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Anomalous scaling in a model of passive scalar advection: Exact results

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Kraichnan's model of passive scalar advection in which the driving velocity field has fast temporal decorrelation is studied as a case model for understanding the appearance of anomalous scaling in turbulent systems. We demonstrate how the techniques of renormalized perturbation theory lead (after exact resummations) to equations for the statistical quantities that also reveal nonperturbative effects. It is shown that ultraviolet divergences in the diagrammatic expansion translate into anomalous scaling with the inner length acting as the renormalization scale. In this paper, we compute analytically the infinite set of anomalous exponents that stem from the ultraviolet divergences. Notwithstanding these computations, nonperturbative effects furnish a possibility of anomalous scaling based on the outer renormalization scale. The mechanism for this intricate behavior is examined and explained in detail. We show that in the language of L'vov, Procaccia, and Fairhall [Phys. Rev. E **50**, 4684 (1994)], the problem is "critical," i.e., the anomalous exponent of the scalar primary field $\Delta = \Delta_c$. This is precisely the condition that allows for anomalous scaling in the structure functions as well, and we prove that this anomaly must be based on the outer renormalization scale. Finally, we derive the scaling laws that were proposed by Kraichnan for this problem and show that his scaling exponents are consistent with our theory.

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I. INTRODUCTION

The model of passive scalar advection with rapidly decorrelating velocity field which was introduced some time ago by Kraichnan [1] was suggested recently [2] as a case model for understanding multiscaling in the statistical description of nonlinear field theories. The model is for a scalar field $\Theta(\mathbf{r},t)$ where $\mathbf{r}$ is a point in $\mathbb{R}^d$. This field satisfies the equation of motion

$$\left[\partial_t - \kappa\nabla^2 + \mathbf{u}(\mathbf{r},t)\cdot\boldsymbol{\nabla}\right]\Theta(\mathbf{r},t) = f(\mathbf{r},t). \tag{1.1}$$

In this equation $f(\mathbf{r},t)$ is the forcing and $\mathbf{u}(\mathbf{r},t)$ is the velocity field which is taken to be a Gaussian stochastic field, rapidly varying in time. The forcing is taken to be δ correlated in time and space homogeneous,

$$\langle f(\mathbf{r},t)f(\mathbf{r}',t')\rangle = \Phi_0(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{1.2}$$

We will study this model in the limit of large Peclet (Pe) number, which is defined as the dimensionless ratio $U_L L/\kappa$, where U_L is the scale of the velocity fluctuations on the outer scale L of the system. The most important property of the driving velocity field from the point of view of the scaling properties of the passive scalar is the tensor "eddy diffusivity" [1]

$$h_{ij}(\mathbf{R}) \equiv \int_0^\infty d\tau \langle [u_i(\mathbf{r}+\mathbf{R},t+\tau) - u_i(\mathbf{r},t+\tau)] \times [u_j(\mathbf{r}+\mathbf{R},t) - u_j(\mathbf{r},t)]\rangle, \tag{1.3}$$

where the symbol $\langle\,\rangle$ in Eq. (1.3) stands for an ensemble average with respect to the statistics of $\mathbf{u}$ which is given *a priori*. The scaling properties of the scalar depend sensitively on the scaling exponent ζ_h that characterizes the R dependence of $h_{ij}(\mathbf{R})$:

$$h_{ij}(\mathbf{R}) = h(R)\left[\frac{\zeta_h + d - 1}{d-1}\delta_{ij} - \frac{\zeta_h}{d-1}\frac{R_i R_j}{R^2}\right], \tag{1.4}$$

$$h(R) = H\left(\frac{R}{\mathcal{L}}\right)^{\zeta_h}, \qquad 0 < \zeta_h < 2 \tag{1.5}$$

where $\mathcal{L}$ is some characteristic outer scale of the driving velocity field. We take the scaling exponent ζ_h of the eddy diffusivity to lie in the interval $(0,2)$. The value of ζ_h that would correspond to Kolmogorov *spatial* scaling in the velocity field is $4/3$. We will assume throughout that the velocity field has an inner scale that is much smaller than the dissipative scale of the scalar. This corresponds to the limit of a low Prandtl number. The structure of the tensor h_{ij} is dictated by the incompressibility of the velocity field, and H is a free dimensional parameter. By "scaling properties" we mean power law dependence of the various correlation and response functions of $\Theta(\mathbf{r},t)$ and its gradients on separation distances. For example, the structure functions of $\Theta(\mathbf{r},t)$ are

$$S_{2n}(\mathbf{R}) \equiv \langle[\Theta(\mathbf{r}+\mathbf{R},t) - \Theta(\mathbf{r},t)]^{2n}\rangle\ . \tag{1.6}$$

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In writing this equation we assumed that the statistics of the velocity field leads to a stationary and space homogeneous ensemble of the scalar Θ. If the statistics is also isotropic, then S_{2n} becomes a function of R only, independent of the direction of $\mathbf{R}$. The scaling exponents of the structure functions $S_{2n}(R)$ characterize their R dependence in the limit of large Pe,

$$S_{2n}(R) \sim R^{\zeta_{2n}}, \tag{1.7}$$

when R is in the "inertial" interval of scales that will be discussed later in this paper.

In Ref. [2] Kraichnan showed that when the driving velocity field $\mathbf{u}(\mathbf{r},t)$ is δ correlated in time one can derive the exact "balance" equations for the structure functions $S_{2n}(R)$:

$$\hat{B}(R)S_{2n}(R) = J_{2n}(R) . \tag{1.8}$$

In this equation $\hat{B}(R)$ is the linear operator that will be used below repeatedly:

$$\hat{B}(R) = R^{1-d}\frac{\partial}{\partial R}R^{d-1}h(R)\frac{\partial}{\partial R}, \tag{1.9}$$

with d being the space dimension. On the right-hand side of the balance equations we have

$$J_{2n}(R) = -4n\kappa\left\langle[\Theta(\mathbf{r}+\mathbf{R})-\Theta(\mathbf{r})]^{2n-1}\nabla^2\Theta(\mathbf{r})\right\rangle . \tag{1.10}$$

Kraichnan conjectured that the scaling dependence of $J_{2n}(R)$ on R when R is in the inertial range is given by the law

$$J_{2n}(R) = nJ_2(R)S_{2n}(R)/S_2(R) \sim R^{\zeta_{2n}-\zeta_2} . \tag{1.11}$$

This scaling law led Kraichnan to far reaching conclusions regarding the scaling exponents ζ_{2n}. Once inserted in the balance equations this conjecture resulted in the prediction that the scaling exponents satisfy the equation

$$\zeta_{2n}(\zeta_{2n}-\zeta_2+d) = nd\zeta_2 . \tag{1.12}$$

If so, this model may be the first nonlinear nonequilibrium case where "multiscaling" can be explicitly demonstrated. Further, Kraichnan *et al.* proposed in [3] that if this model *is* multiscaling, then (1.12) is the unique solution for ζ_{2n}. We will see that among other things our considerations lead to precisely the scaling law $J_{2n}(R) \sim R^{\zeta_{2n}-\zeta_2}$ as conjectured by Kraichnan. We will see how the possibility of multiscaling is indeed realized due to the analytic structure of the theory.

In a previous work on this model [4] (referred to hereafter as paper I) renormalized perturbation theory was employed to study the scaling behavior. It is appropriate to present first a short summary of the results of this paper.

The main point of paper I was that various statistical quantities exhibit anomalous exponents that stem from ultraviolet divergences in their diagrammatic expansion. For example, there exists a quantity called the nonlinear Green's function (that is defined in Sec. II and is a four-point quantity) and which is denoted as $\mathcal{G}_2(0|\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$. It was shown that for $r_1 \simeq r_2 \simeq r$, $r_3 \simeq r_4 \simeq R$, and $R \gg r$ [cf. I, Eqs. (4.11), (4.12), and (4.18)]

$$\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}\mathcal{G}_2(0|\mathbf{r}_1,r_2,r_3,r_4) \sim r^{-\Delta} , \tag{1.13}$$

where Δ is an anomalous exponent that characterizes the leading divergence when $r \to 0$. Moreover, the value of this exponent is important in determining much of the scaling behavior in this model. It appears prominently in J_{2n} and also in the correlation function of passive dissipation fluctuations. The dissipation field is defined here as

$$\epsilon(x) \equiv \kappa|\boldsymbol{\nabla}\Theta(x)|^2, \tag{1.14}$$

where the $(d+1)$-dimensional vector $x \equiv (\mathbf{r},t)$. The correlation function $K(R)$ is

$$K(R) = \langle\epsilon(\mathbf{r}+\mathbf{R},t)\epsilon(\mathbf{r},t)\rangle - \langle\epsilon(x)\rangle^2 . \tag{1.15}$$

It was shown in paper I that the ultraviolet divergences resulted in a dependence on the inner cutoff of the theory, denoted as η [and defined below in Eq. (2.25)] which is written as

$$K(R)/\kappa^2 \sim \eta^{-2\Delta}. \tag{1.16}$$

It was also explained in paper I that the theory indicates that if Δ reaches the critical value $\Delta_c = \zeta_h$ special considerations must be made. For $\Delta < \zeta_h$ and ζ_h in the interval $(0,2)$ it was shown that the perturbation theory for S_4 and higher order structure functions converges order by order both in the ultraviolet and in the infrared limits. Accordingly, there is no external renormalization length scale in the theory and there is no (perturbative) mechanism to ruin simple scaling, i.e., $\zeta_{2n} = n\zeta_2$. In this case the correlation of dissipation fluctuations can be shown to decay in the inertial range of scales as

$$K(R) \sim \langle\epsilon(x)\rangle^2\left(\frac{\eta}{R}\right)^{2\zeta_h-2\Delta} . \tag{1.17}$$

Indeed, as long as $\Delta < \zeta_h$ the correlation decays in the inertial range as it must for a mixing field. On the contrary, if $\Delta = \zeta_h$ Eq. (1.17) cannot continue to hold since it predicts that the correlations do not decay. This is precisely where the need for anomalous scaling of the structure functions comes in. We will see below that the value of Δ is precisely ζ_h, and we will get instead of Eq. (1.17) the following prediction:

$$K(R) \sim \langle\epsilon(x)\rangle^2\left(\frac{L}{R}\right)^{2\zeta_2-\zeta_4} . \tag{1.18}$$

Here L is the outer renormalization scale for this model, which is in general not identical with the outer scale $\mathcal{L}$ of Eq. (1.5). This result is in agreement with Kraichnan's conjectures. The detailed explanation of how this phenomenon takes place is one of the major aims of this paper. It is an important mechanism that indicates how at least in this example ultraviolet divergences coupled with nonperturbative effects may conspire to give at the end anomalous corrections which are carried by the outer renormalization scale. Whether or not such a mechanism operates in other hydrodynamic systems will be discussed in separate presentations. At any rate, the example treated here serves as a powerful demonstration of the fact that renormalized perturbation theory as applied to field theories of the hydrodynamic type allows one, after proper resummations, to capture subtle nonperturbative effects.

It needs to be stressed that the anomalous exponent Δ discussed above is just the leading divergence associated with scalar anomalous fields. We will explain below that there exists a full spectrum of anomalous exponents which are associated with the inner length, and they have to do with anomalous fields of different irreducible representation of the rotation group [7,8]. We will compute below analytically the whole spectrum of these exponents. For the model at hand this is an easy task, but it serves to demonstrate the rich scaling properties of hydrodynamic systems. This rich scaling structure has not been considered by the fluid mechanics community until now.

