that is necessary for the whole picture with the fluxes to exist. This means that we are talking only about such solutions of the Euler equation, that are limits of the respective solutions of the Navier–Stokes equation at $\nu \rightarrow 0$.

Then the Euler equation for incompressible fluid [11, 14] is

$$\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0, \tag{1}$$

Here $\mathbf{v}(\mathbf{x}, t)$ is the (Eulerian) velocity field.

By virtue of Kelvin's theorem, the velocity circulation around a fluid path is conserved. This allows us to represent the vorticity lines as an intersection of the level surfaces of two scalar functions $\lambda(\mathbf{x}, t)$ and $\mu(\mathbf{x}, t)$ [15–17]: $\operatorname{rot}\mathbf{v} = [\nabla\lambda, \nabla\mu]$. This is possible if the vorticity lines are not knotted, that is the helicity integral

$$H = \int (\mathbf{v} \cdot \operatorname{rot}\mathbf{v})d\mathbf{r} \tag{2}$$

is identically equal to zero [6]. Under such an assumption, one may represent equation (1) in a canonical Hamiltonian form [17–21]

$$\frac{\partial\lambda(\mathbf{x}, t)}{\partial t} = \frac{\delta\mathcal{H}}{\delta\mu(\mathbf{x}, t)}, \qquad \frac{\partial\mu(\mathbf{x}, t)}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\lambda(\mathbf{x}, t)}. \tag{3}$$

The Hamiltonian $\mathcal{H}$ represents the kinetic energy expressed via the Clebsch canonical variables $\lambda(\mathbf{x}, t)$ and $\mu(\mathbf{x}, t)$,

$$\mathcal{H} = \frac{1}{2}\int d^3x |\mathbf{v}(\mathbf{x}, t)|^2, \qquad \mathbf{v} = \lambda\nabla\mu - \nabla\phi. \tag{4}$$

The potential $\phi(\mathbf{x}, t)$ is determined from the incompressibility condition.

The general theory of turbulence in the Hamiltonian systems is developed in terms of normal canonical variables that reveal all the conservation laws governing turbulence cascade [13].

Following [22, 27, 28] let us go over from the pair of the real variables $\lambda(\mathbf{x}, t)$ and $\mu(\mathbf{x}, t)$ to the complex ones $a(\mathbf{x}, t)$ and $a^*(\mathbf{x}, t)$, $\sqrt{2}a(\mathbf{x}, t) = \lambda(\mathbf{x}, t) + i\mu(\mathbf{x}, t)$. In these normal variables equations (3, 4), after transition to $\mathbf{k}$-representation, have the form

$$i\frac{\partial a(\mathbf{k}, t)}{\partial t} = \frac{\delta\mathcal{H}}{\delta a^*(\mathbf{k}, t)}, \tag{5}$$

$$\mathcal{H} = \frac{1}{4}\int T(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4)a_1^*a_2^*a_3a_4 \mathrm{d}^3k_1\mathrm{d}^3k_2\mathrm{d}^3k_3\mathrm{d}^3k_4, \tag{6}$$

$$T(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) = (\psi_{13} \cdot \psi_{24} + \psi_{14} \cdot \psi_{23})\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4), \tag{7}$$

$$\psi_{ij} = \psi(\mathbf{k}_i, \mathbf{k}_j) = \frac{1}{2(2\pi)^{3/2}}\left(\mathbf{k}_i + \mathbf{k}_j - (\mathbf{k}_i - \mathbf{k}_j)\frac{(k_i^2 - k_j^2)}{|\mathbf{k}_i - \mathbf{k}_j|^2}\right), \tag{8}$$

if $\mathbf{k}_i \neq \mathbf{k}_j$, otherwise $\psi_{ij} = 0$ in the reference system where mean flow is absent. In (6) $a_j = a(\mathbf{k}_j, t)$. Fluid velocity in the $\mathbf{k}$-representation is expressed as follows:

$$\mathbf{v}(\mathbf{k}, t) = \int \psi_{12}a_1^*a_2\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2)\mathrm{d}^3k_1\mathrm{d}^3k_2. \tag{9}$$

It should be pointed out that the 'Hamiltonian of turbulence' (6) includes only the four-particle interaction Hamiltonian $\mathcal{H} = \mathcal{H}_4$. It has no two-particle term of the type $\mathcal{H}_2 \sim \omega aa^*$ describing the noninteracting field. Therefore, the dimensionless interaction factor $\mathcal{H}_4/\mathcal{H}_2$ is equal to infinity (or to Re with viscosity considered). It reflects the fact that turbulence is a many-point problem involving exceedingly strong interaction.

2.2. *Conservation laws and phenomenology of spectra*

The Euler equation (1) conserves kinetic energy (4), momentum $\mathbf{P} = \int \rho \mathbf{v}(\mathbf{x}, t) \mathrm{d}^3 x$ (which is zero in the reference system moving with the mean flow), and helicity (2). Single-valued Clebsch variables describe only flows without helicity. Turbulence with nonzero helicity will be discussed below in Section 4. The Hamiltonian $\mathcal{H}$ [that is the kinetic energy (4)] is evidently an integral of motion of the canonical equations (3). It is very important for our discussion that the very possibility of writing the Euler equation in the Hamiltonian form (5–7) gives us immediately two additional integrals of motion in addition to the energy $\mathcal{H}$.

The first additional conserved value is the total 'number' of quasi-particles $\int |a(\mathbf{k}, t)|^2 \mathrm{d}^3 k$. Indeed, the Hamiltonian (6) contains two 'operators of creation' $a^* a^*$ and two 'operators of annihilation' $a\, a$ of quasiparticles. So it describes scattering processes $2 \rightarrow 2$ that preserve number of quasiparticles at any act of interaction. For homogeneous turbulence, one can introduce the double correlation function $n(\mathbf{k}, t)\delta(\mathbf{k} - \mathbf{k}') = \langle a(\mathbf{k}, t) a^*(\mathbf{k}', t) \rangle$ so that the number of waves per unit volume

$$N = \int n(\mathbf{k}, t) \mathrm{d}^3 k \tag{10}$$

is an integral of motion, as can be easily seen from (5, 6). Dimension of N is *energy* $\times$ *time* and we will call it the action.

The second integral of motion follows from the spatial homogeneity of the problem under consideration:

$$\mathbf{\Pi} = \int \mathbf{k}\, n(\mathbf{k}, t) \mathrm{d}^3 k. \tag{11}$$

It is the total momentum of the quasiparticles with 'occupation numbers' $n(\mathbf{k}, t)$. Conservation of $\mathbf{\Pi}$ formally follows from the presence of the δ-function in (7) provided by spatial homogeneity of the problem. Note that the very possibility of the ensemble of Clebsch quasiparticles having a nonzero momentum is due to the possibility that $n(\mathbf{k}) \neq n(-\mathbf{k})$. These quasiparticles are thus similar to usual waves which could have nonzero momentum with respect to a medium that is in rest. What is important is that the velocity field does not possess this property: because of the identity $v(\mathbf{k}) = v^*(-\mathbf{k})$, the double correlator of the velocity field is always even. Obviously, this identity follows from the fact that any flow of an incompressible fluid is completely defined by the single real quantity $\mathbf{v}(\mathbf{x}, t)$. This means the absence of waves propagating in an incompressible fluid since any wave is described by two real variables (or one complex variable).

