LETTERS

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The universal scaling exponents of anisotropy in turbulence and their measurement

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Correlation functions of non-scalar fields in isotropic hydrodynamic turbulence are characterized by a set of universal exponents. These exponents also characterize the rate of decay of the effects of anisotropic forcing in developed turbulence. These exponents are important for the general theory of turbulence, and for modeling anisotropic flows. We propose methods for measuring these exponents by designing new laboratory experiments. *© 1996 American Institute of Physics.* [S1070-6631(96)02610-4]

Fundamental studies of turbulence tend to stress the model of locally isotropic, homogeneous turbulence, and most theories and experiments since Kolmogorov's seminal work of 1941 (K41)[1] considered the universal (anomalous) exponents that characterize isotropic turbulent flows (see Refs. 2 and 3 for recent reviews). In fact, most turbulent flows are not forced isotropically, and moreover even in isotropic flows there are important fields that are constructed from velocity derivatives that transform under rotation as vectors or tensors rather than scalars. It has been known for some time that the second-order structure function (that depends on one separation vector) becomes more isotropic as the separation scale decreases. Moreover, the rate of this isotropization process is governed by an exponent.[4–7] In recent papers[8,9] it was pointed out that this exponent is one of an infinite family of anomalous scaling exponents. This family was never considered (beyond the lowest order exponent) in experiments and numerical simulations. In addition to being of fundamental interest these universal exponents are also of importance in modeling realistic flows which are not isotropic. In this Letter we propose how to measure these universal exponents in physical experiments.

The simplest statistical quantity that is built from the velocity field $\mathbf{u}(\mathbf{r},t)$ that displays anisotropic contributions is the second-order structure functions:

$$S_2(\mathbf{R}) \equiv \langle |\mathbf{w}(\mathbf{r}_0|\mathbf{r},t)|^2 \rangle, \quad \mathbf{R} \equiv \mathbf{r} - \mathbf{r}_0, \tag{1}$$

where $\mathbf{w}(\mathbf{r}_0|\mathbf{r},t) \equiv \mathbf{u}(\mathbf{r},t) - \mathbf{u}(\mathbf{r}_0,t)$ and $\langle \ldots \rangle$ stands for a suitably defined ensemble average. Due to space-time homogeneity S_2 is a time independent function of $\mathbf{R}$. In locally homogeneous and isotropic turbulence the scaling properties of $S_2(R)$ were widely discussed[2,3]

$$S_2(R) \simeq (\overline{\epsilon} R)^{2/3} (R/L)^{\delta}, \tag{2}$$

where $\overline{\epsilon}$ is the mean energy flux per unit time per unit mass, and δ is the deviation of the scaling exponent ζ_2 of the structure function from the K41 prediction, $\zeta_2 \equiv 2/3 + \delta$. In anisotropic turbulence $S_2(\mathbf{R})$ depends also on the direction of $\mathbf{R}$. We can use the fact that the Navier-Stokes equations are invariant to the transformations of the SO(3) symmetry group, i.e. all rotations, in order to represent the solutions in terms of the spherical harmonics $Y_{\ell,m}(\hat{\mathbf{R}})$ where $\hat{\mathbf{R}}$ is a unit vector in the direction of $\mathbf{R}$: $S_2(\mathbf{R}) = \Sigma_{\ell=0}^{\infty} S_{2,\ell}(\mathbf{R})$ where

$$S_{2,\ell}(\mathbf{R}) = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{R}}) \int S_2(R\hat{\boldsymbol{\xi}}) Y_{\ell m}(\hat{\boldsymbol{\xi}}) d\hat{\boldsymbol{\xi}}. \tag{3}$$

