EUROPHYSICS LETTERS 20 March 1995

Europhys. Lett., **29** (9), pp. 681-686 (1995)

Anomalous Scaling in Kolmogorov-1941 Turbulence.

V. S. L'vov (*)(**) and V. V. Lebedev (*)(***)

(*) *Department of Physics, Weizmann Institute of Science - Rehovot, 76100, Israel*
(**) *Institute of Automation and Electrometry, Academy of Sciences of Russia 630090, Novosibirsk, Russia*
(***) *Landau Institute for Theoretical Physics, Academy of Sciences of Russia 117940, GSP-1, Moscow, Kosygina 2, Russia*

(received 5 July 1994; accepted in final form 28 February 1995)

PACS. 47.10+g – Fluid dynamics: General theory.
PACS. 47.27.Gs – Isotropic turbulence; homogeneous turbulence.

Abstract. – We show that the Kolmogorov-1941 picture of fully developed hydrodynamic turbulence (with the scaling of the structure functions $S_n(R) \propto R^{n/3}$) necessarily leads to an anomalous scaling for correlation functions which include the rate of energy dissipation $\varepsilon(t, \boldsymbol{r})$, these correlation functions being described by an independent index. The mechanism for anomalous scaling, suggested on the basis of the Navier-Stokes equation, is the multi-step interaction of eddies from the inertial interval with eddies at the viscous scale via a set of eddies of intermediate scales.

Introduction. – The modern theory of hydrodynamic turbulence originates from the Kolmogorov's concept of turbulence (hereafter K41) as a step-by-step cascade of energy over scales [1,2]. According to K41 in the inertial range of scales between the outer (pumping) scale L and inner (viscous) scale $\eta = (\nu^3/\bar{\varepsilon})^{1/4}$ the only relevant parameter is $\bar{\varepsilon}$, the mean value of the energy dissipation rate

$$\varepsilon(\boldsymbol{r}) = \frac{1}{2}\nu[\nabla_\alpha v_\beta(\boldsymbol{r}) + \nabla_\beta v_\alpha(\boldsymbol{r})]^2. \tag{1}$$

For the structure functions with $L \gg R \gg \eta$ K41-dimensional reasoning yields

$$S_n(R) \equiv \langle |\boldsymbol{v}(\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R})|^n \rangle \sim (\bar{\varepsilon} R)^{n/3}. \tag{2}$$

The widely spread belief is that intermittency (fluctuations of ε) modifies (2) leading to $S_n \propto R^{\zeta_n}$, $\zeta_n \neq n/3$ [3]. To characterize intermittency one introduces the correlation function of ε-fluctuations $\tilde{\varepsilon}(\boldsymbol{r}) = \varepsilon(\boldsymbol{r}) - \bar{\varepsilon}$ which is believed to have a scaling form in the inertial range

$$K_{\varepsilon\varepsilon}(R) = \langle \tilde{\varepsilon}(\boldsymbol{r}) \tilde{\varepsilon}(\boldsymbol{r}+\boldsymbol{R}) \rangle \propto R^{-\mu}, \tag{3}$$

with a phenomenological constant μ [3]. Numerous measurements to determine the exponent

μ (see, *e.g.* [4]) have resulted in a set of values ranging from 0.15 to 0.4. Various phenomenological models of intermittency give different relationships between ζ_n and μ; in some models [5-7] the «bridge» $\zeta_6 = 2 - \mu$ arises. However, this constraint is questionable [8, 9] and really one cannot solidly relate ζ_n and μ using only phenomenology. A diagrammatic analysis based on the Navier-Stokes (NS) equation [10-12] shows that in the limit $Re \to \infty$ fluctuations of ε would not change (2). Thus we accept the K41 scaling and try to examine the correlation function (3) at this assumption.