We do not know yet what the conservation of action and of momentum mean in terms of velocities. Neither can we express these integrals through the velocity field. Probably they are related to the Kelvin theorem which is implied in the Clebsch variables. Nevertheless, we are going to use these nonunderstood symmetries to obtain new solutions that can be expressed in terms of velocities.

Note that the Clebsch variables are equivalent to velocity variables, but it is not one-to-one correspondence: the gauge freedom allows for different Clebsch fields corresponding to a single velocity field. The integrals of motion in question surely can be formulated in velocity variables; our point is that such an integral cannot be expressed via the double velocity correlator. Studying that gauge freedom is a complicated task which we were not going to undertake in this paper. Our aim in this paper is to derive some new result using a proper gauge, namely such that the respective steady solution is scale-invariant and could be interpreted as carrying constant flux of some integral. The value of that integral is not gauge-invariant, and it thus has no direct physical meaning but helps to find a new solution. Moreover, in another gauge this solution is not scale-invariant. Still, the distribution in terms of velocities is the same for the whole family of Clebsch fields, so that it can be obtained by using the most convenient gauge.

As was mentioned in Section 1, knowledge of the integrals of motion allows one to suggest phenomenologically the expressions for the respective correlation functions. Energy is the Hamiltonian in the Clebsch variables, so its density is expressed via the fourth-order correlator J_{1234}. The energy flux can thus be expressed by the usual means [28] via the simultaneous six-order correlator J_{123456}, and by requiring the flux to be constant one can get the scaling exponent of $J^{(6)}$ which is $y_6 = -19$. So $J^{(6)} \propto Pk^{-19}$ with P being the energy flux. We designate $J^{(n)}(\lambda k_1, \ldots, k_n) = \lambda^{y_n} J^{(n)}(k_1, \ldots, k_n)$. Assuming simple scaling $y_n = An + B$, we obtain $y_n = (11n - 9)/3$. The double correlator is thus $n_K(k) \propto P^{1/3}k^{-13/3}$ which was first obtained in [27]. The fourth correlator $J_K^{(4)} \propto P^{2/3}k^{-35/3}$ gives the energy density $E(k) \propto P^{2/3}k^{-5/3}$ that can be easily recognized as the Kolmogorov 41 spectrum.

One may argue that this way of obtaining the spectrum is not better than the analogous speculations in terms of velocities. However, further analysis of the anisotropic part of the spectrum is possible only in Clebsch variables, where the momentum integral is present explicitly.

If the pumping is nonisotropic then it generates momentum of the quasiparticles. Viscous damping should absorb both integrals, energy and momentum. If the interaction of eddies is local in k-space (see below), then there are two fluxes in the inertial interval of scales. According to the revised universality concept [4], the spectrum should be determined by the whole set of the fluxes of integrals of motion that flow at the same direction. We thus assume the spectrum in the inertial interval to have a universal form defined by two values of the fluxes. Note that in the anisotropic case the energy flux is not necessary parallel to $\mathbf{k}$, so that it should be described by some tensor. For the scales of the order of the scale of anisotropic pumping, the angular dependence might be rather complicated. But since our additional integral of motion is a vector $\mathbf{R}$, then it is natural to believe that deep in the inertial interval the universal spectrum is axially symmetric around the direction of $\mathbf{R}$. Of course, this is true also for an axially symmetric pumping. From the dimensional analysis, the two-flux spectrum can thus be written as

$$n(\mathbf{k}, P, \mathbf{R}) = \lambda P^{1/3} k^{-13/3} f(\xi), \tag{12}$$

where the dimensionless ratio of the fluxes is $\xi = (\mathbf{R} \cdot \mathrm{k}) T_{kkkk} k^d n_k / Pk^2$. Defining the function $f(\xi)$ for all ξ is beyond our current abilities (hitherto, the two-flux spectrum was explicitly found only for acoustic turbulence [30]). We can, nevertheless, find the form of the stationary spectrum in a weakly anisotropic limit when $\xi \leqslant 1$. Expanding the function $f(\xi)$ at $\xi \leqslant 1$, we get a correction to the pair correlator in the form of the first angular harmonic

$$\frac{\delta n(\mathbf{k})}{n_K(k)} \propto \xi \propto k^{-1/3} \cos\theta_k. \tag{13}$$

This formula was first proposed by Kuznetsov and L'vov [31] using an analogy with wave turbulence. In the same way one can write the corrections to the arbitrary simultaneous correlator: $\delta J/J \propto \xi$. For example, the fourth-order correlation function is written as follows

$$\delta J_{1234} \propto k_1^{-1/3} \cos\theta_1 I_{1,234} + k_2^{-1/3} \cos\theta_2 I_{2,134} + k_3^{-1/3} \cos\theta_3 I^*_{3,124} + k_4^{-1/3} \cos\theta_4 I^*_{4,123} \tag{14}$$

where I has the same scaling properties as $J^{(4)}$ and is symmetric with respect to the last three arguments.

In the next subsection, this can be shown to be an exact steady solution of the linearized equation for the correlation functions.

2.3. *Exact stationary solution for the first angular harmonic*

Multiplying (5) by $a^*(\mathbf{k}, t)$ and averaging, we obtain the equation

$$\frac{\partial n(\mathbf{k}, t)}{\partial t} = \mathrm{St}(\mathbf{k}, t), \qquad \mathrm{St}(\mathbf{k}, t) = \mathrm{Im}\int T_{k123} J_{k123} \mathrm{d}^3k_1 \mathrm{d}^3k_2 \mathrm{d}^3k_3 \tag{15}$$

that governs the time evolution of the pair correlator. Substituting $J = J_K + \delta J$, one can show that (15) is zero for J_K [27]. To show that the r.h.s. of (15) is zero for δJ too, we divide this integral into four identical parts, and then make in three of them the transformations that consist of the conformal dilatations invented independently by Kraichnan [32] and Zakharov [33] and rotations in $\mathbf{k}$-space suggested by Kats and Kontorovich [12]. For the first term this transformation $\hat{G}_1$ looks as follows (here initial integration variables are temporarily denoted by $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ so that $\mathbf{k} + \mathbf{q}_1 = \mathbf{q}_2 + \mathbf{q}_3$)

$$\hat{G}_1 \colon \mathbf{q}_1 = \hat{G}_1^2 \mathbf{k}_1, \quad \mathbf{q}_2 = \hat{G}_1 \mathbf{k}_3, \quad \mathbf{q}_3 = \hat{G}_1 \mathbf{k}_2. \tag{16}$$