The structure of this paper is as follows. Section II is devoted to the derivation of the differential equations satisfied by the n-point time correlation functions and the n-point simultaneous correlation functions, as well as the equations for the two-point and four-point Green's functions. The derivation is based on the diagrammatic expansion of paper I, and the main objective of this section is to demonstrate that this technique yields the exact equations, and that the resulting equations contain aspects of the problem which are not perturbative. The results of the calculations of this section are identical to alternative derivations which are based on standard stochastic methods. Readers who are not interested in the method of derivation can start to read this paper from Sec. III, which begins with a catalog of the equations that are analyzed in the rest of the paper. Section III begins with the exact solution of the Green's function and the two-point correlation function. These solutions are important since they introduce the homogeneous solutions of the operator $\hat{\mathcal{B}}(R)$ which then appears importantly in all the solutions of the higher order quantities. Section II B deals with the exact solution of the two-point correlation function. This solution depends on the nature of the forcing. We are able to study in detail how the effects of nonisotropic forcing on the large scales decay in the inertial range as the scale of observation decreases. The exponents that govern the "law of isotropization" are important in determining the scaling behavior of correlation functions of anomalous fields whose presentation under the symmetry groups is different from scalars. Finally, in Sec. II C we discuss the four-point nonlinear Green's function which allows us the evaluation of the anomalous exponent Δ. Section IV is devoted to the calculation of $K(R)$, $J_4(R)$, and other correlations that expose nonscalar anomalous fields. The section is based on analyzing the equation for $\mathcal{F}_4$, and the strategy is to extract the leading divergence that is characterized by the anomalous exponent Δ. Section V presents a brief calculation of J_{2n} and Sec. VI collects all the results together in order to compute the scaling exponents of $S_{2n}(R)$. We show that the requirement of flux equilibrium in which the energy intake is balanced by diffusive dissipation leads to only two possibilities: one is simple scaling, and the other is anomalous scaling in agreement with Kraichnan's conjectures. Simple scaling is possible only if the dissipation field is not mixing, which seems a nonphysical condition. One has to stress here that there may be more than one scaling solution in the inertial range. Of course, the physical system is characterized by one scaling solution. The question is how to select this solution theoretically. We believe that this is done by demanding the existence of nonvanishing flux. Other demands, such as analyticity in the limit $\zeta_h \to 0$, may lead to other solutions which may be nonphysical. Section VII offers a summary of the paper.

II. DIAGRAMMATIC DERIVATION OF THE EQUATIONS FOR CORRELATION AND RESPONSE FUNCTIONS

In this section we demonstrate that for this model the exact resummation of the diagrammatic expansion for the various n-point correlation and response functions of the theory results in exact differential equations. In paper I the perturbative analysis gave rise to anomalous exponents that stemmed from order-by-order ultraviolet logarithmic divergences. We refer to such anomalies as "perturbative." Here we will find anomalous exponents that arise from the solution of the differential "fully resummed" operators. We refer to these as "nonperturbative" anomalies. The analysis is based on paper I, in which the first step is the Belinicher-L'vov (BL) transformation [4–6]. This is done by allowing the center of the coordinate system to move along the Lagrangian trajectory of a particular fluid point. The reference point is at position $\mathbf{r}_0$ at time t_0. The trajectory of this point with respect to $\mathbf{r}_0$ is

$$\boldsymbol{\rho}(\mathbf{r}_0, t_0|t) = \int_{t_0}^{t} dt'\, \mathbf{u}(\mathbf{r}_0 + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t'), t'). \tag{2.1}$$

Let us denote the transformed variables as

$$T(\mathbf{r}_0, t_0|\mathbf{r}, t) = \Theta(\mathbf{r} + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t), t), \tag{2.2}$$

$$\mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t) = \mathbf{u}(\mathbf{r} + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t), t), \tag{2.3}$$

$$\phi(\mathbf{r}_0, t_0|\mathbf{r}, t) = f(\mathbf{r} + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t), t). \tag{2.4}$$

In terms of these new variables the equation of motion (1.1) reads

$$\left[\partial_t - \kappa\nabla^2 - \mathbf{w}(\mathbf{r}_0, t_0|\mathbf{r}, t)\cdot\boldsymbol{\nabla}\right] T(\mathbf{r}_0, t_0|\mathbf{r}, t) = \phi(\mathbf{r}_0, t_0|\mathbf{r}, t), \tag{2.5}$$

where

$$\mathbf{w}(\mathbf{r}_0, t_0|\mathbf{r}, t) = \mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t) - \mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}_0, t). \tag{2.6}$$

The motivation to use these coordinates is to separate the trivial sweeping effects from the desired scaling behavior.

The sweeping causes infrared divergences in the diagrammatics that are avoided within this coordinate system. One should point out that for the passive scalar problem one can formulate the differential equations for the simultaneous correlations without recourse to this approach. For the Navier-Stokes problem the BL variables are indispensable in formulating a finite theory. Anticipating a possible correspondence between multiscaling behavior in the passive scalar and the Navier-Stokes problems, it is worthwhile to treat them in a similar fashion.

We now turn to the discussion of the statistical quantities that are defined in terms of these variables.

A. The two-point quantities

The two-point functions discussed here are the two-point correlation function

$$\mathcal{F}(\mathbf{r}_0|x_1,x_2) \equiv \langle T(x_0|x_1)T(x_0|x_2)\rangle \tag{2.7}$$

and the Green's function that is defined as

$$\mathcal{G}(\mathbf{r}_0|x_1,x_2) \equiv \left\langle \frac{\delta T(x_0|x_1)}{\delta \xi(x_0|x_2)} \right\rangle\Bigg|_{\xi\to 0} , \tag{2.8}$$

where we recall that the $(d+1)$-dimensional vector $x \equiv (\mathbf{r},t)$, and ξ is the Belinicher-L'vov transformation of a forcing that is added to the right-hand side of Eq. (2.5). The dependence on t_0 disappeared from the left-hand side of Eqs. (2.7) and (2.8) because one can prove [7] that all the average quantities are time translationally invariant.

In this section we present the derivation of the equations for $\mathcal{F}(\mathbf{r}_0|x_1,x_2)$ and $\mathcal{G}(\mathbf{r}_0|x_1,x_2)$. In paper I we derived the Dyson-Wyld equations for these two-point functions. For $t>0$ they read

$$(\partial_t - \kappa\nabla^2)\mathcal{G}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t) = \delta(\mathbf{r}_1-\mathbf{r}_2)\delta(t) + \int d\mathbf{r}'\Sigma(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}')\mathcal{G}(\mathbf{r}_0|\mathbf{r}',\mathbf{r}_2,t), \tag{2.9}$$

$$\mathcal{F}(\mathbf{r}_0\mathbf{r}_1,\mathbf{r}_2,t) = \int d\mathbf{r}'d\mathbf{r}''\int_0^\infty dt'\mathcal{G}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}',t+t') \times[\Phi_0(\mathbf{r}',\mathbf{r}'')+\Phi(\mathbf{r}',\mathbf{r}'')] \times\mathcal{G}(\mathbf{r}_0|\mathbf{r}_2,\mathbf{r}'',t'). \tag{2.10}$$

For negative times the Green's function is zero, and the correlation function is symmetric to inverting time $t\to -t$ and the coordinates $\mathbf{r}\to-\mathbf{r}$ together. In the Dyson equation (2.9) the mass operator $\Sigma(\mathbf{r}_0|\mathbf{r},\mathbf{r}')$ can be written explicitly

$$\Sigma(\mathbf{r}_0|\mathbf{r},\mathbf{r}') = \frac{\partial}{\partial r_i}\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t=0)H_{ij}(\mathbf{r}_0|\mathbf{r},\mathbf{r}')\frac{\partial}{\partial r'_j}, \tag{2.11}$$

whereas in the Wyld equation (2.10) the mass operator Φ takes the form

$$\Phi(\mathbf{r}_0|\mathbf{r},\mathbf{r}') = H_{ij}(\mathbf{r}_0|\mathbf{r},\mathbf{r}')\frac{\partial}{\partial r_i}\frac{\partial}{\partial r'_j}\mathcal{F}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t=0). \tag{2.12}$$

In (2.11) and (2.12)

$$H_{ij}(\mathbf{r}_0|\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty} dt\langle w_i(\mathbf{r}_0,t_0|\mathbf{r},t)w_j(\mathbf{r}_0,t_0|\mathbf{r}',0)\rangle. \tag{2.13}$$

In the case considered in this paper in which the velocity field is δ correlated in time the expression (2.13) can be related to the eddy diffusivity (1.3), expressed in terms of Eulerian correlations as

$$H_{ij}(\mathbf{r}_0|\mathbf{r},\mathbf{r}') = h_{ij}(\mathbf{r}-\mathbf{r}_0)+h_{ij}(\mathbf{r}'-\mathbf{r}_0)-h_{ij}(\mathbf{r}-\mathbf{r}'). \tag{2.14}$$

That the mass operators (2.11) and (2.12) have an explicit form corresponding to the one-loop diagram rather than an infinite series representation stems from the fact that the velocity field decorrelates on an infinitely short time scale. This simple form of the mass operators will be lost if we relax this fast decay of the velocity time correlation functions.

Equations (2.9) and (2.10) can be turned into differential equations for the two-point functions $\mathcal{F}$ and $\mathcal{G}$. Substituting (2.11) in (2.9) we find that

$$(\partial_t-\kappa\nabla^2)\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t) = \delta(\mathbf{r}-\mathbf{r}')\delta(t) + \int d\mathbf{r}''\frac{\partial}{\partial r_i}\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}'',0) \times H_{ij}(\mathbf{r}_0|\mathbf{r},\mathbf{r}'')\frac{\partial}{\partial r''_j} \times\mathcal{G}(\mathbf{r}_0|\mathbf{r}'',\mathbf{r}',t). \tag{2.15}$$

Remember that $\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t)$ is zero for negative times. Integrating (2.15) over time from $t=-\tau$ to $t=\tau$ and taking the limit $\tau\to 0$ one finds that

$$\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t=0^+) = \delta(\mathbf{r}-\mathbf{r}') . \tag{2.16}$$

We will choose symmetrical regularization at $t=0$ and by convention write

$$\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t=0) = \frac{1}{2}\delta(\mathbf{r}-\mathbf{r}'). \tag{2.17}$$

Using this evaluation the integration may be performed leading to

$$\left[\partial_t + \hat{\mathcal{D}}_1(\mathbf{r}-\mathbf{r}_0)\right]\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t) = \delta(\mathbf{r}-\mathbf{r}')\delta(t). \tag{2.18}$$

Here we introduced the generalized diffusion operator

$$\hat{\mathcal{D}}_1(\mathbf{r}) = -\kappa\nabla^2 + \hat{\mathcal{B}}(\mathbf{r}) , \tag{2.19}$$

where the operator $\hat{\mathcal{B}}$ is a key operator that appears repeatedly below. We will distinguish between an operator $\hat{\mathcal{B}}(\mathbf{r}_\alpha,\mathbf{r}_\beta)$ which acts on functions of two variables $\mathbf{r}_\alpha$ and

$\mathbf{r}_\beta$ and an operator $\hat{\mathcal{B}}(\mathbf{R})$ acting on functions of one variable $\mathbf{R}$:

$$\hat{\mathcal{B}}(\mathbf{r}_\alpha,\mathbf{r}_\beta) \equiv \hat{\mathcal{B}}_{\alpha,\beta} = h_{ij}(\mathbf{r}_\alpha - \mathbf{r}_\beta)\frac{\partial^2}{\partial r_{\alpha,i}\partial r_{\beta,j}} , \qquad (2.20)$$

$$\hat{\mathcal{B}}(\mathbf{R}) = -h_{ij}(R)\frac{\partial^2}{\partial R_i \partial R_j} . \qquad (2.21)$$

Clearly $\hat{\mathcal{B}}(\mathbf{r}_\alpha,\mathbf{r}_\beta)$ is equivalent to $\hat{\mathcal{B}}(\mathbf{R})$ on the class of functions depending on the difference $\mathbf{R} = \mathbf{r}_\alpha - \mathbf{r}_\beta$ only. In spherical coordinates the $\hat{\mathcal{B}}$ operator can be represented as the sum of two contributions:

$$\hat{\mathcal{B}}(\mathbf{R}) = -\hat{B}(R) + \frac{(\zeta_h + d - 1)}{(d-1)}\frac{h(R)}{R^2}\hat{L}^2 . \qquad (2.22)$$

Here $\hat{B}$ is the operator (1.9) and $\hat{L}$ is the angular momentum operator $-i\mathbf{R}\times\boldsymbol{\nabla}$ which depends only on the direction of $\mathbf{R}$.