The naive estimation of the correlation function $K_{\varepsilon\varepsilon}(R)$ in the framework of K41 gives [13]

$$K_{\varepsilon\varepsilon}^{\mathrm{G}}(R) \sim \nu^2 (\mathrm{d}^2 S_2(R)/\mathrm{d}R^2)^2 \sim \bar{\varepsilon}^2 (\eta/R)^{8/3} \tag{4}$$

in obvious contradiction with the experiment. The conventional way out (see *e.g.* [14]) is that one should take into account the nonlinear interaction by replacing in (4) the molecular viscosity ν with the turbulent viscosity $\nu_{\mathrm{T}}(R)$. In K41 $\nu_{\mathrm{T}}(R) \propto R^{4/3}$; this yields $\mu = 0$. However, this procedure is not justified. In this Letter we develop an analytical theory of ε-ε correlations based on the diagrammatic approach which predicts a strong renormalization of (4) which has nothing in the replacement $\nu \to \nu_{\mathrm{T}}(R)$. In our theory μ remains an independent parameter. In order to elucidate the physical basis of the involved diagrammatic expansion we first describe our findings using the popular handwaving language of cascades, eddies and their interactions before turning to an overview of the cumbersome technicalities of the presented theory.

Telescopic Multi-Step Eddy Interaction. – For our analysis it is useful to present the velocity field as a superposition of eddies of various sizes x. In K41 the velocity gradient scales like $x^{-2/3}$, therefore the main contribution to $\bar{\varepsilon}$ itself is expected to come from eddies of the size of the order of η, while the main contribution to the correlation on the distance R is expected from the eddies which bridge the gap between $\boldsymbol{r}$ and $\boldsymbol{r} + \boldsymbol{R}$. This means that scales larger than R can be neglected in our consideration. The direct effect of the R-eddies produces the naive estimation (4), while the indirect influence (via the eddies of intermediate scales) leads to the strong renormalization of the index μ.

To this end, we portion the range from η to R into N subintervals $\eta < x_1 < x_2 < \ldots < x_N$, where R is within the N-th subinterval. This can be thought of as a partition of k-space into shells separated by wave vectors $k_n = 2\pi/x_n$. The velocity of n-eddies $\boldsymbol{v}_n(\boldsymbol{x})$ is the sum of Fourier harmonics with k between k_{n-1} and k_n. To find the correlation function $K_{\varepsilon\varepsilon}$, we should perform averaging over the statistics of eddies in all subintervals. The crucial point is that the statistics of eddies in different subintervals is not independent: the statistics of small eddies is influenced by the strain field of larger eddies.

The correlation between fluctuations on different scales can be described by a conditional probability density. We will designate as P_n the probability density to find near $\boldsymbol{r}$ the velocity gradient $\nabla_\alpha \boldsymbol{v}$ of n-eddies. The quantity P_n is a function of $\nabla_\alpha \boldsymbol{v}_{>n}$, which is the sum of gradients of the velocity fields of larger eddies: $\boldsymbol{v}_{>n} = \boldsymbol{v}_{n+1}(\boldsymbol{r}) + \boldsymbol{v}_{n+2}(\boldsymbol{r}) + \ldots$. K41 scaling implies that the velocity gradients of larger scales are comparatively small. Hence, we will make use of the expansion of P_n for small large-scale gradients:

$$P_n = P_n^{(0)} + P_n^{(2)} \cdot (\nabla_\alpha \boldsymbol{v}_{>n})^2 + \ldots, \tag{5}$$

where $P_n^{(0)}$, $P_n^{(2)}$ are functions of $\nabla_\alpha \boldsymbol{v}_n$ only. The linear term of the expansion over $\nabla_\alpha \boldsymbol{v}_{>n}$ is absent in (5) because of the incompressibility condition $\nabla \cdot \boldsymbol{v}_{>n} = 0$.