Here $\hat{\boldsymbol{\xi}}$ is a unit vector. Such a decomposition is particularly useful when the turbulent flow is only weakly anisotropic. Then one can linearize the equation for the anisotropic corrections to $S_2(\mathbf{R})$. The kernel of the linearized equation is invariant under rotations; as a result the equations for the different ℓ-components decouple. Each equation has its own eigenvalue and leads in a scale invariant situation to a generalization of (2) for $S_{2,\ell}$:

$$S_{2,\ell}(\mathbf{R}) \sim (\overline{\epsilon} R)^{2/3} (R/L)^{\delta_\ell} \propto R^{\beta_\ell}, \tag{4}$$

where $\delta_\ell \equiv \beta_\ell - 2/3$. Comparing with Eq. (2) we recognize that in this notation $\delta = \delta_0$. If we accept that homogeneous turbulence enjoys universal statistics in the inertial interval, then the kernels of the above linearized equations are all universal. This leads to the understanding that the exponents of anisotropy β_ℓ are universal. The full spectrum of exponents β_ℓ was found analytically[8,10] in the context of Kraichnan's model of passive scalar advection.[11] For Navier-Stokes turbulence β_2 can be computed using perturbation theory[6,7] (which disregards the nonperturbative effects leading to anomalous scaling[12]) with the result $\beta_2 = 4/3$. The corresponding result for $\beta_0 \equiv \zeta_2$ is 2/3, but experimentally $\zeta_2 \approx 0.7$.[13,14] The theory indicates that such deviations from the naive predictions stem from non-perturbative effects. It is likely that the perturbative result for β_2 holds to a similar accuracy. The significant difference $\beta_2-\beta_0\simeq 2/3$ explains why isotropic scaling may be observed in anisotropic experiments; the contribution of $S_{2,2}$ to S_2 peels off like $(R/L)^{\beta_2-\beta_0}$. We do not possess any numerical estimates for the higher order values of β_ℓ in the case of Navier-Stokes turbulence, but the exact results in the case of passive advection lead us to expect them to be all positive and increasing with ℓ.

In the context of passive scalar advection we demonstrated that the very same exponents β_ℓ have an important role in the context of *isotropic* turbulence for statistical quantities that depend on more than two coordinates.[8] The same is true for Navier-Stokes turbulence. Consider the correlation function of four velocity differences

$$S_4(\mathbf{R}_1,\mathbf{R}_2)\equiv\langle|\mathbf{w}(\mathbf{r}_0|\mathbf{r}_1)|^2|\mathbf{w}(\mathbf{r}_0|\mathbf{r}_2)|^2\rangle, \tag{5}$$

where $\mathbf{R}_1=\mathbf{r}_1-\mathbf{r}_0$ and $\mathbf{R}_2=\mathbf{r}_2-\mathbf{r}_0$. As usual we assume that this, and all other correlators, are scale invariant. Mathematically this means that they are all homogeneous functions of their arguments as long as these are in the ''inertial range.'' In other words $S_4(\lambda\mathbf{R}_1,\lambda\mathbf{R}_2)=\lambda^{\zeta_4}S_4(\mathbf{R}_1,\mathbf{R}_2)$, where ζ_4 is the scaling exponents of the fourth-order structure function: $\langle|\mathbf{w}(\mathbf{r}_0|\mathbf{r}_1)|^4\rangle \propto R_1^{\zeta_4}$. In isotropic turbulence $S_4(\mathbf{R}_1,\mathbf{R}_2)$ depends on the separations R_1,R_2 and on the angle $\theta_{1,2}$ between these two vectors. We are interested in the limit $R_1\ll R_2$, but R_1 and R_2 are both in the inertial interval. For turbulent systems that enjoy scale invariance in the inertial interval, and under the assumption of weak universality the asymptotic behavior of $S_4(\mathbf{R}_1,\mathbf{R}_2)$ is determined by the fusion rules.[12,15] According to the fusion rules in the limit $R_1\ll R_2$ the leading dependence of $S_4(\mathbf{R}_1,\mathbf{R}_2)$ on R_1 and R_2 is independent of $\theta_{1,2}$. The scaling law is

$$S_4(\mathbf{R}_1,\mathbf{R}_2)\propto(R_1/R_2)^{\zeta_2}R_2^{\zeta_4}, \quad R_1\ll R_2. \tag{6}$$