For future purposes we also need the equation of the time-integrated Green's function $\mathcal{G}(\mathbf{r}-\mathbf{r}_0,\mathbf{r}'-\mathbf{r}_0)$, defined as

$$\mathcal{G}(\mathbf{r}-\mathbf{r}_0,\mathbf{r}'-\mathbf{r}_0) = \int dt \mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t). \qquad (2.23)$$

This function satisfies the equation

$$\hat{\mathcal{D}}_1(\mathbf{R})\mathcal{G}(\mathbf{R},\mathbf{R}') = \delta(\mathbf{R}-\mathbf{R}') , \qquad (2.24)$$

which follows from Eq. (2.18).

The equation of motion (2.24) allows us to introduce the inner scale of this model, denoted by η. By definition η is the scale for which the advective term $\hat{\mathcal{B}}$ is of the order of the dissipative term $\kappa\nabla^2$. Equating the two terms we get

$$\eta \simeq \left(\frac{\kappa}{H}\right)^{1/\zeta_h}\mathcal{L}. \qquad (2.25)$$

We proceed now to determine an equation of motion for the two-point correlator $\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t)$. It is clear from Eq. (2.18) that one may define an inverse operator for the Green's function $\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t)$ according to

$$\mathcal{G}_1^{-1}(\mathbf{r}-\mathbf{r}_0,t) \equiv \partial_t + \hat{\mathcal{D}}_1(\mathbf{r}-\mathbf{r}_0) . \qquad (2.26)$$

Note from Eq. (2.18) that the equation of motion for $\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t)$ only depends on the first coordinate $\mathbf{r}$, so that $\mathcal{G}_1^{-1} \equiv \mathcal{G}_1^{-1}(\mathbf{r}-\mathbf{r}_0,t)$. Operating with $\mathcal{G}^{-1}$ we may rewrite the Wyld equation (2.10) as

$$\left[\partial_t + \hat{\mathcal{D}}_1(\mathbf{r}_1-\mathbf{r}_0)\right]\left[-\partial_t + \hat{\mathcal{D}}_1(\mathbf{r}_2-\mathbf{r}_0)\right]\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t)$$
$$= \delta(t)\left[\Phi(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2) + \Phi_0(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2)\right], \qquad (2.27)$$

where we have used the fact that $\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t)$ is only a function of the time difference $t = t_1 - t_2$. We have also the "boundary" condition

$$\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t) \to 0 \quad \text{for} \quad t\to\pm\infty . \qquad (2.28)$$

Next we want to derive the differential equation satisfied by the simultaneous two-point correlator $\mathcal{F}(\mathbf{R})$:

$$\mathcal{F}(\mathbf{R}) = \mathcal{F}(\mathbf{r}_1,\mathbf{r}_2,t=0), \qquad (2.29)$$

where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$. The derivation is described in detail in Appendix A with the final result

$$\hat{\mathcal{D}}_2(\mathbf{R})\mathcal{F}(\mathbf{R}) = \Phi_0(\mathbf{R}), \qquad (2.30)$$

where generally the operator $\hat{\mathcal{D}}_2(\mathbf{r}_\alpha,\mathbf{r}_\beta)$ operates on two coordinates:

$$\hat{\mathcal{D}}_2(\mathbf{r}_\alpha,\mathbf{r}_\beta) = -\kappa[\nabla_\alpha^2 + \nabla_\beta^2] + \hat{\mathcal{B}}(\mathbf{r}_\alpha,\mathbf{r}_\beta) - \hat{\mathcal{H}}(\mathbf{r}_\alpha,\mathbf{r}_\beta) , \qquad (2.31)$$

and the operator $\hat{\mathcal{H}}$ is given by

$$\hat{\mathcal{H}}(\mathbf{r}_\alpha,\mathbf{r}_\beta) \equiv \left(h_{ij}(\mathbf{r}_\alpha)\frac{\partial}{\partial r_{\alpha 1}} + h_{ij}(\mathbf{r}_\beta)\frac{\partial}{\partial r_{\beta 1}}\right) \times \left(\frac{\partial}{\partial r_{\alpha 1}} + \frac{\partial}{\partial r_{\beta 1}}\right) . \qquad (2.32)$$

In the case where the operand is a function only of the difference $\mathbf{R} = \mathbf{r}_\alpha - \mathbf{r}_\beta$ the operator $\hat{\mathcal{H}}$ disappears and this equation reduces to

$$\hat{\mathcal{D}}_2(\mathbf{R}) \equiv -2\kappa\nabla^2 + \hat{\mathcal{B}}(\mathbf{R}) . \qquad (2.33)$$

B. The derivation of the differential equations for higher order correlations and response functions

In this section we present the equations for the four-point and higher order correlation functions and for the nonlinear Green's function (2.8). The solutions are deferred to the next section.

1. The four-point Green's function

In paper I we presented the diagrammatic series for the nonlinear Green's function $\mathcal{G}_2(\mathbf{r}_0|x_1,x_2,x_3,x_4)$. This quantity is defined as

$$\mathcal{G}_2(\mathbf{r}_0|x_1,x_2,x_3,x_4) \equiv \left\langle \frac{\delta T(x_0|x_1)}{\delta\xi(x_0|x_3)}\frac{\delta T(x_0|x_2)}{\delta\xi(x_0|x_4)}\right\rangle\Bigg|_{\xi\to 0} . \qquad (2.34)$$

The diagrammatic expansion of this quantity is an infinite series of ladder diagrams that can be resummed exactly. In Fig. 1 we recall the notation for the diagrammatic elements, and in Fig. 2 is reproduced the diagrammatic resummed equation for this function. In analytic form this equation reads

$$\mathcal{G}_2(0|x_1,x_2,x_3,x_4) = \mathcal{G}_2^{\mathrm{G}}(0|x_1,x_2,x_3,x_4) + \int d\mathbf{r}'d\mathbf{r}'' \int_{t_m}^{\infty} dt'\, \mathcal{G}_2^{\mathrm{G}}(0|x_1,x_2,\mathbf{r}',t',\mathbf{r}'',t') H_{ij}(\mathbf{r}',\mathbf{r}'') \frac{\partial}{\partial r'_i}\frac{\partial}{\partial r''_j} \mathcal{G}_2(0|\mathbf{r}',t',\mathbf{r}'',t',x_3,x_4), \quad (2.35)$$

where $t_m = \min\{t_1, t_2\}$ and

$$\mathcal{G}_2^{\mathrm{G}}(0|x_1,x_2,x_3,x_4) \equiv \mathcal{G}(0|x_1,x_3)\mathcal{G}(0|x_2,x_4). \quad (2.36)$$

We want now to derive a differential equation for this quantity. We can proceed as in the case of the correlation function by applying the product of the inverse operators $\tilde{\mathcal{G}}_1^{-1}$ of Eq. (2.26) to obtain a differential equation for $\mathcal{G}_2(0|x_1,x_2,x_3,x_4)$ as a function of three time differences. We may, however, make use of the structure of Eq. (2.35) to directly derive an equation in only one time difference. The availability of such a reduction follows from the δ-correlated velocity field in much the same way as the availability of a differential operator for $\mathcal{F}(0|\mathbf{r}_1,\mathbf{r}_2,t=0)$ as we saw before. We can choose at will to consider Eq. (2.35) for the times $t_1 = t_2 = t$ and $t_3 = t_4 = 0$ to derive a differential equation in one time difference for the quantity

$$\mathcal{G}_2(0|\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,t) \equiv \mathcal{G}_2(0|\mathbf{r}_1,t,\mathbf{r}_2,t,\mathbf{r}_3,0,\mathbf{r}_4,0). \quad (2.37)$$

The derivation closely resembles the derivation of the equation for the two-point correlator described in Appendix A. Applying the operator (2.27) to Eq. (2.35) with the above choice of times we get

$$[\partial_t + \hat{\mathcal{D}}_2(\mathbf{r}_1,\mathbf{r}_2) - \hat{\mathcal{H}}(\mathbf{r}_1,\mathbf{r}_2)]\mathcal{G}_2(0|\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,t) = \delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_2-\mathbf{r}_4)\delta(t)\,, \quad (2.38)$$

where $\hat{\mathcal{D}}_2(\mathbf{r}_1,\mathbf{r}_2)$ is the operator defined in Eq. (2.33) and $\hat{\mathcal{H}}$ is an operator defined in Appendix A that arises from the derivation of the equation for the two-point correlator.

For the time-integrated four-point Green's function

$$\mathcal{G}_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = \int dt \mathcal{G}_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,t) \quad (2.39)$$

one has the equation

$$\left[\hat{\mathcal{D}}_2(\mathbf{r}_1,\mathbf{r}_2) - \hat{\mathcal{H}}(\mathbf{r}_1,\mathbf{r}_2)\right]\mathcal{G}_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = \delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_2-\mathbf{r}_4)\,. \quad (2.40)$$

The nonlinear Green's function is the kernel of the response to some forcing at points $\mathbf{r}_3$ and $\mathbf{r}_4$, and we are interested in this response. In order to study this, we introduce a new function

$$\Psi(\mathbf{r}_1,\mathbf{r}_2) = \int d\mathbf{r}_3 d\mathbf{r}_4 \mathcal{G}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)A(\mathbf{r}_3,\mathbf{r}_4)\,, \quad (2.41)$$

where $A(\mathbf{r}_3,\mathbf{r}_4)$ is an arbitrary function. From Eq. (2.40) one may determine that

$$\left[\hat{\mathcal{D}}_2(\mathbf{r}_1,\mathbf{r}_2) + \hat{\mathcal{H}}\right]\Psi(\mathbf{r}_1,\mathbf{r}_2) = A(\mathbf{r}_1,\mathbf{r}_2). \quad (2.42)$$

If one chooses now to restrict $A(\mathbf{r}_1,\mathbf{r}_2)$ to the space of functions depending only upon the difference $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$, the equation simplifies (as in the derivation in Appendix A) to the form

$$\hat{\mathcal{D}}_2(\mathbf{R})\Psi(\mathbf{R}) = A(\mathbf{R}). \quad (2.43)$$

We will return to an analysis of this equation in Sec. III C.

2. The simultaneous higher order correlations

The equations of motion for the time-dependent $2n$th order correlation functions are derived in Appendix B. Here we will derive the equations for the simultaneous correlations. The simultaneous $2n$-point correlator is

$$\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n}) = \langle T(0|\mathbf{r}_1,t)T(0|\mathbf{r}_2,t),\ldots,T(0|\mathbf{r}_{2n},t)\rangle\,. \quad (2.44)$$

This same time quantity is identical in the Eulerian frame

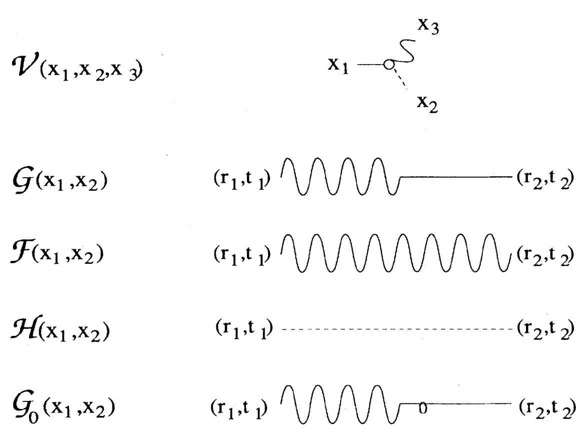

FIG. 1. Notation for the diagrammatic representation.