The operation $\hat{G}_1$ is determined by the condition $\hat{G}_1 \mathbf{k}_1 = \mathbf{k}$, and it transforms the quadrangle $\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$ from T_{k123} into a similar quadrangle $\mathbf{k} + \mathbf{q}_1 = \mathbf{q}_2 + \mathbf{q}_3$. The transformation thus relabels $\mathbf{k}_1$ and $\mathbf{k}$ and dilates all arguments of T and δJ with a factor $\lambda_1 = k/k_1$. Similar transformations $\mathbf{q}_1 = \hat{G}_2\mathbf{k}_3$, $\mathbf{q}_2 = \hat{G}_2^2\mathbf{k}_2$, $\mathbf{q}_3 = \hat{G}_2\mathbf{k}_1$, $(\hat{G}_2\mathbf{k}_2 = \mathbf{k})$, $\mathbf{q}_1 = \hat{G}_3\mathbf{k}_2$, $\mathbf{q}_2 = \hat{G}_3\mathbf{k}_1$, $\mathbf{q}_3 = \hat{G}_3^2\mathbf{k}_3$, $(\hat{G}_3\mathbf{k}_3 = \mathbf{k})$ should be done in the second and third terms. The scattering amplitude (7) is a homogeneous function with the index -1, i.e. $T(\lambda\mathbf{k}_1, \lambda\mathbf{k}_2; \lambda\mathbf{k}_3, \lambda\mathbf{k}_4) = \lambda^{-1} T(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4)$. The correction to the fourth correlator (14) is also a homogeneous function with its index being $-35/3 - 1/3 = -12$. The transformation Jacobian gives λ_i^{12} so in all the transformed terms acquire factors λ_i^{-1}. Therefore, after the transformations, the equation (15) has the following form

$$\begin{aligned}\frac{\partial n(\mathbf{k}, t)}{\partial t} = \frac{\mathrm{Im}}{4\mathbf{k}}\int & \mathrm{d}^3k_1 \mathrm{d}^3k_2 \mathrm{d}^3k_3 T_{k123}(I_{k,123} + I_{1,k23} + I_{2,1k3} + I_{k,123}) \\ & \times (k\cos\theta_k + k_1\cos\theta_1 - k_2\cos\theta_2 - k_3\cos\theta_3)\end{aligned} \tag{17}$$

that is equal to zero by virtue of the δ-function in T_{k123}. Note that transformations like (16) interchange the zero and infinity of the integration domain in the $\mathbf{k}$-space. Therefore, they are possible for convergent integrals only. In Section 2.5 below we will show that integral (15) converges with the correlator (14). The perturbation (14) is thus a steady solution of the linearized problem. Let us emphasize that this has been shown independently of the form of the function $I_{1,234}$.

What is accessible to experimental measurements is velocity (and its correlation functions) but not the correlation functions in the Clebsch variables. To obtain the law of turbulence approach to isotropy in terms of the velocity correlators, one should know the behaviour of even angular harmonics. This follows from the identity $v(\mathbf{k}) = v^*(-\mathbf{k})$. Expressing, for instance, the double velocity correlation function

$$F(\mathbf{k})\delta(\mathbf{k} - \mathbf{q}) = \langle v(\mathbf{k}) v^*(\mathbf{q}) \rangle = \int \psi_{12}^{-2} J_{12,34} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_3) \mathrm{d}^3k_1 \mathrm{d}^3k_2 \mathrm{d}^3k_3 \mathrm{d}^3k_4, \tag{18}$$

one can see that the first angular harmonic of δJ gives no contribution. We thus ought to find the next, second, harmonic of our universal solution. This cannot be done in the framework of the linearized approach since by multiplying, for instance, two first harmonics in the product of two second-order correlators one obtains the second harmonic in the fourth correlator.

Diagrammatic perturbation approach should be implemented both to show that (13) can be a solution as well as (14), and in order to find the behaviour of the second angular harmonics.

2.4. *Perturbation theory*

The natural scheme to develop perturbation approach for a nonequilibrium system with a strong interaction is the diagrammatic technique of the type first suggested by Wyld [36] for hydrodynamic turbulence. This technique was later generalized by Martin, Siggia and Rose [37] who demonstrated that it may be used for investigating the nonlinear dynamics of any condensed matter system. Then Zakharov and L'vov [27] extended the Wyld technique to the statistical description of hydrodynamics in the Clebsch variables. In fact, this technique is also a classical limit of the Keldysh technique [38] which is applicable to any physical system described by interacting Fermi and Bose fields. The detailed description of the technique can be found in [39].

The natural objects in the Wyld diagrammatic technique are the dressed propagators, which are the Green's function $G(\mathbf{k}, \omega)$ and correlator $n(\mathbf{k}, \omega)$. The Green's function is defined as the average response of the Clebsch field $a(\mathbf{x}, t)$ to a vanishingly small external 'force' $f(\mathbf{x}, t)$ which should be added to the r.h.s. of the equation of motion (5). In ω-representation

$$G(\mathbf{k}, \omega)\delta(\mathbf{k} - \mathbf{k}')\delta(\omega - \omega') = \left\langle \frac{\delta a(\mathbf{k}, \omega)}{\delta f(\mathbf{k}', \omega')} \right\rangle. \tag{19}$$

As a consequence of the causality principle, the function $G(\mathbf{k}, \omega)$ has to be analytic in the upper half of ω-plane. The correlator $n(\mathbf{k}, \omega)$ is the second correlation function of the Clebsch field $a(\mathbf{k}, t)$. In ω-representation

$$n(\mathbf{k}, \omega)\delta(\mathbf{k} - \mathbf{k}')\delta(\omega - \omega') = \langle a(\mathbf{k}, \omega)a(\mathbf{k}', \omega')\rangle. \tag{20}$$

Using the Wyld technique one may derive the system of equations for the dressed propagators [36, 27], known as the Dyson–Wyld equations:

$$G(\mathbf{k}, \omega) = \frac{1}{\omega - \Sigma(\mathbf{k}, \omega)}, \tag{21}$$

$$n(\mathbf{k}, \omega) = |G(k, \omega)|^2[\Phi_0(\mathbf{k}, \omega) + \Phi(\mathbf{k}, \omega)]. \tag{22}$$

In these equations $\Phi_0(\mathbf{k}, \omega)$ is the correlator of the external force concentrated in the energy-contained interval. The mass operators $\Sigma(\mathbf{k}, \omega)$ and $\Phi(\mathbf{k}, \omega)$ are the *self-energy* and *intrinsic noise functions* respectively. These are given by infinite series of one-particle irreducible diagrams:

$$\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 + \cdots, \qquad \Phi = \Phi_2 + \Phi_3 + \Phi_4 + \cdots.$$

In these expressions Σ_m is a functional of m vertices T, $(m-1)$ Green's functions $G(\mathbf{k}_j, \omega_j)$, and m correlators $n(\mathbf{k}_j, \omega_j)$; Φ_m is a functional of m vertices, $(m+1)$ correlators and $m-1$ Green's functions. Analytical expressions for Σ_2 and for Φ_2 have the form

$$\begin{aligned} \Sigma_2(\mathbf{k}, \omega) = \int |T(\mathbf{k}, \mathbf{k}_1; \mathbf{k}_2, \mathbf{k}_3)|^2[n_1 n_2 G_3 + \tfrac{1}{2}G_1^* n_2 n_3]\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \\ \times \delta(\omega + \omega_1 - \omega_2 - \omega_3)\mathrm{d}^3k_1\mathrm{d}^3k_2\mathrm{d}^3k_3\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3, \end{aligned} \tag{23}$$

$$\begin{aligned} \Phi_2(\mathbf{k}, \omega) = \tfrac{1}{2}\int |T(\mathbf{k}, \mathbf{k}_1; \mathbf{k}_2, \mathbf{k}_3)|^2[n_1 n_2 n_3]\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \\ \times \mathrm{d}^3k_1\mathrm{d}^3k_2\mathrm{d}^3k_3\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3. \end{aligned}$$

Here we used shorthand notations n_j and G_j which represent $n(k_j, \omega_i)$ and $G(\mathbf{k}_j, \omega_i)$.