In order to extract the contributions that depend on the angle of $\mathbf{R}_1$ we use a multipole decomposition of S_4, $S_4(\mathbf{R}_1,\mathbf{R}_2)=\Sigma_{\ell=0}{}^{\infty}S_{4,\ell}(\mathbf{R}_1,\mathbf{R}_2)$ in a way similar to (3). For small anisotropy one can again consider ℓ-decoupled equations that lead, similarly to (4), to the prediction

$$S_{4,\ell}(\mathbf{R}_1,\mathbf{R}_2)\propto(R_1/R_2)^{\beta_\ell}R_2^{\zeta_4}, \quad R_1\ll R_2. \tag{7}$$

Note that two statements are being made here: (i) The overall exponent for S_4 is ζ_4. This directly follows from the property of scale invariance. (ii) The scaling exponents characterizing the R_1-dependence of $S_{4,\ell}$ are the exponents of $S_{2,\ell}$. We note that the observation of the exponents β_ℓ in the context of $S_2(\mathbf{R})$ requires anisotropic driving of the flow. In contrast, using $S_4(\mathbf{R}_1,\mathbf{R}_2)$ the exponents β_ℓ are observable even in fully isotropic flows. The reason is that here we have a built-in direction $\mathbf{R}_2$. When R_1 is of the order of R_2 the dependence on the angle $\theta_{1,2}$ is all important. When R_1 decreases this dependence weakens at a rate determined by $\beta_\ell-\beta_0$.

Next in order of complication we consider

$$S_4(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0)\equiv\langle|\mathbf{w}(\mathbf{r}_0|\mathbf{r}_1)|^2|\mathbf{w}(\mathbf{r}_0'|\mathbf{r}_2)|^2\rangle, \tag{8}$$

where $\mathbf{R}_0=\mathbf{r}_0'-\mathbf{r}_0$. This is a function of three separation and the three angles $\theta_{1,0}$, $\theta_{2,0}$ and $\theta_{1,2}$. As before represent this function as a double multipole-expansion with respect to the directions of $\mathbf{R}_1$ and $\mathbf{R}_2$: $S_4(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0)=\Sigma_{\ell_1,\ell_2}S_{4,\ell_1,\ell_2}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0)$. In the limit $R_1,R_2\ll R_0$ these functions exhibit a universal scaling form similar to (7)

$$S_{4,\ell_1,\ell_2}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0)\propto(R_1/R_0)^{\beta_{\ell_1}}(R_2/R_0)^{\beta_{\ell_2}}R_0^{\zeta_4}. \tag{9}$$

At this point we want to discuss how to set up possible experiments to measure the new universal exponents β_ℓ. Given direct numerical simulations with anisotropic forcing, one can simply compute $S_{2,\ell}(\mathbf{R})$ from the definitions (1) and (3) and then to plot log-log plots of $S_{2,\ell}$ vs. R, or, following the ideas[14] of ''extended self-similarity,'' of $S_{2,\ell}$ vs. $S_{2,\ell=0}$. It is impossible to follow this route in standard laboratory experiments since the detailed angular information is not usually available. One can estimate β_2 in anisotropic flows by measuring for example the longitudinal and transverse components of the second-order structure function, and form a combination that vanishes in isotropic flows. Such a combination scales with R and the leading contribution is R^{β_2}. This type of measurement was performed (see Ref. 16) and discussed in detail by Nelkin.[3] The experimental evidence is that the numerical value of β_2 is indeed rather close to 4/3. Our point in this Letter is that the very same exponents β_ℓ play an important role also in isotropic flows.