Let us start with the hypothetical case where only N- and n-eddies ($n < N$) are excited. Then designating by halfsquare brackets the averaging over the statistics of corresponding eddies, we find

$$K_{\varepsilon\varepsilon}(R) = \lfloor \lfloor \tilde{\varepsilon}(\boldsymbol{r})\tilde{\varepsilon}(\boldsymbol{r}+\boldsymbol{R}) \rfloor_n \rfloor_N = \lfloor \lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor_n \lfloor \tilde{\varepsilon}(\boldsymbol{r}+\boldsymbol{R}) \rfloor_n \rfloor_N, \tag{6}$$

because the correlation length under the conditional n-average is much shorter than R. Expanding the conditional probability, we obtain

$$\lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor_n = \tilde{\varepsilon}_N(\boldsymbol{r}) + B_n \tilde{\varepsilon}_N(\boldsymbol{r}), \tag{7}$$

where $\tilde{\varepsilon}_N(\boldsymbol{r}) = \nu(\nabla v_N(\boldsymbol{r}))^2 - \nu\lfloor(\nabla v_N(\boldsymbol{r}))^2\rfloor_N$. The first term on the r.h.s. of (7) is the direct contribution of the N-eddies to $\lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor_n$, the second term derives immediately from the expansion of P_n. The largest contribution to $\lfloor \varepsilon(\boldsymbol{r}) \rfloor_n$ comes from the n-eddies themselves, this is $\nu\lfloor[\nabla v_n(\boldsymbol{r})]^2\rfloor_n$. However, this contribution is independent of time and space coordinates and is cancelled by subtracting $\bar{\varepsilon}$.

The coefficient B_n in (7) can be estimated applying physical reasoning based on NS equations. The dimensionless parameter describing the relative change of velocity $\boldsymbol{v}_n(\boldsymbol{r})$ with varying $\nabla_\alpha \boldsymbol{v}_N$ is $\tau_n \nabla v_N(\boldsymbol{r})$, where τ_n is the lifetime of n-eddies, hence $B_n \sim \tau_n^2 \lfloor |\nabla v_n|^2 \rfloor_n$. Since in K41 the lifetime τ_n coincides with the turnover time of the n-eddies, we conclude $B_n \sim r_n^0$, which means that B_n is independent of the scale r_n. This statement is of crucial importance for understanding the origin of the anomalous scaling: all scales in the interval from R to η contribute equally to $K_{\varepsilon\varepsilon}$. Therefore, we have to take into account not only contributions from two scales, as we did up to now, but contributions of all the scales from the above interval.

In the case of three groups of eddies of scales $x_N > x_n > x_m$, which are depicted in 1, instead of the contribution $\lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor_n$ in (6) one has now

$$\lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor_{m,n} = \tilde{\varepsilon}_N(\boldsymbol{r})(1 + B_n + B_m + B_n B_m). \tag{8}$$

The first term on the r.h.s of (8) describes the direct contribution of the R-eddies, the second and the third terms are associated with the influence of R-eddies on the n- and m-eddies, respectively. The last term ($\propto B_n B_m$) is due to the indirect effect of the largest scale, the R-eddies, on the smallest scale x_m via the intermediate n-eddies. To obtain this term, one has to repeat twice the above expansion. Instead of the product $(1 + B_n)(1 + B_m)$ figuring in the

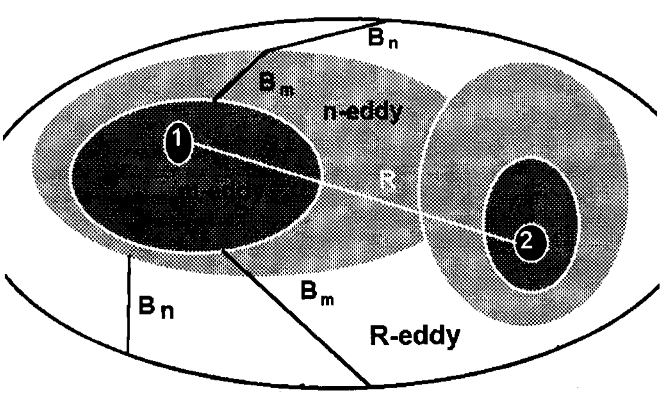

Fig. 1. – Telescopic eddy interaction of three groups of eddies of scales R, x_n, and x_m. Ellipses 1 and 2 show eddies of the scale η, separated by the distance R. For contributions of various interactions in the «telescope», see eq. (8).