Any correlation function can be expressed through the propagators n and G. We are interested in solving an inverse problem; using our knowledge of the simultaneous fourth-order correlator (14) we are going to find anisotropic corrections to n and G. Then we shall find any quantity of interest, for instance, the k-dependencies of higher angular harmonics.

The lowest-order contribution to the simultaneous fourth-order correlation function can be represented as follows

$$J_{k1,23} = \frac{1}{2}\int d\omega_1 d\omega_2 d\omega_3 T_{k1,23}\delta(\omega + \omega_1 - \omega_2 - \omega_3) \times [n_q n_1 n_2 G_3 + n_q n_1 G_2 n_3 + n_q G_1^* n_2 n_3 + G_q^* n_1 n_2 n_3] \tag{24}$$

Here index q means $(\mathbf{k}, \omega)$, j means $\mathbf{k}_j, \omega_j$. This first nontrivial expression for $J^{(4)}$ appears by using equations (23) for Σ and Φ. If we restricted ourselves to this term, this would be similar to a direct interaction approximation [40] for the Clebsch variables. We are going, however, to analyse the entire perturbation series for J. A diagram of pth order contains p vertexes T and the appropriate number of the propagators.

2.5. *Angular expansion and higher harmonics*

Assuming weak anisotropy we expand the propagators in Legendre polynomials P_l:

$$n(\mathbf{k}, \omega) = \sum_{l=0}^{\infty} P_l(\cos\theta)(kL)^{z_l} n_l(k, \omega), \qquad G(\mathbf{k}, \omega) = \sum_{l=0}^{\infty} P_l(\cos\theta)(kL)^{z_l} G_l(k, \omega) \tag{25}$$

Here L is the energy containing scale and z are the scaling exponents for dimensionless anisotropic corrections proportional to P_l, $z_0 = 0$.

Considering the different-time propagators $G(\mathbf{k}, \omega)$ and $n(\mathbf{k}, \omega)$ one necessarily finds infrared divergencies of integrals even in the direct interaction approximation (23). The reason for this is the sweeping effect which, however, does not contribute to the results for the simultaneous correlator if the entire perturbation series is considered. The procedure of eliminating the sweeping in each order of perturbation theory is rather cumbersome, but is sometimes necessary [28]. In the isotropic case, the sweeping has been excluded in [29]. Fortunately in the problem of weak anisotropic corrections there is a way to avoid this difficulty. Let us substitute the expansions (25) into the Dyson–Wyld equations (21, 22). Thus we can see that z_l are indeed the same for both n and G and that $n_l(k, \omega)$ and $G_l(k, \omega)$ should have the same scaling properties as $n(\mathbf{k}, \omega)$ and $G(\mathbf{k}, \omega)$ respectively. What is the scaling exponent of the frequency ω is a different question: in the Eulerian approach ω scales as k because of the sweeping, whereas in the Lagrangian or quasi-Lagrangian approach $\omega \propto k^{2/3}$ due to a dynamic interaction of eddies. However, for our goal the only important thing is that the scaling exponents of the frequency are the same for all the terms in the expansions (25).

Let us first consider the terms with $l = 1$, which are $\propto P_1(\cos\theta_k) = \cos\theta_k$. Substituting

$$\begin{aligned} n(\mathbf{k}, \omega) &= n(k, \omega) + (kL)^{z} n_1(k, \omega)\cos\theta_k, \\ G(\mathbf{k}, \omega) &= G(k, \omega) + (kL)^{z} G_1(k, \omega)\cos\theta_k \end{aligned} \tag{26}$$

into equation (24), one can see that in the linear approximation with respect to n_1 and G_1, the anisotropic corrections to the fourth-order correlation function have the form (14) with some analytical expression for $I_{1,234}$ if and only if $z_1 = -1/3$. This is thus the justification of the formula (13) previously obtained from dimensional analysis. Note that corrections linear in $\cos\theta_k$ to the fourth correlator have the form (14) because of symmetry. Therefore it is clear that n_1 and G_1 do have this form in each order of perturbation theory (which also can be directly proved by the simple calculation of powers at any order). The only difference between consideration of the whole series and the direct interaction approximation (24) would be the form of the analytical expression for $I_{1,234}$. In Section 2.3 we proved that expression (14) is an exact solution of the linearized equation for any form of $I_{1,234}$, providing the convergence of integrals in the equation (17).

We thus come to the key point of any turbulence consideration, namely, to locality investigation. This is to be done without any approximations and closures. The direct interaction approximation for the fourth-order correlators (24) does provide the convergency, even when calculated in the Eulerian approach containing sweeping. In order to find asymptotic behaviour of the true fourth-order correlators and to prove the convergency in the whole series one can combine the quasi-Lagrangian approach for Clebsch variables [28] with the method of getting the asymptotics of correlators by sorting diagrams [34, 35]. The main idea that enables one to find the asymptotics of any high-order correlator as some wavenumbers go to zero is as follows: in the framework of Wyld diagram technique one can express any function in terms of series containing a pair correlator and Green function. If the scaling exponent of the pair correlator is positive $y_2 > 0$, then the main contribution stems from the terms containing the pair correlators of small wavenumbers [34]. In our case, this gives the following asymptotics of the fourth correlator with two small wavenumbers κ, κ_1:

$$J^{(4)} \propto \kappa_{\alpha} \kappa_{1\beta} n(\kappa) n(\kappa_1), \tag{27}$$

where the factors κ, κ_1 appear due to vertexes adjacent to the pair correlators. This is the first term in the expansion on small κ, κ_1. Note that these asymptotics are not based on Gaussian-like coupling. Still, the statistics might be so degenerate that the first term in the expansion is cancelled. The locality criterion below will be sufficient (but not necessary) in this case.