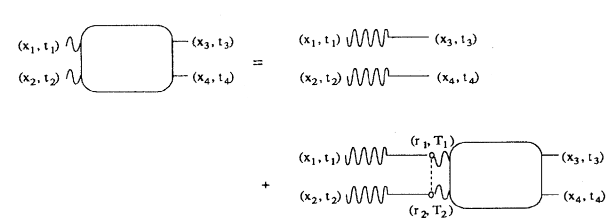

FIG. 2. Nonlinear Green's function $\mathcal{G}_2(\mathbf{r}_0|x_1,x_2,x_3,x_4)$.

of reference and in the transformed reference frame that we use. We can thus forget the $\mathbf{r}_0$ designation, and remember that in homogeneous systems the quantity is a function only of differences of its space arguments. Its time derivative is on the one hand zero and on the other hand

$$\frac{\partial}{\partial t}\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n})$$
$$=\sum_{\alpha=1}^{2n}\left\langle T(0|\mathbf{r}_1,t)\;\cdots\;\frac{\partial}{\partial t}T(0|\mathbf{r}_\alpha,t)\;\cdots\;T(0|\mathbf{r}_{2n},t)\right\rangle . \tag{2.45}$$

Using the equation of motion (2.5) we find that the right-hand side of (2.45) has three types of terms, one with $\mathbf{w}$ (advection), one with κ (dissipation), and the last proportional to the forcing. These terms are denoted as $A_{\rm adv}$, $A_{\rm dis}$, and $A_{\rm for}$ with

$$A_{\rm adv}+A_{\rm dis}+A_{\rm for}=0\,, \tag{2.46}$$

where

$$A_{\rm adv}=\sum_{\alpha=1}^{2n}\boldsymbol{\nabla}_\alpha\cdot\mathbf{F}_{w,2nT}(\mathbf{r}_\alpha,\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) \tag{2.47}$$

and

$$\mathbf{F}_{w,2nT}(\mathbf{r}_\alpha,\mathbf{r}_1\;,\ldots,\;\mathbf{r}_{2n})=\langle\mathbf{w}_\alpha T(0|\mathbf{r}_1,t)\cdots T(0|\mathbf{r}_{\alpha-1},t)T(0|\mathbf{r}_\alpha,t)\cdots T(0|\mathbf{r}_{2n},t)\rangle\,, \tag{2.48}$$

$$A_{\rm dis}=\kappa\sum_{\alpha=1}^{2n}\nabla_\alpha^2\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,\;\ldots,\mathbf{r}_{2n})\;. \tag{2.49}$$

Finally

$$A_{\rm for}=\sum_{\alpha=1}^{2n}\langle T(0|\mathbf{r}_1,t)\cdots T(0|\mathbf{r}_{\alpha-1},t)\phi(0|\mathbf{r}_\alpha,t)\cdots T(0|\mathbf{r}_{2n},t)\rangle\,. \tag{2.50}$$

The diagrammatic representation of $\mathbf{F}_{w,2nT}(\mathbf{r}_\alpha,\mathbf{r}_1,\ldots,\mathbf{r}_{2n})$ was discussed in paper I with a final result [cf. paper I, Eq. (5.7)] which in $\mathbf{r},t$ representation is

$$\mathbf{F}_{w,2nT}(\mathbf{r}_\alpha,\mathbf{r}_1,\ldots,\mathbf{r}_{2n})=\frac{1}{2}\sum_{\beta=1}^{2n}\mathbf{H}(0|\mathbf{r}_\alpha,\mathbf{r}_\beta)\cdot\boldsymbol{\nabla}_\beta\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n})\;. \tag{2.51}$$

Introducing this result into (2.48) and using (2.14) we find

$$A_{\rm adv}=\frac{1}{2}\sum_{\alpha,\beta=1}^{2n}[h_{ij}(\mathbf{r}_\alpha)+h_{ij}(\mathbf{r}_\beta)-h_{ij}(\mathbf{r}_\alpha-\mathbf{r}_\beta)]\frac{\partial^2}{\partial r_{\alpha i}\partial r_{\beta j}}\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n}). \tag{2.52}$$

This form can be rewritten equivalently as

$$A_{\rm adv}=\left\{-\sum_{\alpha>\beta=1}^{2n}\hat{\mathcal{B}}_{\alpha\beta}+\left[\sum_{\alpha=1}^{2n}h_{ij}(\mathbf{r}_\alpha)\frac{\partial}{\partial r_{\alpha i}}\right]\left[\sum_{\beta=1}^{2n}\frac{\partial}{\partial r_{\beta j}}\right]\right\}\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n}). \tag{2.53}$$

Finally we use the fact that $\mathcal{F}_{2n}$ is a function of differences of its spatial arguments to see that the second operator in the first line of the right-hand side gives zero and the line can be omitted. Thus

$$A_{\rm adv}=-\sum_{\alpha>\beta=1}^{2n}\hat{\mathcal{B}}_{\alpha\beta}\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n}). \tag{2.54}$$

Consider next the forcing term. Using the fact that for a Gaussian force

$$\langle T(0|\mathbf{r}_1,t)\cdots T(0|\mathbf{r}_{\alpha-1},t)\phi(0|\mathbf{r}_\alpha,t)\cdots T(0|\mathbf{r}_{2n},t)\rangle$$
$$=\int\left\langle\frac{\delta T(0|\mathbf{r}_1,t)\cdots T(0|\mathbf{r}_{\alpha-1},t)\cdots T(0|\mathbf{r}_{2n},t)}{\delta\phi(0|\mathbf{r}'_\alpha,t')}\right\rangle\Phi_0(\mathbf{r}_\alpha-\mathbf{r}'_\alpha)d\mathbf{r}'_\alpha, \tag{2.55}$$

where we used (1.2) and the fact that we deal with simultaneous correlations. The functional derivative in the integrand is a zero time response, which as usual is computed in the noninteracting limit:

$$\left\langle\frac{\delta T(0|\mathbf{r}_1,t)\cdots T(0|\mathbf{r}_{\alpha-1},t)\cdots T(0|\mathbf{r}_{2n},t)}{\delta\phi(0|\mathbf{r}'_\alpha,t')}\right\rangle=\sum_\beta G^0(\mathbf{r}_\alpha-\mathbf{r}_\beta,t=0)\mathcal{F}_{2n-2}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n}) \tag{2.56}$$

where in $\mathcal{F}_{2n-2}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n})$ the two arguments $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$ are missing. Substituting (2.56) in (2.55) and the result in (2.50), we find

$$A_{\rm for} = \sum_{\alpha>\beta} \Phi_0(\mathbf{r}_\alpha - \mathbf{r}_\beta)\mathcal{F}_{2n-2}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n}). \tag{2.57}$$

Substituting all these results in (2.46) yields

$$\left[-\kappa \sum_\alpha \nabla_\alpha^2 + \sum_{\alpha>\beta}^{2n} \hat{\mathcal{B}}_{\alpha\beta}\right] \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n}) = M(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n}) \ , \tag{2.58}$$

where we define

$$M(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}) = \sum_{\alpha>\beta} \Phi_0(\mathbf{r}_\alpha - \mathbf{r}_\beta)\mathcal{F}_{2n-2}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}). \tag{2.59}$$

This equation, which is our final equation for the simultaneous correlation function, is again identical with Kraichnan's.

3. Equation for the irreducible correlation $\mathcal{F}_4^c$

The irreducible (cumulant) part of the correlation functions is defined as

$$\mathcal{F}_{2n}^c = \mathcal{F}_{2n} - \mathcal{F}_{2n}^G. \tag{2.60}$$

The equation for the simultaneous $\mathcal{F}_4^c$ follows from specializing Eq. (2.58) to the case $2n = 4$, using Eq. (2.30) for $\mathcal{F}_2$,

$$\left[-\sum_\alpha^4 \kappa \nabla_\alpha^2 + \sum_{\alpha>\beta} \hat{\mathcal{B}}_{\alpha\beta}\right] \mathcal{F}_4^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = -M(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4), \tag{2.61}$$

where explicitly,

$$\begin{aligned} M(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) &= -\frac{1}{2} \sum_{\substack{(\alpha_i)\\ \text{perm}(1234)}} \hat{\mathcal{B}}(\mathbf{r}_{alpha1} - \mathbf{r}_{\alpha 3)}\mathcal{F}(\mathbf{r}_1, \mathbf{r}_2)\mathcal{F}(\mathbf{r}_3, \mathbf{r}_4) \\ &\equiv -\Big[(\hat{\mathcal{B}}_{14} + \hat{\mathcal{B}}_{23} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{24})\mathcal{F}_{12}\mathcal{F}_{34} + (\hat{\mathcal{B}}_{14} + \hat{\mathcal{B}}_{23} + \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34})\mathcal{F}_{13}\mathcal{F}_{24} \\ &\quad + (\hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{24} + \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34})\mathcal{F}_{14}\mathcal{F}_{23}\Big] \ . \end{aligned} \tag{2.62}$$

III. ANALYSIS OF THE GREEN'S FUNCTIONS AND THE TWO-POINT CORRELATION

In this section we begin to discuss the exact solution for the various quantities in this model when all the separation distances are in the inertial interval. For the sake of clarity we catalog here all the equations that we are going to solve below, neglecting the diffusive terms which appear in the full equations. These terms will be shown *a posteriori* to be much smaller than the operators retained. Note that in the balance equations (6.5) we retain the diffusive term. The reason for this is that J_{2n} goes to a finite limit when $\kappa \to 0$ since it balances the "energy" intake at any value of κ. The equations are the following, for (1) the time-integrated Green's function, cf. (2.23):

$$\hat{\mathcal{B}}(\mathbf{R})\mathcal{G}(\mathbf{R}, \mathbf{R}') = \delta(\mathbf{R} - \mathbf{R}') \ , \tag{3.1}$$

which follows from (2.24), (2) the two-point simultaneous correlation functions,

$$\hat{\mathcal{B}}(\mathbf{R})\mathcal{F}(\mathbf{R}) = \Phi_0(\mathbf{R}) \ , \tag{3.2}$$

which follow from (2.30), (3) the function Ψ introduced in Sec. II B,

$$\hat{\mathcal{B}}(\mathbf{R})\Psi(\mathbf{R}) = A(\mathbf{R}), \tag{3.3}$$

which is the $\kappa \to 0$ limit of Eq. (2.43), and (4) the cumulant of the four-point simultaneous correlation function

$$\begin{aligned} \Big[\hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{14} + \hat{\mathcal{B}}_{23} + \hat{\mathcal{B}}_{24} + \hat{\mathcal{B}}_{34}\Big]\mathcal{F}^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = &-\Big[\hat{\mathcal{B}}_{14} + \hat{\mathcal{B}}_{23} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{24}\Big]\mathcal{F}(\mathbf{r}_1 - \mathbf{r}_2)\mathcal{F}(\mathbf{r}_3 - \mathbf{r}_4) \\ &-\Big[\hat{\mathcal{B}}_{14} + \hat{\mathcal{B}}_{23} + \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34}\Big]\mathcal{F}(\mathbf{r}_1 - \mathbf{r}_3)\mathcal{F}(\mathbf{r}_2 - \mathbf{r}_4) \\ &-\Big[\hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{24} + \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34}\Big]\mathcal{F}(\mathbf{r}_1 - \mathbf{r}_4)\mathcal{F}(\mathbf{r}_2 - \mathbf{r}_3) \ , \end{aligned} \tag{3.4}$$

which is the $\kappa \to 0$ limit of Eq. (2.60).

It is noteworthy that the same operator $\hat{\mathcal{B}}$, defined in (2.20), (2.22) appears in all these equations. Below we consider the scaling properties that arise from the eigenfunctions of $\hat{\mathcal{B}}$ with zero eigenvalue which are the solutions of the homogeneous equation (3.5).