Consider an experiment with a mean flow (like a wind tunnel or an atmospheric boundary layer). Assign the direction of the mean flow to the x-coordinate. The minimal experimental setup calls for two local probes (like hot wires) positioned at $\mathbf{r}_0=(0,0,0)$ and $\mathbf{r}_1=(0,\Delta,0)$, separated by a distance Δ in the y-direction which is orthogonal to the mean flow. Under the standard Taylor hypothesis differences in time are interpreted as differences along the longitudinal x-direction. This means that one can measure the longitudinal projections $au_x(x,0,0)$ and $bu_x(x,\Delta,0)$. The coefficients a and b are introduced in recognition of the fact that in realistic experiments the two probes cannot be perfectly calibrated. Define now the longitudinal and transverse velocity differences

$$w_{||}(x,\Delta)\equiv u_x(x+\Delta,0,0)-u_x(x,0,0), \tag{10}$$

$$w_{\perp}(x,\Delta)\equiv u_x(x,\Delta,0)-u_x(x,0,0). \tag{11}$$

Next one can measure the corresponding structure functions for Δ-separations $S_{2||}(\Delta)\equiv\langle w_{||}^2(x,\Delta)\rangle$, $S_{2\perp}(\Delta)\equiv\langle w_{\perp}^2(x,\Delta)\rangle$. In isotropic conditions these two quantities are related by $S_{2\perp}(\Delta)=S_{2||}(\Delta)+\Delta dS_{2||}(\Delta)/2d\Delta$, and one can use this relation to assess the degree of isotropy on the scale Δ. Next we introduce the normalized squared of velocity differences in which the calibration constants are eliminated:

$$\begin{aligned} W_{||}^2(x,\Delta)&\equiv w_{||}^2(x,\Delta)/S_{2||}(\Delta)\,,\\ W_{\perp}^2(x,\Delta)&\equiv w_{\perp}^2(x,\Delta)/S_{2\perp}(\Delta)\,. \end{aligned} \tag{12}$$

Finally we define two fields $\Psi_\pm$ according to

$$\begin{aligned} \Psi_+(x,\Delta)&\equiv W_{\perp}^2(x,\Delta)+W_{||}^2(x,\Delta)-2,\\ \Psi_-(x,\Delta)&\equiv W_{\perp}^2(x,\Delta)-W_{||}^2(x,\Delta). \end{aligned} \tag{13}$$

These fields have zero mean, and we will argue that they have different leading order contributions in the multipole expansion. To see this we introduce a generalization of the structure function (8) and the fusion rule (9) which pertains to the case in which tensor indices are kept: $S_4^{\alpha\beta\gamma\delta}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0)\equiv\langle w^\alpha(\mathbf{r}_0|\mathbf{r}_1)w^\beta(\mathbf{r}_0|\mathbf{r}_1)w^\gamma(\mathbf{r}_0'|\mathbf{r}_2)$ $w^\delta(\mathbf{r}_0'|\mathbf{r}_2)\rangle$. In terms of a multipole expansion

$$S_4^{\alpha\beta\gamma\delta}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0)=\sum_{\ell_1,\ell_2} S_{4,\ell_1,\ell_2}^{\alpha\beta\gamma\delta}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0). \tag{14}$$

We write the generalization of (9) when $R_1,R_2\ll R_0$:

$$S_{4,\ell_1,\ell_2}^{\alpha\beta\gamma\delta}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0) \propto S_{2,\ell_1}^{\alpha\beta}(\mathbf{R}_1;\mathbf{R}_0)S_{2,\ell_2}^{\gamma\delta}(\mathbf{R}_2;\mathbf{R}_0)\Psi(\mathbf{R}_0).$$