The first term in the asymptotical expansion of the vertex can be obtained from (7, 8): $T_{\kappa\kappa_1,kk} \propto (\boldsymbol{\kappa}\boldsymbol{\kappa}_1)$ and $T_{k\kappa,k\kappa_1} \propto (\mathbf{k} \cdot \min \boldsymbol{\kappa}, \boldsymbol{\kappa}_1\})$ where κ, $\kappa_1 \ll k$. Substituting this together with (27) into (15), one can see that the infrared convergence is determined by the following integral (to be specific, we put $\kappa < \kappa_1$):

$$\int (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}_1)(\mathbf{k} \cdot \boldsymbol{\kappa}_{\,})\delta n(\kappa) n(\kappa_1) \mathrm{d}^3\kappa \mathrm{d}^3\kappa_1. \tag{28}$$

Substituting here $n(\kappa_1) \propto \kappa_1^{-13/3}$ and $\delta n(\kappa) \propto (\boldsymbol{\kappa} \boldsymbol{b})\kappa^{-17/3}$ one gets that the power counting for (28) gives zero, so that one may expect a logarithmic divergency. One can easily see, however, that this integral vanishes due to angle integration. Taking into account the next terms in the expansions, we find that the infrared contribution converges, being proportional to κ. By the same means one can get the ultraviolet asymptotics at $k_1 \gg k$: $T \propto k_1$ and $\delta J^{(4)} \propto k_1^{-y_3+2y_2-2} = k_1^{-5}$. Ultraviolet contribution $\int_k^\infty k_1^{-41/3} \mathrm{d}^3 k_1$ thus converges as a first power as well (such a coincidence for the law of decreasing nonlocal contributions is called counterbalanced locality [34]). Exactly the same asymptotics are obtained while analysing any diagram in the perturbation expansion.

The above consideration is not a rigorous proof that the steady anisotropic corrections have the form (26). This is rather the basis of concluding that, if the solution analytic in parameters kL and Re exists, then it has the form (26). The diagrammatic approach cannot catch nonanalytic solutions if they exist.

If one substitutes (26) into (22) and takes into account terms quadratic with respect to the anisotropic corrections, one obtains the exponent for the second angular harmonics: $z_2 = -2/3$. The relative part of the anisotropic perturbation corresponding to the second angular harmonic thus decreases with k faster than the first one. Subsequent harmonics can be shown to have $z_l = -l/3$ so they can be neglected in the inertial interval. Substituting then the first two terms of the propagator expansion into (26), and restricting ourselves to the terms linear with respect to the second harmonics and quadratic with respect to the first one, we get the second angular harmonic of the fourth correlator in the following form

$$\begin{aligned}\delta J_{1234} \propto\ & k_1^{-2/3} P_2(\cos\theta_1) U_{1,234} + k_2^{-2/3} P_2(\cos\theta_2) U_{2,134} \\ & + k_3^{-2/3} P_2(\cos\theta_3) U^*_{3,124} + k_4^{-2/3} P_2(\cos\theta_4) U^*_{4,123}. \qquad (29)\end{aligned}$$

Here U is some unknown function of four arguments that has the same scaling properties as $J^{(4)}_{\mathrm{K}}$ and is symmetric with respect to the last three arguments as well as above function I.

Let us emphasize that the second angular harmonic of δJ does not turn the collision integral (15) into zero. It is compensated by the contribution from the pumping region that decays in the inertial interval by the same law $(kL)^{-2/3}$. To show this, one should analyse convergency (i.e. contributions from distant regions) for the second angular harmonic which can be done by the same way as for the first one. One can show that the contribution into (15) from the second harmonics decays as $(kL)^{-2/3}$ and is compensated by the contribution of the pumping region [41]. Note that such a remarkable coincidence happens if the exponent of the isotropic spectrum is exactly equal to 5/3.

2.6. *Anisotropic energy spectrum*

Substituting the second harmonic of the fourth-order correlation function (29) into the expression (18) for the double velocity correlation function, one obtains the anisotropic part of the energy density in **k**-space:

$$\frac{\delta F(\mathbf{k})}{F(k)} \propto k^{-2/3} P_2(\cos\theta_k). \qquad (30)$$

This should be the asymptotic form of the turbulence spectrum in the inertial interval for arbitrary anisotropic large-scale pumping. The same formula was obtained by Leslie [10] (see also [42]) in the framework of the linearized direct interaction approximation for the special case of flow with a constant shear. The data from observations (summarized in [11]) satisfactorily fits (30), which is thus *the universal law of turbulence approach to isotropy.* Indeed, the anisotropic part decreases with k faster than the isotropic spectrum, so our formula is the justification of the isotropy hypothesis of Taylor [43]. Higher $2n$-harmonics should decay with k faster (as $k^{-2n/3}$), so (30) represents the main anisotropic part in the inertial interval. By the way, the presence of the law (30) in the inertial interval means that the turbulent flow actually has a two-flux spectrum there [4]. However, the flux of the second integral (that is $\int \mathbf{k} n_k d\mathbf{k}$ which probably has kinematic rather than dynamic meaning) is hidden at sufficiently large k.

3. INVERSE CASCADE OF 3D ISOTROPIC TURBULENCE

Besides the energy, the Euler equation in the Hamiltonian form also preserves the action $N = \int n_k d\mathbf{k}$, i.e. 'the total number of waves'. Conservation of this integral follows from the very possibility of introducing Clebsch variables, which is a consequence of Kelvin's theorem. The spectrum which carries the constant flux Q of the wave action was suggested by Zakharov and L'vov [27] in the following form:

$$n(k, Q) = \lambda Q^{1/3} k^{-4}. \qquad (31)$$

This form also follows from dimensional consideration. Considering the equation of motion (15) for the isotropic spectrum and making transformation (16) we get the following expression:

$$\frac{\partial n(\mathbf{k}, t)}{\partial t} = \int \mathrm{d}^3 k_1 \mathrm{d}^3 k_2 \mathrm{d}^3 k_3 T_{k123} J_{k123} \left\{ 1 + \left(\frac{k}{k_1}\right)^{11-y_4} - \left(\frac{k}{k_2}\right)^{11-y_4} - \left(\frac{k}{k_3}\right)^{11-y_4} \right\}.$$

The homogeneity index y_4 of the fourth-order correlation function is expressed via the index y_2 of $n(k)$ in the following way: $y_4 = 2y_2 + d$; since $y_2 = 4$ we have $y_4 = -11$. As one can see, the spectrum with a constant action flux is an exact solution as well as the Kolmogorov spectrum. Locality of this spectrum has been proven in [44]: an infrared contribution decreases as $(k/k_1)^{-2}$ while an ultraviolet one decreases as k_1/k. Note that the kinetic equation written in Clebsch variables [45] gives logarithmic divergence, which is an artefact of an uncontrollable approximation.

After recalculation into the energy density, one has

$$E(k, Q) = C_1 Q^{2/3} k^{-1}. \tag{32}$$

Note that the spectrum is k^{-1} independently of the space dimension d. For d = 3, it is possible to prove that our case corresponds to the usual four-wave one: energy flows towards small scales while the action N goes at the opposite direction. This can be done similarly to wave turbulence theory [13] by comparing turbulence spectra with equilibrium ones [4]. For example, the first-order approximation [4, 45] gives the action flux for power solutions $n_k \propto k^{-s}$ as follows:

$$Q = -\int_0^1 \mathrm{d}y \int_{1-y}^1 \mathrm{d}y_1 W(y, y_1)\{1 + (y + y_1 - 1)^x - y^x - y_1^x\}.$$

Here $W(y, y_1)$ is some positive function, $y = (k_2/k)^{5-s}$, $y_1 = (k_3/k)^{5-s}$ and $x = s/(5-s)$. Therefore, sign $Q = -\operatorname{sign} s(2s-5)$. The flux changes sign for $s_1 = 0$ and $s_2 = 5/2$ which correspond to the equilibrium spectra $E_1(k) \propto k^2$ and $E_2(k) \propto k^7$ that give equipartition of the energy and of the wave action respectively. This can be proved in any order of perturbation theory. For the case in question, $s = 4$ and $Q < 0$. Therefore, if the spectrum (31) was stable, it would be realised for scales larger than that of the pump while the Kolmogorov-41 spectrum [11] extends towards smaller scales up to the viscous range.