A. Solutions of the basic homogeneous equation

For two distinct coordinates $\mathbf{R} \neq \mathbf{R}'$ the Green's function $\mathcal{G}(\mathbf{R}, \mathbf{R}')$ satisfies the homogeneous part of the equation (3.1)

$$\hat{\mathcal{B}}(\mathbf{R})\mathcal{G}(\mathbf{R},\mathbf{R}')=0\ , \tag{3.5}$$

which will be referred to hereafter as the "basic homogeneous equation." In light of the representation of Eq. (2.22) of the $\hat{\mathcal{B}}$ operator via the angular momentum operator we represent the solution of (3.5) in three-dimensional space as an expansion over spherical harmonics

$$\mathcal{G}(\mathbf{R},\mathbf{R}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} b_{ll',mm'}g_{lm}(\mathbf{R})g_{l'm'}(\mathbf{R}'), \tag{3.6}$$

where $b_{ll',mm'}$ are coefficients and

$$g_{lm}(\mathbf{R})=R^{\beta_l}Y_{lm}(\theta,\phi)\ . \tag{3.7}$$

The spherical harmonics $Y_{lm}(\theta,\phi)$ are the eigenfunctions of the angular momentum operator:

$$\hat{L}^2 Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi) \tag{3.8}$$

and θ and ϕ are the polar and azimuthal angles of $\mathbf{R}$. Substituting the expansion (3.6) into Eq. (3.5) we find the relationship

$$\beta_l(\beta_l+1+\zeta_h)-l(l+1)(1+\zeta_h/2)=0\ . \tag{3.9}$$

Of the two solutions of this quadratic equation we must select the non-negative branch since the negative branch is unphysical:

$$\beta_l=\frac{1}{2}\left[-\zeta_h-1+\sqrt{(\zeta_h+1)^2+2l(l+1)(2+\zeta_h)}\right]\ ,\qquad (d=3)\ . \tag{3.10}$$

In two dimensions one has a similar representation with the difference that instead of the spherical harmonics we have the eigenfunctions $\exp(il\phi)$ and instead of the eigenvalues $l(l+1)$ we have the eigenvalues l^2. The result for the exponents β_l in two dimensions is

$$\beta_l=\frac{1}{2}\left[-\zeta_h+\sqrt{\zeta_h^2+4l^2(1+\zeta_h)}\right]\qquad (d=2)\ . \tag{3.11}$$

Multipole expansions of the type (3.6) and the scaling exponents β_l play an important role in all our development below. We will see that the basic homogeneous equation reappears in various guises, and in each of them the coefficients $b_{ll',mm'}$ are determined by the symmetry of the particular object involved, the boundary conditions, etc. For example, in this case we know that the Green's function $\mathcal{G}(\mathbf{R},\mathbf{R}')$ is symmetric in $\mathbf{R}$ and $\mathbf{R}'$. Accordingly $b_{ll',mm'}=b_{l'l,m'm}$. If the solution depends on R, R', and the angle between $\mathbf{R}$ and $\mathbf{R}'$, then $b_{ll',mm'}\neq 0$ only for $l=l'$, $m+m'=0$.

B. Exact solution of the two-point correlation function

1. *Isotropic forcing*

The solution of the two-point correlation function depends on the nature of the external forcing $f(\mathbf{r},t)$. It is customary to take $f(\mathbf{r},t)$ to be Gaussian and statistically homogeneous in space and time. The properties of the correlation function $\Phi_0(\mathbf{r}-\mathbf{r}')$ were not determined up to now. Since we are interested in the universal scaling properties of the scalar field we want to choose the forcing such that it has only large scale components. Otherwise the scaling exponent of the two-point correlation may be colored by the functional dependence of the forcing on r. On the other hand, the forcing may be isotropic or nonisotropic. In this section we will deal with the isotropic case, and the nonistropic case will be treated in the next section.

The properties of $f(\mathbf{r},t)$ are best stated in $\mathbf{k}$ space: it is concentrated in the small k region, i.e., $k\leq 1/L$, and it decays quickly to zero for $k\gg 1/L$. In $\mathbf{r}$ space this means that $\langle f(\mathbf{r},t)f(\mathbf{r}+\mathbf{R},t)\rangle$ is constant for $\mathbf{R}\ll L$:

$$\langle f(\mathbf{r},t)f(\mathbf{r}+\mathbf{R},t)\rangle=\Phi_0\qquad (R\ll L)\ . \tag{3.12}$$

Using this form of the forcing correlation function we realize that the inhomogeneous term on the right-hand side of Eq. (2.30) is translationally and rotationally invariant. Accordingly we can seek solutions that have the same symmetry, solving the equation

$$\hat{B}(R)\mathcal{F}(R)=\Phi_0\ . \tag{3.13}$$

The solution of the inhomogeneous equation is

$$\mathcal{F}_{\rm inh}(R)=C_{\rm inh}R^{\zeta_2}\ , \tag{3.14}$$

with $C_{\rm inh}=-\Phi_0 L^{\zeta_h}/Hd\zeta_2$ and

$$\zeta_2=2-\zeta_h\ . \tag{3.15}$$

Of course, the inhomogeneous solution has to be supplemented with the solutions of the homogeneous equation in order to match the boundary conditions. There are two homogeneous solutions: one is a constant which must be taken as $\mathcal{F}(\eta)$, and the other is $\mathcal{F}(R)\propto R^{-\zeta_h}$. We know, however, that the theory for the correlation function itself (without taking derivatives) converges in the ultraviolet regime for $\zeta_h<2$, and see, for example, paper I. Therefore we have a boundary condition $\mathcal{F}(\eta)<\infty$. In the limit $\eta\to 0$ this boundary condition rules out a divergent solution. One should note that the constant homogeneous solution belongs to the family of exponents β_l with $l=0$ as one would expect. The solution that is ruled out is indeed the $l=0$ member of the forbidden branch of solutions of the quadratic equation (3.9).

2. *The law of isotropization*

When the forcing is anisotropic on the large scales we need to apply the full operator $\hat{\mathcal{B}}(\mathbf{R})$. In the regime $R\ll$

L the forcing is again a constant which is independent of the angles. However, for $R \simeq L$ we expect anisotropic forcing, which leads to anisotropic tails for every value of R. We can find the law of isotropization of $\mathcal{F}(\mathbf{R})$ for R small by solving the equation that involves now also the nonisotropic part of the operator $\hat{\mathcal{B}}(\mathbf{R})$:

$$\hat{\mathcal{B}}(\mathbf{R})\mathcal{F}(\mathbf{R}) = \Phi_0 \ , \quad R \ll L. \tag{3.16}$$

In writing this equation we take there to be no anisotropy of the forcing on scales much smaller than L. However, the operator $\mathcal{B}$ allows anisotropic solutions the amplitude of which can be determined by matching the solution at inertial scales with the boundary conditions at $R \simeq L$. The inhomogeneous solutions are the same as before, and in addition we invoke the complete set of homogeneous solutions $g_{lm}(\mathbf{R})$, Eq. (3.7). The solution in three dimensions is written as

$$\mathcal{F}(\mathbf{R}) = \mathcal{F}(0) + R^{\zeta_2}\left[C_{in} + \sum_{l=2}^{\infty}\sum_{m=-l}^{l} a_{lm}\left(\frac{R}{L}\right)^{\beta_l-\zeta_2} \times Y_{lm}(\theta,\phi)\right] . \tag{3.17}$$

Due to inversion symmetry the sum over l contains only positive even l. To understand this result we note that all values of β_l are larger than ζ_2. In particular, β_2 (for both $d = 3$ and $d = 2$) is

$$\beta_2 = \frac{1}{2}[\zeta_2 - d + \sqrt{(\zeta_2 + d)^2 + 24\zeta_h}] \ . \tag{3.18}$$

For positive ζ_h, β_2 is larger than ζ_2. For larger l, β_l is even larger, and dependent on dimension. Thus all the anisotropic terms decay when $R \ll L$. Note that the coefficients a_{lm} are nonuniversal and should be found by matching at $R \sim L$. The law of decay is, however, universal.

3. *Anisotropic structure functions*

The solution (3.17) suggests the introduction of anisotropic structure functions via the definition

$$S_{2,lm}(R) = \int d\theta d\phi Y_{lm}(\theta,\phi) S_2(\mathbf{R}) \ . \tag{3.19}$$

Here l should be even due to the symmetry with respect to the inversion of $\mathbf{R}$. These anisotropic structure functions display "clean" scaling behavior with the exponent β_l:

$$S_{2,lm}(R) \sim S_2(L)\left(\frac{R}{L}\right)^{\beta_l} \ . \tag{3.20}$$

We will see that the same scaling exponents feature prominently below.

C. The nonlinear Green's function and the anomalous exponent Δ

In this section we discuss the nonlinear Green's function and the anomalous exponent Δ which is associated with it, cf. (1.13). To this aim we return to the equation for Ψ, Eq. (3.3). We realize that the solutions of this equation are identical to those discussed already in the context of the two-point correlator. For a constant function A we can write the solution

$$\Psi(\mathbf{R},\mathbf{R}') = \Psi(0) + C|\mathbf{R} - \mathbf{R}'|^{\zeta_2} \ . \tag{3.21}$$

Taking the derivative with respect to $\mathbf{R}$ and $\mathbf{R}'$ as required by Eq. (1.13) we find

$$(\boldsymbol{\nabla}_1 \cdot \boldsymbol{\nabla}_2)\Psi(\mathbf{R},\mathbf{R}') \sim |\mathbf{R} - \mathbf{R}'|^{-\zeta_h} \ . \tag{3.22}$$

Comparing now with the definition of the anomalous exponent Δ in Eq. (1.13) we see that in the limit $|\mathbf{R}-\mathbf{R}'| \to \eta$ this quantity diverges as $1/\eta^{\zeta_h}$ and

$$\Delta = \zeta_h. \tag{3.23}$$

As explained before, this value of Δ is the critical value Δ_c, and the implications of this finding are explored below. We turn now to the appearance of the anomalous exponent in the four-point correlator and related quantities.

IV. THE FOUR-POINT CORRELATOR AND RELATED QUANTITIES: $K(R)$, $J_4(R)$, AND $L(R)$

In this section we present the analysis of the four-point correlator $\mathcal{F}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ and of the quantities that are related to it, i.e., the correlation of dissipation fluctuation $K(R)$, Eq. (1.15) and the right-hand side of the balance equation $J_4(R)$, Eq. (1.10). These last quantities are not exactly four-point correlations, but they are obtained as a limit of $\mathcal{F}_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$. The simpler limit is $K(R)$, which is a centered correlation function and is therefore related to $\mathcal{F}_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$:

$$K(R) = \kappa^2 \lim_{r_{12},r_{34}\to 0}\lim_{r_{13}\to R}(\boldsymbol{\nabla}_1 \cdot \boldsymbol{\nabla}_2)(\boldsymbol{\nabla}_3 \cdot \boldsymbol{\nabla}_4) \times\Big[\mathcal{F}_4^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) + 2\mathcal{F}_2(\mathbf{r}_1,\mathbf{r}_3)\mathcal{F}_2(\mathbf{r}_2,\mathbf{r}_4)\Big] \ . \tag{4.1}$$

Since we found that the second derivative is singular, cf. Eq. (3.22), the above limit must be carefully examined.