r.h.s of (8), in the general case one finds

$$\lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor = \tilde{\varepsilon}_N(\boldsymbol{r}) \prod_{n=1}^{N} (1 + B_n) . \tag{9}$$

The expression (9) should be substituted into (6) instead of $\lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor_n$. Note that the independence of the B_n from n as pointed out above will only hold when the width Δk_n of the shells scales in the same way as the k_n themselves. We choose $k_{n+1}/k_n = \Lambda$ so that neighbouring shells may be considered as almost statistically independent. Such $\Lambda > 1$ does exist because of the locality of energy transfer via the scales [10]. Then one can rewrite (9) as

$$\lfloor \tilde{\varepsilon}(\boldsymbol{r}) \rfloor = \tilde{\varepsilon}_N(\boldsymbol{r})(1 + B_n)^N \sim \tilde{\varepsilon}_N(\boldsymbol{r})(R/\eta)^{\Delta}, \tag{10}$$

where $N = \log_\Lambda (R/\eta)$ and $\Delta = \ln(1 + B)/\ln(\Lambda)$. Following the terminology accepted in the theory of phase transitions, one can call the exponent Δ an *anomalous dimension* of ε. From (6), (10) one obtains finally

$$K_{\varepsilon\varepsilon}(R) \sim \nu^2 (R/\eta)^{2\Delta} \lfloor \tilde{\varepsilon}_N(0) \tilde{\varepsilon}_N(\boldsymbol{R}) \rfloor_N \sim \bar{\varepsilon}^2 (\eta/R)^{\mu} , \tag{11}$$

with $\mu = 8/3 - 2\Delta$ since the average over N-eddies can be estimated naively as in (4). To explain the experimental value of $\mu < 0.5$, the value of Δ should be near 1 which corresponds to the strong renormalization of the naive value.

Diagrammatic equations. – We will use the Wyld diagrammatic technique [15] with the Belinicher-L'vov resummation [10] (see also [11]). This enables us to represent any correlation function characterizing a turbulent flow as a series over the different-time correlation function $F_{\alpha\beta}$ of the quasi-Lagrangian (qL) velocity differences $\boldsymbol{w}(t, \boldsymbol{r}) = \boldsymbol{v}(t, \boldsymbol{r}) - \boldsymbol{v}(t, 0)$ and the corresponding Greens' function $G_{\alpha\beta}$:

$$\begin{cases} F_{\alpha\beta}(t, \boldsymbol{r}_1, \boldsymbol{r}_2) = \langle w_\alpha(t, \boldsymbol{r}_1) w_\beta(0, \boldsymbol{r}_2) \rangle , \\ G_{\alpha\beta}(t, \boldsymbol{r}_1, \boldsymbol{r}_2) = -i\delta \langle w_\alpha(t, \boldsymbol{r}_1) \rangle / \delta \langle f_\beta(0, \boldsymbol{r}_2) \rangle . \end{cases} \tag{12}$$

The qL velocity $\boldsymbol{v}(t, \boldsymbol{r})$ is related to the Eulerian velocity $\boldsymbol{u}(t, \boldsymbol{r})$ according to [10]: $\boldsymbol{u}(t, \boldsymbol{r}) = \boldsymbol{v}\left(t, \boldsymbol{r} - \int^t \boldsymbol{v}(\tau, 0)\, \mathrm{d}\tau\right)$. The G-function is defined here as the susceptibility determining the average $\langle \boldsymbol{w} \rangle$ which appears if the external force $\boldsymbol{f}$ on the r.h.s. of the NS equation has a nonzero average $\langle \boldsymbol{f} \rangle$. The above formulation is an order-by-order Galilei invariant and has no infrared divergences of the integrals in all diagrams. Therefore, the external scale of the turbulence is absent in diagrammatic equations. We showed that K41 is the only scale-invariant solution of these equations [12]. The correlation functions (12) may be evaluated as

$$G(t, \boldsymbol{r}_1, \boldsymbol{r}_2) \propto R^{-3}, \qquad F(t, \boldsymbol{r}_1, \boldsymbol{r}_2) \propto R^{2/3}, \tag{13}$$

where R is the characteristic scale.