After recalculation of the behaviour of the velocity differencies $\delta bfv(\mathbf{l}) = \mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})$, the formulas (31, 32) give the following scaling: $\delta v(l) \propto l^0 = \text{const}$ in the inertial interval [4]. Formally considered for any l (which presumes a pump with $l = 0$), the spectrum (31) thus corresponds to the velocity differences which do not turn into zero while $l \to 0$. Velocity discontinuities can only be tangential due to incompressibility. Any tangential discontinuity is an exact solution of the Euler equation (for $\mathbf{v} \perp \nabla v$, the nonlinear term is zero). And it is natural that a random set of such discontinuities (giving the spectrum (31) might also be a steady solution of the respective statistical equations. Moreover, this physical explanation of the solution (31) gives an insight allowing us to predict (following [4]) that this spectrum considered for all l is inevitably unstable as with any tangential discontinuity in the framework of the Euler equation.

It would be interesting to find this inverse cascade experimentally. It is not surprising that it has not yet been found since one needs steady turbulence at the scales much larger than the pumping scale. Most experimental and natural set-ups violate the second condition while grid turbulence violate the first (steadiness). Indeed, an inverse cascade is formed in a decelerating way (time increases with a scale), so that such a cascade cannot be formed in a decaying turbulence. Most probably, it could be first obtained in numerics (if it exists at all).

Returning to a more realistic picture with a small-scale Kolmogorov spectrum (giving $\delta v(l) \propto l^{1/3}$ at $l \to 0$) and a large-scale spectrum (31), one can imagine large sheets with different nonzero mean velocities and small eddies arising at the boundaries of the sheets due to the Kelvin–Helmholtz instability. Eddies give the Kolmogorov cascade while the regions with large-scale velocity differences being randomly distributed give for small $k \simeq 1/l$ ($l \gg L$) $v_k \approx \int_0^l v(x)dx \propto l^{1/2} \propto k^{-1/2}$ which corresponds to the spectrum (32).

This might be, however, that even the presence of the small-scale Kolmogorov-41 spectrum cannot save the spectrum (32) from being unstable. This is a nonisotropic perturbation which can provide a large-scale instability of the spectrum (32). Indeed, a steady correction carrying a constant momentum flux, according to dimensional consideration looks as follows

$$\frac{\delta n(\mathbf{k}, R)}{n(k, Q)} \propto \vartheta \propto \frac{(\mathbf{Rk})}{k^2}.$$

Using the same technique as in Section 3, one can show that this $\delta n(\mathbf{k}, R)$ is an exact stationary solution of the collision integral, linearized with respect to a small deviation from the spectrum (31). So if the momentum of the auxiliary wave field flows towards small k, then the variation of the energy spectrum should grow quickly as k decreases:

$$\frac{\delta E(k)}{E(k, Q)} \propto k^{-2} \cos^2 \theta_k. \tag{33}$$

This formula implies the structural instability of the isotropic spectrum (32) with respect to the angular modulation of a small-scale pump: even a weakly anisotropic pump produces a strongly anisotropic large-scale turbulence. Such an instability was first predicted by Falkovich [4]. This probably explains both the absence of the spectrum (32) and the presence of substantial large-scale nonisotropy in the experimental data.

Note also that the usual Fourier transform relating correlation functions of the velocity

$$\langle v(\mathbf{r})v(\mathbf{r} + \mathbf{l})\rangle = \int_{1/L}^{1/l} |1 - \exp(i\mathbf{k} \cdot \mathbf{l})|^2 \langle v(\mathbf{k})v(-\mathbf{k})\rangle d^3k$$

diverges logarithmically for spectrum (32).

Whether the spectrum (32) might be observed at the edge of dissipative interval [45] is, in our opinion, a rather controversial and open question.

4. COMPLEMENTARITY OF TURBULENCE DESCRIPTION IN CLEBSCH AND VELOCITY VARIABLES

The traditional approach to the analytical description of turbulence [40, 36] is based on the Euler equation in the $\mathbf{k}$-representation

$$\frac{\partial v_\alpha(t, \mathbf{k})}{\partial t} = \frac{1}{2}\int \Gamma_{\alpha\beta\gamma}(\mathbf{k}|\mathbf{q}, \mathbf{p}) v^*_\beta(t, \mathbf{q}) v^*_\gamma(t, \mathbf{p}) \mathrm{d}^3q\, \mathrm{d}^3p + f(t, \mathbf{k}) \tag{34}$$

with an external random Gaussian force f. Here the vertex

$$\Gamma_{\alpha\beta\gamma}(\mathbf{k}|\mathbf{q}, \mathbf{p}) = \Gamma_{\alpha\beta\gamma}(\mathbf{k}) = P_{\alpha\alpha'}(\mathbf{k}) P_{\beta\beta'}(\mathbf{q}) P_{\gamma\gamma'}(\mathbf{p}) i(k_{\beta'}\delta_{\alpha'\gamma'} + k_{\gamma'}\delta_{\alpha'\beta'})\delta(\mathbf{k} + \mathbf{q} + \mathbf{p}) \tag{35}$$

is expressed via the transverse projector $P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2$.

Statistical description can be provided in the $(\mathbf{k}, \omega)$-representation in terms of the double correlation function $F_{\alpha\beta}(\mathbf{k}, \omega)\delta(\mathbf{k} - \mathbf{q})\delta(\omega - \omega') = \langle v_\alpha(\mathbf{k}, \omega) v^*_\beta(\mathbf{q}, \omega')\rangle$ and the Green's function $G_{\alpha\beta}(\omega, \mathbf{k})\delta(\mathbf{k} - \mathbf{q})\delta(\omega - \omega') = \langle \delta v_\alpha(\mathbf{k}, \omega)/\delta f_\beta(\mathbf{q}, \omega')\rangle$ giving the response of the velocity to an external force f (per unit mass). Here the angular brackets denote ensemble averaging.

Probably the mostly powerful tool used hitherto is the Direct Interaction Approximation (DIA) introduced by Kraichnan [40]. The collision integral governing the evolution of the double correlator $F_{ij}(\mathbf{k}, \omega)$ has the following form in DIA:

$$
\begin{aligned}
I_{ij} = \int &\delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\delta(\omega + \omega_1 + \omega_2)\{G_{l\alpha}(\omega_1, \mathbf{k}_1)\Gamma_{\alpha\beta\gamma}(\mathbf{k}_1)F_{m\beta}(\omega_2, \mathbf{k}_2)F_{j\gamma}(\omega, \mathbf{k}) \\
&+ G_{m\alpha}(\omega_2, \mathbf{k}_2)\Gamma_{\alpha\beta\gamma}(\mathbf{k}_2)F_{l\beta}(\omega_1, bfk_1)F_{i\gamma}(\omega, \mathbf{k}) \\
&+ G_{j\alpha}(\omega, \mathbf{k})\Gamma_{\alpha\beta\gamma}(\mathbf{k})F_{l\beta}(\omega_1, \mathbf{k}_1)F_{m\gamma}(\omega_2, bfk_2)\}\Gamma_{ilm}(\mathbf{k}|\mathbf{k}_1, \mathbf{k}_2)\mathrm{d}^3k_1\mathrm{d}^3k_2\mathrm{d}\omega_1\mathrm{d}\omega_2.
\end{aligned}
\tag{36}
$$

Further details and the equation for G can be found in the monograph [10] completely devoted to DIA.