The quantity $J_4(R)$ has a Gaussian decomposition $J_4^G(R)$ and a cumulant part $J_4^c(R)$. The Gaussian decomposition is trivially computed as

$$J_4^G(R) = 24\bar{\epsilon}S_2(R) - 6\kappa\nabla^2 S_2^2(R) \ , \tag{4.2}$$

where

$$\bar{\epsilon} = \kappa\left\langle|\boldsymbol{\nabla}\Theta(\mathbf{r})|^2\right\rangle \ . \tag{4.3}$$

Note that $\bar{\epsilon}$ remains finite in the limit $\kappa \to 0$. Accordingly the first term in Eq. (4.2) remains finite whereas the second vanishes in this limit. Thus the second term in Eq. (4.2) is smaller than the first for R in the inertial interval and can be neglected. The cumulant part of J_4 is written as

$$J_4^c(R) = 24\kappa\langle\langle|\nabla_1\Theta(\mathbf{r}_1)|^2\Theta^2(\mathbf{r}_2)\rangle\rangle -48\kappa\langle\langle|\nabla_1\Theta(\mathbf{r}_1)|^2\Theta(\mathbf{r}_1)\Theta(\mathbf{r}_2)\rangle\rangle +24\kappa\langle\langle|\nabla_1\Theta(\mathbf{r}_1)|^2\Theta^2(\mathbf{r}_1)\rangle\rangle -2\kappa\langle\langle\nabla^2 S_4(R)\rangle\rangle , \quad (4.4)$$

where double brackets denote the cumulant part and $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$. We will see that the sum of the first three terms remains finite in the limit $\kappa \to 0$, whereas the last goes to zero as in Eq. (4.2). For large Pe we can also neglect this last term throughout the inertial interval. The sum of the remaining terms vanishes exactly for $R = 0$. This means that all the R-independent constants cancel in the combination of these terms. We will also see later that there is no constant contribution to the leading order terms and therefore we will also neglect the third term. This allows us to write $J_4^c(R)$ as the sum of two terms that we denote as $J_{4,1}^c(R)$ and $J_{4,2}^c(R)$, which may be considered as the following limits of $\mathcal{F}^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$:

$$J_{4,1}^c(R) = 24\kappa \lim_{r_{12},r_{34}\to 0} \lim_{r_{13}\to R} (\nabla_1 \cdot \nabla_2)\mathcal{F}_4^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) , \quad (4.5)$$

$$J_{4,2}^c(R) = -48\kappa \lim_{r_{12},r_{13}\to 0} \lim_{r_{14}\to R} (\nabla_1 \cdot \nabla_2) \times\mathcal{F}_4^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) . \quad (4.6)$$

Note that in Eq. (4.5) we have two pairs of coalescing points, i.e., (1, 2) and (3, 4), which are separated by a large distance R. In Eq. (4.6) we have three coalescing points, i.e., 1, 2, 3, and this group is separated from point 4 by R. One should also note that J_4 is obtained from the full $\mathcal{F}_4$ and not from the cumulant part.

In the next sections we are going to make strong use of the divergence with respect to small distances. Our strategy will be to expose the leading exponent in the divergence with respect to small separations and to compute it exactly. Then we will find the exponent of the dependence on R by power counting, knowing the overall scaling exponent ζ_4 of $\mathcal{F}_4^c$. In other words, our basic assumption is that the correlator $\mathcal{F}_4^c$ is a homogeneous function of its arguments as long as all separations are in the inertial range:

$$\mathcal{F}_4^c(\lambda\mathbf{r}_1, \lambda\mathbf{r}_2, \lambda\mathbf{r}_3, \lambda\mathbf{r}_4) = \lambda^{\zeta_4}\mathcal{F}_4^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) , \quad (4.7)$$

where ζ_4 is the unknown scaling exponent that characterizes the structure function $S_4(R)$. We will not make any assumption about the numerical value of ζ_4.

A. Two coalescing pairs of points

1. The effective equation for $\mathcal{F}_4^c$

Consider Eq. (2.61) in the limit $r_{12}, r_{34} \ll R$, but all separations in the inertial interval. This allows us, first, to neglect the diffusion terms. Furthermore, one can now expand the operators using the relative smallness of r_{12} and r_{34} and see that $\hat{\mathcal{B}}_{12} \propto r_{12}^{-\zeta_2}$ and $\hat{\mathcal{B}}_{34} \propto r_{34}^{-\zeta_2}$. Each of the other binary operators will have a stronger divergence. For example, $\hat{\mathcal{B}}_{13}$ can be evaluated as $R^{\zeta_h}/(r_{12}r_{34})$. However, *in the sum* of the operators acting between coordinates separated by R, i.e., $\hat{\mathcal{B}}_{13}+\hat{\mathcal{B}}_{14}+\hat{\mathcal{B}}_{23}+\hat{\mathcal{B}}_{24}$, there appear cancellations in the two leading orders. What remains is proportional to $R^{-\zeta_2}$ and therefore much smaller. For the derivation of this see Appendix C. This suggests rewriting the equation in the form

$$\left[\hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34} + \sum \hat{\mathcal{B}}\right]\mathcal{F}_4^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = M(\mathbf{r}_{12}, \mathbf{r}_{34}, R), \quad (4.8)$$

where the sum on the left-hand side is on all the four $\hat{\mathcal{B}}$ operators other than the ones explicitly displayed. The calculation of $M(\mathbf{r}_{12}, \mathbf{r}_{34}, R)$ [Eq. (2.62)] is elementary since we know everything explicitly. The result of the calculation is, to first order,

$$M(\mathbf{r}_{12}, \mathbf{r}_{34}, R) = R^{\zeta_2}\left[A_0 + A_1\left(\frac{r_{12}^{\zeta_h} + r_{34}^{\zeta_h}}{R^{\zeta_h}}\right) + A_2\left(\frac{r_{12}r_{34}}{R^2}\right)^{\zeta_2}\right] , \quad (4.9)$$

where A_0, A_1, and A_2 are some angle-dependent dimensionless functions. The last term in the square brackets comes from the first group of terms in the right-hand side of Eq. (2.62). Indeed, the product of two $\mathcal{F}_2$'s is proportional to $(r_{12}r_{34})^{\zeta_2}$. The naive evaluation of the sum of $\hat{\mathcal{B}}$'s is $R^{\zeta_h}/r_{12}r_{34}$. However, the cancellations of Appendix C result in the replacement of $r_{12}r_{34}$ by R^2. The sum of $\hat{\mathcal{B}}$'s becomes proportional to $R^{-\zeta_2}$. The leading contribution in the last two groups of terms is regular in r_{12} and r_{34} and therefore can be evaluated as R^{ζ_2}. This is the first term in the square brackets. The second term in the square brackets is contributed by the sum of $\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}$ in the second and third groups of terms. Higher order terms exist but they are proportional to the second power of the small distances.

To discuss Eq. (4.8) further we note that in a space homogeneous situation $\mathcal{F}_4^c$ is a function of six differences in three dimensions and five differences in two dimensions. In light of our strategy it is convenient to choose the variables as $\mathbf{R}$, $\mathbf{r}_{12}$, and $\mathbf{r}_{34}$ where $\mathbf{R} = [\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_4]/2$. We will fix the axis of the coordinate system along $\mathbf{R}$, so that we are left only with a dependence on R. In doing so, we are using seven rather than the needed number of variables, but we will see that the dependence on the extra angle variable disappears. We can now group the right-hand side of Eq. (4.8) together with $\sum \hat{\mathcal{B}}\mathcal{F}_4^c$ into a new function, say, $E(\mathbf{r}_{12}, \mathbf{r}_{34}, R)$,

$$E(\mathbf{r}_{12}, \mathbf{r}_{34}, R) = M(\mathbf{r}_{12}, \mathbf{r}_{34}, R) - \sum \hat{\mathcal{B}}\mathcal{F}_4^c . \quad (4.10)$$

In the limits $r_{12}, r_{34} \ll R$ we can expand $E(\mathbf{r}_{12}, \mathbf{r}_{34}, R)$ in orders of r_{12} and r_{34},

$$E(\mathbf{r}_{12}, \mathbf{r}_{34}, R) = E^{(0)}(R) + E^{(1)}(\mathbf{r}_{12}, \mathbf{r}_{34}, R) + \cdots , \quad (4.11)$$

where

$$E^{(0)}(R) = \lim_{r_{12},r_{34}\to 0} E(\mathbf{r}_{12}, \mathbf{r}_{34}, R) . \quad (4.12)$$

The equation that we need to analyze takes on the form

$$\left[\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}\right]\mathcal{F}_4^c(\mathbf{r}_{12},\mathbf{r}_{34},R)$$

$$=E^{(0)}(R)+E^{(1)}(\mathbf{r}_{12},\mathbf{r}_{34},R)+\cdots \quad (4.13)$$

and the explicit expression for $E^{(1)}(\mathbf{r}_{12},\mathbf{r}_{34},R)$ will be presented below.

2. *Solutions*

To leading order we have the effective equation

$$\left[\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}\right]\mathcal{F}_4^c(\mathbf{r}_{12},\mathbf{r}_{34},R)=E^{(0)}(R)\ . \quad (4.14)$$

Our aim in the following analysis is to determine the solutions order by order in the small separation distances. These all contain positive powers but when substituted into the equations for $K(R)$ and J_4^c, Eqs. (4.1), (4.5), (4.6), the derivatives may produce negative powers which lead to singularities as the small separations tend to zero. Notice that $K(R)$ contains products of derivatives of r_{12} and r_{34} so that any solution contributing to $K(R)$ must contain products of r_{12} and r_{34}.

The leading inhomogeneous solution of Eq. (4.14) is found by inspection:

$$\mathcal{F}_{4,\mathrm{inh}}^{(1)}(\mathbf{r}_{12},\mathbf{r}_{34},R)=C_1E^{(0)}(R)(r_{12}^{\zeta_2}+r_{34}^{\zeta_2})\ , \quad (4.15)$$

where C_1 is a dimensionless constant. Using the overall scaling exponent ζ_4 that was introduced in Eq. (4.7) we can rewrite this solution as

$$\mathcal{F}_{4,\mathrm{inh}}^{(1)}(\mathbf{r}_{12},\mathbf{r}_{34},R)\sim S_4(R)\frac{r_{12}^{\zeta_2}+r_{34}^{\zeta_2}}{R^{\zeta_2}}\ . \quad (4.16)$$

Note that this solution is not exact; there are terms generated by the derivatives with respect to R, but as we are focusing on the dependence on the small distances, and the extra terms are of higher order in these, they may be safely neglected. This inhomogeneous solution will be important in the calculation of $J_4(R)$. It does not contribute, however, to the evaluation of $K(R)$: recall that this quantity contains a product of derivatives over r_{12} and r_{34} and so we need to continue until we find the next order solution containing *products* of r_{12} and r_{34}. There are two next order terms on the right-hand side of Eq. (4.13) which we denote $E_a^{(1)}$ and $E_b^{(1)}$. $E_a^{(1)}$ corresponds to the A_1 term in Eq. (4.9). This term is produced by an inhomogeneous solution which is

$$\mathcal{F}_{4,\mathrm{inh}}^{(2a)}(r_{12},r_{34},R)=\Phi_a(R)\frac{r_{12}^2+r_{34}^2}{R^2}\ , \quad (4.17)$$

where $\Phi_a(R)$ is some function of R. $E_b^{(1)}$ appears as a result of the operation of $\sum\hat{\mathcal{B}}$ on the first order inhomogeneous solution (4.16):

$$E_b^{(1)}(r_{12},r_{34},R)=E_1(R)\frac{r_{12}^{\zeta_2}+r_{34}^{\zeta_2}}{R^{\zeta_2}}\ . \quad (4.18)$$

Substituting (4.18) in Eq. (4.13) one infers the respective inhomogeneous solutions

$$\mathcal{F}_{4,\mathrm{inh}}^{(2b)}(\mathbf{r}_{12},\mathbf{r}_{34},R)=\Phi_b(R)\left(\frac{r_{12}r_{34}}{R^2}\right)^{\zeta_2}\ , \quad (4.19)$$

$$\mathcal{F}_{4,\mathrm{inh}}^{(2c)}(\mathbf{r}_{12},\mathbf{r}_{34},R)=\Phi_c(R)\frac{r_{12}^{2\zeta_2}+r_{34}^{2\zeta_2}}{R^{2\zeta_2}}, \quad (4.20)$$

where again $\Phi_b(R)$ is a determinable function. We use again Eq. (4.7) to rewrite these solutions in the form

$$\mathcal{F}_{4,\mathrm{inh}}^{(2a)}(\mathbf{r}_{12},\mathbf{r}_{34},R)\sim S_4(R)\frac{r_{12}^2+r_{34}^2}{R^2}\ , \quad (4.21)$$

$$\mathcal{F}_{4,\mathrm{inh}}^{(2b)}(\mathbf{r}_{12},\mathbf{r}_{34},R)\sim S_4(R)\left(\frac{r_{12}r_{34}}{R^2}\right)^{\zeta_2}\ , \quad (4.22)$$

$$\mathcal{F}_{4,\mathrm{inh}}^{(2c)}(\mathbf{r}_{12},\mathbf{r}_{34},R)\sim S_4(R)\frac{r_{12}^{2\zeta_2}+r_{34}^{2\zeta_2}}{R^{2\zeta_2}}\ . \quad (4.23)$$

The solutions so far obtained will be shown to give the leading order contributions to $J_4(R)$ and $K(R)$, and therefore these orders of the inhomogeneous solution will suffice for our analysis. The inhomogeneous solutions producing the A_2 term in Eq. (4.9) are of higher order than those we have already obtained.