In the last equation $S_{2,\ell_1}^{\alpha\beta}(\mathbf{R}_1;\mathbf{R}_0)$ is a function of $\mathbf{R}_1$ that depends on the relative orientation of $\mathbf{R}_1$ with respect to $\mathbf{R}_0$. It scales with R_1 according to (4). In particular for $\ell_1=0$ the scaling exponent is $\beta_0=\zeta_2$ and the tensor structure is that of $S_2^{\alpha\beta}(\mathbf{R}_1)$ in isotropic flows. With this background we can consider the correlation functions of the fields Ψ_+ and Ψ_- that were introduced in (13):

$$K_{++}(R)\equiv\langle\Psi_+(x+R,\Delta)\Psi_+(x,\Delta)\rangle\propto R^{-\mu_{++}}, \tag{15}$$

$$K_{+-}(R)\equiv\langle\Psi_+(x+R,\Delta)\Psi_-(x,\Delta)\rangle\propto R^{-\mu_{+-}}, \tag{16}$$

$$K_{--}(R)\equiv\langle\Psi_-(x+R,\Delta)\Psi_-(x,\Delta)\rangle\propto R^{-\mu_{--}}. \tag{17}$$

In all these equations $\eta<\Delta<R<L$. All these correlations can be understood as particular components and combinations of $S_{4,\ell_1,\ell_2}^{\alpha\beta\gamma\delta}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_0)$ with $R_1=R_2=\Delta$ and $R_0=R$. We note that the correlation function K_{--} is constructed such that the $\ell_1=\ell_2=0$ component which appears in $S_{4,\ell_1,\ell_2}^{\alpha\beta\gamma\delta}$ exactly cancel. The lowest order contribution to this correlation is $\ell=2$. The correlation function K_{++} is generic, having all the ℓ components, and it is therefore dominated by $\ell=0$. The correlation K_{+-} is mixed. The theoretical prediction is

$$\mu_{++}=2\beta_0-\zeta_4,\mu_{+-}=\beta_0+\beta_2-\zeta_4,\mu_{--}=2\beta_2-\zeta_4.$$

Note that the correlations in (15)-(17) are all dimensionally identical; yet we predict very different scaling exponents. This is just another way to explore the breakdown of dimensional analysis in fully developed turbulence. Using known experimental data[13,14] $\zeta_2\simeq0.7$, $\zeta_4\simeq1.2$ and our guess that $\beta_2\simeq4/3$ we expect:

$$\mu_{++}\simeq0.2,\mu_{--}\simeq1.4-1.5,\mu_{+-}=\tfrac{1}{2}(\mu_{++}+\mu_{--}). \tag{18}$$

The last relation exact for Re$\to\infty$. Note, however, that for a finite extent of the inertial interval sub-leading contributions may be important and have to be carefully assessed. Nevertheless, the wide disparity between these scaling exponents promises a worthwhile experiment even if the inertial range is of the order of one decade.

The exponent β_1 is not available from the rate of isotropization of $S_2(\mathbf{R})$. It can be seen in the flow field without inversion symmetry by forming a nonsymmetric second-order correlation function like

$$\langle u_\alpha(\mathbf{r}+\mathbf{R})u_\beta(\mathbf{r})\rangle-\langle u_\alpha(\mathbf{r}-\mathbf{R})u_\beta(\mathbf{r})\rangle\propto R^{\beta_1}. \tag{19}$$

Since this object is odd in $\mathbf{R}$ it vanishes when there exists inversion symmetry. Otherwise its leading scaling exponent is β_1. This exponent is related to the existence of the flux of helicity and standard K41 arguments lead to the value $\beta_1=1$, see for example Ref. 17. This holds probably to the same accuracy as other K41 arguments.

In summary, there exists an infinite set of exponents that characterize the rate of isotropization of the second-order structure function under non-isotropic forcing. There is another set of exponents that determines the scaling behavior of tensorial correlation functions in isotropic turbulence. The first message of this Letter is that these two sets of exponents are identical. The central role that these exponents play warrants their measurement in laboratory experiments. We thus offered some simple ways to measure the low order exponents β_1 and β_2 in realistic experiments. We presented an estimate of the numerical values of these two exponents. The calculation of these exponents from first principles is a different task that is outside the scope of this Letter.

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