Being a finite closure of the initially infinite chain of equations, DIA is an uncontrolled approximation. It preserves, nevertheless, some basic symmetries (though not the whole set as we show in this section) of the Euler equation. The most important one is Jakobi identity for the vertex

$$
[\Gamma_{\alpha\beta\gamma}(\mathbf{k}) + \Gamma_{\beta\alpha\gamma}(\mathbf{q}) + \Gamma_{\gamma\beta\alpha}(\mathbf{p})]\delta(\mathbf{k} + \mathbf{q} + \mathbf{p}) = 0 \tag{37}
$$

that gives energy conservation: $\int I_{ii}(\mathbf{k}, \omega)\mathrm{d}^3k = 0$. Therefore, the Kolmogorov spectrum carrying a constant energy flux should be an exact stationary solution in the framework of DIA i.e. it should make $I(\mathbf{k}, \omega)$ disappear. Let us now demonstrate this following [27]. Looking for solutions in the form $F_{ij}(\mathbf{k}, \omega) = P_{ij}k^{-y-x}f(\omega/k^x)$ and $G_{ij}(\mathbf{k}, \omega) = P_{ij}k^{-x}g(\omega/k^x)$ one can see that this is possible if $2x + y = 5$. Executing then in the second term of (36) a conformal transformation similar to (16)

$$
k = k'k/k', \quad k_1 = k^2/k_1, \quad k_2 = k''k/k' \tag{38}
$$

and accordingly in the third term with $k_1 \leftrightarrow k_2$, we obtain the collision integral in the factorized form

$$
\begin{aligned}
I(k) = \int I(\mathbf{k}, \omega)d\omega = \int &\mathrm{d}\omega\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}^3k\mathrm{d}^3k_1\mathrm{d}^3k_2\delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\delta(\omega + \omega_1 + \omega_2) \\
&\times G_{l\alpha}(\omega_1, \mathbf{k}_1)\Gamma_{\alpha\beta\gamma}(\mathbf{k}_1)F_{m\beta}(\omega_2, \mathbf{k}_2)F_{i\gamma}(\omega, \mathbf{k}) \\
&\times \left[\Gamma_{ilm}(\mathbf{k}|\mathbf{k}_1, \mathbf{k}_2) + \left(\frac{k}{k_1}\right)^{8-x-2y}\Gamma_{lim}(\mathbf{k}_1|\mathbf{k}, \mathbf{k}_2) + \left(\frac{k}{k_2}\right)^{8-x-2y}\Gamma_{mli}(\mathbf{k}_2|\mathbf{k}_1, \mathbf{k})\right].
\end{aligned}
\tag{39}
$$

For the Kolmogorov indices $x = 2/3$ and $y = 11/3$, this integral turns into zero. We shall not dwell upon the problem of divergencies since they can be completely swept away by passing to Lagrange variables [46] or using the improved DIA approximation [31].

Another integral of motion that can be expressed in terms of the double velocity correlator is the helicity $H = \int(\mathbf{v} \cdot \mathbf{rot}\, \mathbf{v})d\mathbf{r}$ that characterizes the 'knottedness' of the flow [6] (remember that the Clebsch variables thus describe only flows with zero helicity). For homogeneous turbulence, the spectral density of this integral can be expressed in terms of the pseudoscalar quantity $\langle \mathbf{v} \cdot \mathbf{rot}\, \mathbf{v}\rangle = \int kA(k)dk$ such that the double correlator acquires the term $F^{\mathrm{H}}_{ij}(k, t) = i\epsilon_{ijl}(k_l/k)A(k, t)$. The time evolution of $A(k, t)$ is defined by the same collision integral (36) with the vertex Γ_{ijl} replaced by $R_{ijl}(\mathbf{k}, bfk_1, \mathbf{k}_2) = i\epsilon_{imn}k_m\Gamma_{njl}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$. Helicity conservation is provided by the identity

$$
[R_{\alpha\beta\gamma}(\mathbf{k}) + R_{\beta\alpha\gamma}(\mathbf{q}) + R_{\gamma\beta\alpha}(\mathbf{p})]\delta(\mathbf{k} + \mathbf{q} + \mathbf{p}) = 0 \tag{40}
$$

that makes it possible to find an exact solution of the linearized DIA [7]. Looking for the solution in the form of the Kolmogorov spectrum with the small pseudoscalar correction

$$F_{ij}(\mathbf{k},\omega) = P_{ij}k^{-13/3} f\left(\frac{\omega}{k^{2/3}}\right) + i\epsilon_{ijl}\frac{k_l}{k}k^{-s-2/3}h\left(\frac{\omega}{k^{2/3}}\right)$$

and making the same conformal transformation (38) we get similarly to (39)

$$\delta I(k) = \int \delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\delta(\omega+\omega_1+\omega_2)\left[R_{ilm}(\mathbf{k}) + \left(\frac{k}{k_1}\right)^p R_{lim}(\mathbf{k}_1) + \left(\frac{k}{k_2}\right)^p R_{mli}(\mathbf{k}_2)\right]$$
$$\times \{\delta[G_{l\alpha}(\omega_1,\mathbf{k}_1)\Gamma_{\alpha\beta\gamma}(\mathbf{k}_1)F_{m\beta}(\omega_2,\mathbf{k}_2)F_{i\gamma}(\omega,\mathbf{k})]\}\mathrm{d}\omega\mathrm{d}\omega_1\mathrm{d}\omega_2\mathbf{dkdk}_1\mathbf{dk}_2. \tag{41}$$

Here $\delta[G\Gamma F_2F] = \delta G\Gamma F_2F + G\Gamma\delta F_2F + G\Gamma F_2\delta F$ and $p = 14/3 - s$. The stationary solution corresponds to $s = 14/3$ [7]. This solution describes Kolmogorov turbulence with large energy flux and small helicity flux.

As it can be seen from (39, 41), these spectra exhaust all universal power solutions to be obtained in the framework of DIA. By universal, we mean solutions whose scaling exponent does not depend on the precise form of the ω-dependencies. Indeed, one can recalculate the solutions (32) with the action flux obtained in the previous section for the double velocity correlator, then substitute them into (39) and find the first bracket to be nonzero (and to have a definite sign). Still, there exists a possibility that other factors (which have different signs in different regions of $\mathbf{k}$-space) can turn the collision integral into zero for spectra with scaling exponents x, y, s depending on the ω-dependencies, but this seems unlikely.