The correlation function $K_{\varepsilon\varepsilon}$ is represented by an infinite series of diagrams. In the spirit of the Keldysh diagrammatic technique [16] we will utilize only one type of line designating both F- and G-functions. Interaction vertices on diagrams are determined by the nonlinear term in the qL version of NS equation [10, 11]. Diagrams representing contributions to $K_{\varepsilon\varepsilon}$ are depicted in fig. 2. A circle on these diagrams corresponds to (1). A rectangle on these

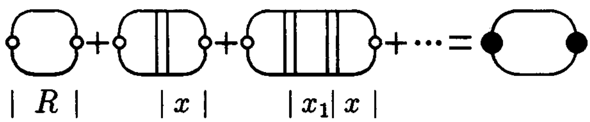

Fig. 2. – The sum of the diagrams for $K_{\varepsilon\varepsilon}(R)$.

diagrams represents the sum of diagrams which cannot be cut into two parts along two horizontal lines. The first contributions to the rectangle are determined simply by the F- or G-line. An analysis of higher-order contributions analogous to that produced in [10, 11] shows that the dimension estimate of the rectangle given by the first contributions is reproduced for the entire sum of diagrams and is consequently determined by eq. (13).

Based on this result, we conclude that the expression corresponding to the right loop in the second diagram in fig. 2 contains a logarithmic factor originating from scales $R \gg x \gg \eta$. The third diagram in fig. 2 will give the second power of the logarithm originating from the region of scales $R \gg x \gg x_1 \gg \eta$ (we would like to stress that the region $R \gg x_1 \gg x \gg \eta$ does not give the second power of the logarithm). Therefore the problem of summing the main logarithmic terms appears. To solve it let us take the cut corresponding to the largest separation of the order of R. Then the parts of the diagrams on the left and on the right of this cut will give after summation a three-leg object which we depict as a disk. As a result we come to the representation for $K_{\varepsilon\varepsilon}(R)$ depicted in fig. 2.

The three-leg object differs from the bare object by a factor $\mathcal{N}(R)$. Resumming the «ladder» sequence of diagrams, we find for $\mathcal{N}(x)$ the integral equation depicted in fig. 3. In analytical form it is

$$\mathcal{N}(x) = 1 + \int_{\eta}^{x} \mathrm{d}y\, A(x, y)\, \mathcal{N}(y)\,, \tag{14}$$

where $A(x, y)$ is a uniform function with the index -1. This allows a powerlike solution

$$\mathcal{N}(x) \sim (x/\eta)^{\Delta}\,, \tag{15}$$

where Δ is the index coinciding with the ones introduced in the preceding section. This law is the consequence of the asymptotic behavior of the qL vertex [10, 11].

Using the diagrammatic representation for $K_{\varepsilon\varepsilon}$ given in fig. 2 (the last term), we can proceed to the analysis of the scaling behavior of $K_{\varepsilon\varepsilon}$. If one takes the bare value $\mathcal{N} = 1$ the last diagram in fig. 2 will reproduce the expression (4). If one takes the dressed functions $\mathcal{N}$, we conclude that the last diagram in fig. 2 gives the contribution $\propto R^{2\Delta - 8/3}$. The above diagrammatic analysis obviously leads to the same results (11) as the semi-qualitative analysis given in the previous section.