In addition to the inhomogeneous solutions all the homogeneous solutions found below (3.6) are available to us, since the homogeneous solutions of $\hat{\mathcal{B}}_{12}$ and of $\hat{\mathcal{B}}_{34}$ are also homogeneous solutions of $\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}$. We can therefore write the homogeneous solution as

$$\mathcal{F}_{4,\mathrm{hom}}(\mathbf{r}_{12},\mathbf{r}_{34},R)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l}f_{ll',mm'}(R)\times g_{lm}(\mathbf{r}_{12})g_{l'm'}(\mathbf{r}_{34})\ . \quad (4.24)$$

We recall our choice of the coordinate system such that the z axis is directed along the $\mathbf{R}$ axis. Accordingly the dependence on the sum of azimuthal angles ϕ_{12} and ϕ_{34} should disappear due to the symmetry of the problem. This requirement is met if all coefficients $f_{ll',mm'}$ vanish when $m+m'\neq 0$. In addition, our correlation function is symmetric with respect to the exchange of any pair of points. Accordingly all odd values of l and l' are excluded from the sums in (4.24). Finally, we can find the R dependence of these coefficients from the overall power counting. Since $g_{l,m}(\mathbf{r})\propto r^{\beta_l}$ we can write

$$F_{4,hom}^{ll',mm'}(r_{12},r_{34},R)\sim S_4(R)\left(\frac{r_{12}}{R}\right)^{\beta_l}\left(\frac{r_{34}}{R}\right)^{\beta_{l'}}\ . \quad (4.25)$$

In principle we need to check that the inhomogeneous solution resulting from the term obtained by substituting (4.24) back into the definition of E, Eq. (4.10), is not more important than those we found already. Indeed, the sum in (4.24) cannot include r_{12} or r_{34} to negative exponents, since the homogeneous solution has to satisfy the boundary conditions. Accordingly the inhomogeneous solutions generated by it cannot be more important than those considered so far by terms starting with a constant function of r_{12} and r_{34}. In addition to this homogeneous solution one can consider also the homogeneous solutions of the sum of the two $\hat{\mathcal{B}}$ operators on the left-hand side of (4.14). It can be shown that also these homogeneous solutions do not add any information that is not contained in the solutions described above.

B. Three coalescing points

For the calculation of J_4 we need also to consider the geometry of three coalescing points, see Eq. (4.6). Accordingly we focus on the limit $r_{12}, r_{13}, r_{23} \ll R \simeq r_{14} \simeq r_{24} \simeq r_{34}$. Instead of Eq. (4.13) we now have the equation

$$\left[\hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{23}\right] \mathcal{F}_4^c(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R) = \tilde{E}^{(0)}(R) + \tilde{E}^{(1)}(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R), \quad (4.26)$$

where now the function $\tilde{E}(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R)$ is given by Eq. (4.10) but with the sum on $\hat{\mathcal{B}}$ containing one less operator. We seek solutions that are symmetric with respect to permuting the pairs 12, 13, 23. The leading inhomogeneous solution can be found as before, cf. (4.16),

$$\mathcal{F}^{(1)}_{4,\text{inh}}(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R) \sim S_4(R) \frac{r_{12}^{\zeta_2} + r_{13}^{\zeta_2} + r_{23}^{\zeta_2}}{R^{\zeta_2}} . \quad (4.27)$$

The homogeneous solutions will be of the form (4.24), but with additional summations over l'' and m'' for the third factor $g_{l''m''}$ as a function of the third small vector distance. This is all the information needed for the calculations that follow.

C. Calculation of $K(R)$

According to our strategy we need now to evaluate the limits shown in Eq. (4.1). We first compute the derivatives operating on the products of $\mathcal{F}_2$ and discover that the limits are not singular. In fact, these contributions decay rapidly in R like $R^{-2\zeta_h}$. In computing the derivatives of the cumulant of $\mathcal{F}_4$ we need to consider all the contributions to the solution of $\mathcal{F}_4$ for two pairs of coalescing points that were examined in Sec. IV A 2, and seek the leading one. We will argue that the leading one is the inhomogeneous solution (4.22). Upon performing the derivatives on this solution one obtains

$$K(R) \sim \frac{S_4(R)}{R^{2\zeta_2}} \lim_{r_{12}, r_{34} \to 0} \frac{\kappa^2}{(r_{12} r_{34})^{\zeta_h}} , \quad (4.28)$$

where we have used the fact that $\zeta_h = 2 - \zeta_2$. The other solutions are less singular. The $l = l' = 0$ term in (4.24) is constant in r_{12} and r_{34} and so is only affected by the derivative in R; as this distance is taken to be within the inertial range it is nonsingular. The terms with $l = 0$, $l' = 2$ or vice versa are less singular than the inhomogeneous solution (4.22) since $\beta_2 > \zeta_2$. The derivatives $\nabla_3 \cdot \nabla_4$ acting upon the r_{12}-dependent part of the solution (4.15) act upon the R-dependent multiplier contributing a term proportional to $r_{12}^{-\zeta_h} R^{\zeta_4 - \zeta_2 - 2}$. Again as R is taken to be always within the inertial range, no divergence occurs with respect to the R factors and this contribution is of order $r_{12}^{-\zeta_h}$. One can easily check that after derivation the two remaining inhomogeneous terms (4.21) and (4.23) also are left after derivation with larger exponents.

The singular limit in (4.28) has to be understood in light of the full equation for $\mathcal{F}_4^c$, Eq. (2.59), in which the κ-diffusive terms are explicit. The role of these terms is precisely to truncate the divergence that is implied by (4.28). As a consequence the divergence is only applicable in the inertial range with $r_{12}, r_{34} > \eta$, whereas in the dissipative regime the divergence disappears. Thus in evaluating $K(R)$ in the inertial range we must replace the limit $r_{12}, r_{34} \to 0$ by $r_{12} = r_{34} = \eta$:

$$K(R) \sim \frac{S_4(R)}{R^{2\zeta_2}} \frac{\kappa^2}{\eta^{2\zeta_h}} . \quad (4.29)$$

Comparing to Eq. (1.16) we see that we recover the result that $\Delta = \Delta_c = \zeta_h$.

We can rewrite (4.29) in a final form by rewriting $\bar{\epsilon}$ as

$$\bar{\epsilon} = -\kappa \lim_{r_{12} \to \eta} \nabla_1 \nabla_2 \mathcal{F}(r_{12}) \propto \kappa / \eta^{\zeta_h} , \quad (4.30)$$

where we used the fact that $\mathcal{F}(r_{12}) \sim r_{12}^{\zeta_2}$. Using the last equation in the preceding one we find the final result:

$$K(R) \simeq \bar{\epsilon}^2 S_4(R) / S_2(R)^2 . \quad (4.31)$$

D. Calculation of the correlation functions $L_{ll',m}$

In this section we turn to the calculation of correlation functions that expose the scaling properties of anomalous fields with other irreducible representations of the rotation group. We can do this by introducing the following correlation functions:

$$L_{l,l',m}(r_{12}, r_{34}, R) = \int d\cos\theta_{12} d\cos\theta_{34} d(\phi_{12} - \phi_{34}) \times \mathcal{F}_4(\mathbf{r}_{12}, \mathbf{r}_{34}, R) Y_{l,m}(\theta_{12}, \phi_{12}) \times Y_{l',-m}(\theta_{34}, \phi_{34}) . \quad (4.32)$$

For $l = l' = 0$ there is a contribution arising from the $l = l' = 0$ term in $\mathcal{F}_{4,\text{hom}}$. The next order arises from (4.16), and

$$L_{0,0,0}(r_{12}, r_{34}, R) \simeq \text{const} + \mathcal{F}^{(1)}_{4,\text{inh}} \propto C + R^{\zeta_4 - \zeta_2}(r_{12}^{\zeta_2} + r_{34}^{\zeta_2}) . \quad (4.33)$$

For $l, l' \geq 2$ a contribution arises from (4.24). Using (4.25) we write the final result

$$L_{l,l',m}(r_{12}, r_{34}, R) \sim S_4(R) \left(\frac{r_{12}}{R}\right)^{\beta_l} \left(\frac{r_{34}}{R}\right)^{\beta_l'} . \quad (4.34)$$

Finally, for $l = 0$ and $l' \geq 2$ or vice versa the leading contribution to $L_{0,l',0}(r_{12}, r_{34}, R)$ can be obtained from (4.34) by replacing $(r_{12}/R)^{\beta_0}$ ($\beta_0 = 0$) by $C_{l'} + (r_{12}/R)^{\zeta_2}$. One should point out that this analysis is not complete. There are angular contributions which stem from the inhomogeneous part of $\mathcal{F}_4$ which originate from $E(\mathbf{r}_{12},\mathbf{r}_{34},R)$ of Eq. (4.10). These were not computed explicitly in this paper, and it is possible that they lead to relevant contributions that are of the order of those found above or even more important ones in some domain of ζ_h.

It is interesting to note that Eq. (4.34) can be the starting point for the introduction of a set of anomalous local fields. By this we mean fields defined at a single point whose two-point correlation functions scale with anomalous scaling exponents that are determined by the same exponents β_l appearing in (4.34). Such local fields are related to the same irreducible representations of the rotation (plus inversion) group as the corresponding eigenfunctions $Y_{l,m}$ of the angular momentum operator. We have already discussed the scalar field $|\nabla\Theta(x)|^2$ and its correlation. When we rewrite its correlation by fusing two pairs of coordinates of F_4, we pick only the $l=0$ components in the spherical harmonics expansion of F_4. We can define local fields whose correlations will select other spherical harmonics. For example, the traceless tensor field $\boldsymbol{\nabla}_\alpha\Theta(x)\boldsymbol{\nabla}_\beta\Theta(x)-\frac{1}{3}\delta_{\alpha\beta}|\boldsymbol{\nabla}\Theta(x)|^2$ has a correlation function that, once represented through fusing pairs of coordinates, has components on $Y_{l,m}$ with $l=2$ only. Taking four gradients one can produce anomalous fields with components only on spherical harmonics with $l=4$, and so on. This understanding is the starting point for the development of an operator algebra that will be discussed elsewhere [8].