As far as the weakly anisotropic spectrum (26) is concerned, it carries the flux of momentum of auxiliary wave field. As well as the action, this integral of motion cannot be expressed as a linear function of the double velocity correlator. To show that this spectrum does not satisfy finite closure in terms of velocities, one can write down the linearized anisotropic DIA [10, 42] and for the second angular harmonic after the conformal transformation one has (41) with the first square bracket being replaced by

$$P_2(\cos\theta)\Gamma_{ilm}(\mathbf{k}) + P_2(\cos\theta_1)\left(\frac{k}{k_1}\right)^{-2/3}\Gamma_{lim}(\mathbf{k}_1) + P_2(\cos\theta_2)\left(\frac{k}{k_2}\right)^{-2/3}\Gamma_{mli}(\mathbf{k}_2).$$

This is nonzero, demonstrating again that not every exact solution found in Clebsch variables can be obtained as a universal solution in the perturbation approach in terms of velocities. The same consideration can be done in any (finite) order of perturbation theory. We therefore conclude that exact stationary solutions corresponding to motion integrals that are not linear functions of the double velocity correlator cannot be obtained by the traditional perturbation approach in terms of moments of the velocity field.

5. CONCLUSION

Small steady anisotropic corrections (13, 30) found analytically in the present paper solve the problem of structural stability of three-dimensional turbulence spectra with respect to the pumping variation. Note that by the same means the structural instability of the inverse energy cascade in two-dimensional turbulence was recently predicted [4]. These solutions also provide a basis for solving the general stability problem of turbulence spectra in hydrodynamics, including the temporal behaviour of perturbations. Such a steady correction is called a neutrally stable mode in stability theory [13] and it defines the asymptotics of arbitrary perturbation either at $k \to \infty$ or at $k \to 0$. By analogy with the stability theory of the spectra of wave turbulence [13] one may suggest also the following picture of the temporal evolution of any anisotropic perturbation arising in the inertial interval at some instant of time on top of the Kolmogorov spectrum; the first angular harmonic of the perturbation quickly (during a few turnover times) acquires the asymptotical form (30) at large wavenumbers. The left edge of the perturbation will move upscale (towards small k) by the law $k \propto t^{-2/3}$.

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REFERENCES

1. A. N. Kolmogorov, *Dokl. Akad. Nauk SSSR* **30**, 229 (1941).
2. A. M. Obukhov, *Dokl. Akad. Nauk SSSR* **32**, 22 (1941).
3. R. H. Kraichnan, *Phys. Fluids* **9**, 1937 (1967).
4. G. Falkovich, in *Nonlinear Waves and Weak Turbulence*, edited by H. Brezis. Series in Nonlinear Partial Differential Equations. Birkhauser, Boston, MA (1993).
5. P. G. Saffman, *Phys. Fluids* **10**, 1349 (1967).
6. H. K. Moffat, *J. Fluid Mech* **35**, 117 (1969).
7. E. Kuznetsov and N. Noskov *Sov. Phys. JETP* **48**(4), 660 (1978).
8. J. Rotta, Statistische Theorie nichthomogener Turbulenz, *Z. Phys.* **129**, 547–572 (1951).
9. J. Herring, *Phys. Fluids* **17**, 859–872 (1974).
10. D. C. Leslie, *Developments in the Theory of Turbulence*. Clarendon Press, Oxford (1973).
11. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vol. II, Fig. 60, MIT Press, Cambridge MA (1975).
12. A. Kats and V. Kontorovich, *Sov. Phys. JETP*, **27**, 80 (1973); **38**(1), 102 (1974).
13. V. Zakharov, V. L'vov and G. Falkovich, *Kolmogorov Spectra of Turbulence*. Springer, Heidelberg (1992).
14. L. D. Landau and E. M. Lifschitz, *Fluid Mechanics*. Pergamon Press, Oxford (1987).
15. A. Clebsch, *Crelle, J. reine angew. Math.* **54**, 293 (1857).
16. A. Clebsch, *Crelle, J. reine angew. Math.* **56**, 1 (1859).
17. H. Lamb, *Hydrodynamics*, Cambridge University Press, Cambridge (1932).
18. H. Bateman, *Proc. Roy. Soc. A*, **125**, 598 (1929).
19. H. Bateman, *Partial Differential Equations of Mathematical Physics*. Cambridge University Press, Cambridge (1932).
20. J. W. Herivel, *Proc. Camb. Phil. Soc.* **51**, 344 (1955).
21. C. C. Lin, *Proc. Int. School of Physics, Enrico Fermi*, Course XXI, Academic Press, New York 1963).
22. S. Grossmann, *Z. Phys. B* **21**, 403 (1975).
23. E. A. Kuznetsov and A. V. Mikhailov, *Phys. Lett. A* **77**, 37–38 (1980).
24. V. E. Zakharov and E. A. Kuznetsov, *Sov. Sci. Rev. C* **4**, 167–220 (1984).
25. R. Salmon, Hamiltonian fluid mechanics, *Ann. Rev. Fluid Mech.* **20**, 225–256 (1988).
26. A. Pumir, B. I. Shraiman and E. D. Siggia, *Phys. Rev. A* **45**, R5351–R5354 (1992).
27. V. E. Zakharov and V. S. L'vov, *Radiophys. Quantum Electro.* **18** 1084–1097 (1975).
28. V. S. L'vov, *Phys. Rep.* **207**, 1 (1991).
29. V. I. Belinicher and V. S. L'vov, *Zh. Eksp. Teor. Fis.* **93**, 533 (1987), [*Sov. Phys. JETP* **66**, 303 (1987)].
30. V. L'vov and G. E. Falkovich, *Sov. Phys. JETP* **69**, 1148–1150 (1989).
31. E. Kuznetsov and V. L'vov, *Physica D* **2**, 203 (1981).
32. R. H. Kraichnan, *Phys. Fluids* **9**, 1728 (1966).
33. V. Zakharov, *Sov. Phys. JETP* **24**, 457 (1967).
34. V. L'vov and G. E. Falkovich, *Phys. Rev. A* **46**, 4762 (1992).
35. V. L'vov, *Phys. Rev. E* **48**, R669 (1993).
36. H. W. Wyld, *Ann. Phys. (N.Y.)* **14**, 143 (1961).
37. P. C. Martin, E. D. Siggia and H. A. Rose, *Phys. Rev. A* **8**, 423 (1973).
38. L. V. Keldysh, *Sov. Phys. - JETP* **20**, 1018 (1965).
39. W. D. McComb, *The Physics of Fluid Turbulence*. Clarendon Press, Oxford (1992).
40. R. H. Kraichnan, *J. Fluid Mech.* **6**, 497 (1959).
41. M. Chertkov and G. Falkovich, to be published.
42. J. R. Herring, *J. Atm. Sci.* **32**, 2254 (1975); *Phys. Fluids* **17**, 859 (1974).
43. G. I. Taylor, *Proc. Roy. Soc. A*, **151**, 184, 421–78 (1935).
44. G. Falkovich, in *Singularities in Fluids, Plasmas and Optics* edited by R. Caflisch, NATO ASI Series, Kluwer, London (1993).
45. V. E. Zakharov and V. Yakhot, *Physica D* **64**, 379 (1992)
46. R. H. Kraichnan, *Phys. Fluids* **8**, 575 (1965).