Discussion. – We have demonstrated that assuming K41 scaling (2) one finds an essential renormalization of the naive K41 value $\mu = 8/3$ (the experimental value is $\mu < 0.5$) due to the multi-step eddy interaction. Although we cannot find the renormalization explicitly, we see that μ is determined by complex integrals of correlation functions. Therefore one could not

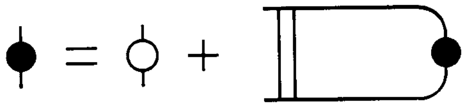

Fig. 3. – The equation for the renormalization factor $\mathcal{N}(x)$.

expect a simple relation between μ and ζ_n like the «bridge» $\zeta_6 - \mu$. Note also that the same renormalized exponent μ should determine the scaling of other objects, *e.g.* the asymptotic behavior of the four-point correlation function of velocities [17]. Let us stress that in our theory both the structure functions S_n and the correlation functions of $\tilde{\varepsilon}$ do not depend on the outer scale of turbulence L at $Re \to \infty$, whereas old experiments seem to demonstrate L-dependence of different correlation functions. On the contrary recent data reported in [18] do not show a Re-dependence of both skewness and flatness at large Re which corresponds to our concept. In any case (as the physics of second-order phase transitions shows) in order to compare adequately experiment and theory, one needs suitable interpolation formulae. Obtaining such formulae is a separate problem which is out of the scope of this letter.

* * *

Numerous discussion with R. Kraichnan, I. Procaccia and S. Thomae were very illuminative for us. We are grateful to A. Fairhall for help. This work has been supported by the Minerva Center for Nonlinear Physics of Complex Systems and the Landau-Weizmann Program.

REFERENCES

[1] A. N. Kolmogorov, *Dokl. Akad. Nauk SSSR*, **30** (1941) 229; **32** (1941) 19.
[2] Obukhov A. M., *Dokl. Akad. Nauk SSSR*, **32** (1941) 22.
[3] Monin A. S., Yaglom A. M., *Statistical Fluid Mechanics*, Vol. I, II (MIT Press, Cambridge) 1975.
[4] Sreenivasan K. R., Meneveau C., *Phys. Fluids A*, **5** (1993) 512 and references therein.
[5] Kolmogorov A. N., *J. Fluid Mech.*, **13** (1962) 82.
[6] Novikov E., Steward R., *Izv. Geophys.*, **3** (1964) 408.
[7] Frisch U., Sulem P.-L., and Nelkin M., *J. Fluid Mech.*, **87** (1978) 719.
[8] Meneveau C. and Sreenivasan K. R., *J. Fluid Mech.*, **224** (1991) 429.
[9] Frisch U., Private communication, Corsica, 1993.
[10] Belinicher V. I. and L'vov V. S., *Ž. Èksp. Teor. Fiz.*, **93** (1987) 533 (*Sov. Phys. JETP*, **66** (1987) 303).
[11] L'vov V. S., *Phys. Rep.*, **207** (1991) 1.
[12] L'vov V. S. and Lebedev V. V., *Phys. Rev. E*, **47** (1993) 1794.
[13] Golitsyn S. G., *Dokl. Akad. Nauk SSSR*, **144** (1962) 520.
[14] Yakhot V. and Zakharov V., *Physica D*, **64** (1193) 379.
[15] Wyld H. W., *Ann. Phys. (N.Y.)*, **14** (1961) 143.
[16] Keldysh L. V., *Ž. Èksp. Teor. Fiz.*, **47** (1964) 1515 (*Sov. Phys. JETP*, **20** (1965) 1018).
[17] Lebedev V. V. and L'vov V. S., *Pis'ma Ž. Èksp. Teor. Fiz.*, **59** (1994) 546 (*JETP Lett.*, **59** (1994) 577).
[18] Belin F., Maurer J., Tabeling P., Zocchi G. and Wilaime H., *Probability density functions, skewness and flatness in large Reynolds number turbulence*, submitted to *Phys. Rev. E*.