E. Calculation of the cumulant $J_4^c(R)$

The quantity $J_4(R)$ has reducible contributions which were computed above, and a cumulant part which is obtained from $\mathcal{F}_4^c$. The calculation of $J_4^c(R)$ follows very much the lines of the calculation of $K(R)$, except that one needs to find again which of the solutions found in Sec. IV A 2 contributes most to Eqs. (4.5) and (4.6). It turns out that the leading contribution to (4.5) stems from the solution (4.15), whereas the leading contribution to (4.6) arises from the solution (4.27). As before, the derivatives with respect to r_1 and r_2 result in a singular limit when $r_{12}\to 0$, and we have to take $r_{12}=\eta$ after computing the derivatives. On the other hand, the other limits are regular and they do not require special care. The evaluations of $J_{4,1}(R)$ and $J_{4,2}(R)$ turn out to be the same up to constants. The result is

$$J_4^c(R)\sim\kappa S_4(R)/R^{\zeta_2}\eta^{\zeta_h}\ . \tag{4.35}$$

Using Eq. (4.30) this can be written finally as

$$J_4^c(R)=C_4\bar\epsilon S_4(R)/S_2(R)\ , \tag{4.36}$$

where C_4 is a dimensionless constant that will be determined below. Together with the reducible contributions that were computed above we can write

$$\begin{aligned}J_4(R)=&\,24\bar\epsilon S_2(R)-12\kappa S_2(R)\nabla^2 S_2(R)\\&+C_4\bar\epsilon S_4(R)/S_2(R)\ .\end{aligned} \tag{4.37}$$

It is obvious that the second term is small compared with the first and it can be neglected.

V. THE CALCULATION OF $J_{2N}(R)$

The correlations (1.10) which enter $J_{2n}(R)$ have Gaussian decompositions and a cumulant part. The leading contribution to the Gaussian decomposition is

$$J_{2n}^G(R)=4n(2n-1)\bar\epsilon S_{2n-2}(R), \tag{5.1}$$

which corresponds to the first term in (4.37). To evaluate the cumulant part one needs to compute correlation functions of the type

$$J_{p,q}(R)=\langle\!\langle|\nabla_1\Theta(\mathbf{r}_1)|^2\Theta^{p-2}(\mathbf{r}_1)\Theta^q(\mathbf{r}_2)\rangle\!\rangle, \tag{5.2}$$

with $\mathbf{r}_1=-\mathbf{R}/2$ and $\mathbf{r}_2=\mathbf{R}/2$. The calculation of $J_{p,q}(R)$ follows from the equation for $\mathcal{F}_{2n}^c$ upon coalescing a group of p points (denoted below as the α group) into the position $-\mathbf{R}/2$ and a group of q points (denoted as the β group) into the position $\mathbf{R}/2$. We start with Eq. (2.58), which in the inertial interval takes the form

$$\begin{aligned}\Big[\sum_{\alpha>\alpha'=1}^{p}\hat{\mathcal{B}}_{\alpha\alpha'}+\sum_{\beta>\beta'=p+1}^{2n}\hat{\mathcal{B}}_{\beta\beta'}+\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{2n}\hat{\mathcal{B}}_{\alpha\beta}\Big]\\ \times\mathcal{F}_{2n}^c(\mathbf{r}_1,\dots,\mathbf{r}_{2n})=M(\mathbf{r}_1,\dots,\mathbf{r}_{2n})\ .\end{aligned} \tag{5.3}$$

As before, the effective equation is obtained by grouping together a quantity $E(\{r_{\alpha\alpha'}\},\{r_{\beta\beta'}\},R)$=$M(\mathbf{r}_1,\dots,\mathbf{r}_{2n})$ $-\sum\sum\hat{\mathcal{B}}_{\alpha\beta}\mathcal{F}_{2n}^c(\mathbf{r}_1,\dots,\mathbf{r}_{2n})$. To find the form of the solution we note that when we had one pair of coalescing points this led to the solution (3.14). Two pairs of coalescing points led to (4.16), whereas three pairs of coalescing points resulted in (4.27). In the present case we have $[p(p-1)+q(q-1)]/2$ coalescing pairs and the solution which belongs to the same family is written as

$$\mathcal{F}_{2n,\mathrm{inh}}\sim S_{2n}(R)\Big[\sum_{\alpha>\alpha'=1}^{p}r_{\alpha\alpha'}^{\zeta_2}+\sum_{\beta>\beta'=p+1}^{2n}r_{\beta\beta'}^{\zeta_2}\Big]\Big/R^{\zeta_2}\ . \tag{5.4}$$

Next we need to compute the derivative with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$ and take the limit of all pairs of coalescing distances going to zero. The only divergence will be associated with $r_{\alpha\alpha'}^{-\zeta_h}$, whereas all the other limits are trivial. According to our strategy we have to cut the divergence at $r_{\alpha\alpha'}=\eta$ and thus we calculate the leading contribution to J_{2n}^c, which for $n>1$ is $J_{2n}^c\sim\kappa S_{2n}(R)/(\eta^{\zeta_h}R^{\zeta_2})$. Using again Eq. (4.30) we have

$$J_{2n}^c(R)=nC_{2n}\bar\epsilon S_{2n}(R)/S_2(R)+\cdots\quad\text{for } n>1\ , \tag{5.5}$$

where C_{2n} is an unknown dimensionless coefficient that will be determined soon. This equation is the generalization of (4.36) for any $n>1$.

VI. SCALING EXPONENTS OF THE STRUCTURE FUNCTIONS

In this section we collect all the results obtained above with the aim of reaching conclusions about the scaling exponents of the structure functions. The first result that we need to pay attention to is (4.29) or (4.31) for $K(R)$. This result shows that $K(R) \sim R^{\zeta_4 - 2\zeta_2}$. Since $K(R)$ cannot be an increasing function of R we conclude immediately that

$$\zeta_4 \leq 2\zeta_2 \ . \tag{6.1}$$

Next we need to consider (4.37) for $J_4(R)$, rewriting it after neglecting the second term as

$$J_4(R) \simeq \bar{\epsilon} S_2(R) \left[1 + C_4' \frac{S_4(R)}{S_2^2(R)} \right] . \tag{6.2}$$

Note that in this equation we have included the reducible and the leading contribution to the irreducible part of J_4. If the irreducible part turns out to be much larger than the reducible part (multiscaling) then it may occur that a subdominant term in the irreducible part becomes larger than the reducible contribution that is explicit in (6.2). Therefore this equation does not necessarily display the two largest terms in J_4. Nevertheless this does not affect our considerations and conclusions.

The second term in the parentheses in (6.2) is dimensionless, and assuming the simplest possibility of one renormalization scale may be written as $(\ell/R)^{2\zeta_2 - \zeta_4}$. Now there are two possibilities.

(i) $\zeta_4 = 2\zeta_2$ and the scaling is normal. The first and second terms have the same scaling and they are of the same order.

(ii) $\zeta_4 < 2\zeta_2$, and the scaling is anomalous. If so, by self-consistency the second term in (6.2) must be larger than the first, or else we recover (i). For that to happen the renormalization scale *must* be the outer scale L.

The implications of the first possibility are somewhat strange. For example, if there is normal scaling the correlation function $K(R)$ does not decay in R. This means that the dissipation field is not mixing. On the contrary, if there is anomalous scaling, then the correlation $K(R)$ decays, as is expected from a random field. We will now explore the second possibility and show that if there is anomalous scaling then the scaling exponents can be computed in agreement with Kraichnan's arguments.

If we accept anomalous scaling, then the leading term in J_{2n} is the irreducible contribution (5.5). Generally speaking $J_{2n}(R)$ may be written (following Kraichnan *et al.* [3]) as

$$J_{2n}(R) = -4n\kappa \int d\Delta\Theta P(\Delta\Theta)[\Delta\Theta]^{2n-1} \left\langle \nabla^2 \Delta\Theta | \Delta\Theta \right\rangle \ , \tag{6.3}$$

where $\left\langle \nabla^2 \Delta\Theta | \Delta\Theta \right\rangle$ is the average of $\nabla^2 \Delta\Theta$ *conditional* on a given value of $\Delta\Theta$, where $\Delta\Theta$ is a temperature difference between two R-separated points. $P(\Delta\Theta)$ is the probability to observe a given value of $\Delta\Theta$. Equation (6.3) is exact. The question now is what is the dependence of $\left\langle \nabla^2 \Delta\Theta | \Delta\Theta \right\rangle$ on $\Delta\Theta$. In order to recover our result (5.5) for the leading contribution of J_{2n}, this conditional average *must* satisfy

$$\left\langle \nabla^2 \Delta\Theta | \Delta\Theta \right\rangle = C \bar{\epsilon}(\Delta\Theta) / \kappa S_2(R). \tag{6.4}$$

This means that the coefficient C_{2n} in (5.5) is n independent. We can determine this coefficient from the particular case $n = 1$. Since $J_2(r) = 4\bar{\epsilon}$ the conclusion is that $C_{2n} = 4$. Equipped with this we consider again the balance equation (1.8)

$$R^{1-d} \frac{\partial}{\partial R} R^{d-1} h(R) \frac{\partial}{\partial R} S_{2n}(R) = J_{2n}(R) \ . \tag{6.5}$$

Using now the definition of the scaling exponents $S_{2n}(R) \sim R^{\zeta_{2n}}$ we retrieve from (6.5) the result (1.12) which can also be written as

$$\zeta_{2n} = \frac{1}{2} \left[\zeta_2 - d + \sqrt{(\zeta_2 + d)^2 + 4\zeta_2(n-1)} \right] . \tag{6.6}$$

This is Kraichnan's anomalous scaling.

It should be reiterated here that Eq. (6.5) is a statement of balance between a transfer term and a dissipative term. This balance means that there exists a flux (here of T^2) and the system is maintained out of thermodynamic equilibrium. We refer to such a state as "flux equilibrium." The anomalous exponents (6.6) are consistent with the existence of such a flux equilibrium. It is clear that there may be other inertial range scaling solutions in this problem. By imposing other demands on the solution one may select other exponents. We believe that the requirement of flux equilibrium results in the physical solution.

VII. SUMMARY AND CONCLUSIONS

The first conclusion of this paper is that renormalized perturbation theory for hydrodynamic fields has the potential to describe nonperturbative effects. Exact resummations of the diagrammatic series result in exact equations for the statistical quantities that contain not only perturbative but also nonperturbative effects.

Second, the passive scalar problem with rapidly decorrelating velocity field displays a particularly simple analytic structure in which all the scaling properties flow from one differential operator $\hat{\mathcal{B}}(R)$. As a result this theory becomes "critical" in the sense that the anomalous exponent Δ that was introduced in paper I is exactly critical. The reason for this is that both Δ and Δ_c come from the same operator $\hat{\mathcal{B}}$, and thus they must be the same. In such a situation the subcritical scenario that was suggested in [9] is untenable.

Third, the simplicity of the theory allowed us to compute the whole spectrum of anomalous exponents that are associated with the ultraviolet divergence. It was shown that these exponents are related to the spectrum β_l (3.11) which appears in the law of isotropization of the two-point correlation function. The same exponents appear in the R dependence of the correlation functions $L_{ll',m}$ that were introduced in (4.32).

Finally we showed that as far as the structure functions $S_n(R)$ are concerned, the requirement of flux equilibrium leaves only two possibilities for the inertial range scaling. Either the exponents satisfy $\zeta_{2n} = n\zeta_2$ and the scaling is normal or the scaling is anomalous with the outer renormalization scale and with the law (6.6). We noted that normal scaling also implies that the dissipation field is not mixing. We reiterate that other requirements may lead to additional inertial range scalings. We believe that the present one selects the physical solution.

There exists an interesting question regarding the criticality of the theory in the sense that $\Delta = \Delta_c$. Is this a result of the peculiarities of the model, or is this a structurally stable property of this type of problem? In Refs. [10,11] it was found that flux equilibria result in this property also for Navier-Stokes turbulence. In fact, it is structurally stable, and should exist in the passive scalar problem also after relaxing many of the restrictions of the model.

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APPENDIX A: DERIVATION OF THE EQUATION FOR THE SIMULTANEOUS TWO-POINT CORRELATOR

Here we obtain the equation of motion for the simultaneous two-point correlator, $\mathcal{F}(\mathbf{r}_1,\mathbf{r}_2,t=0)$. We make use of Eq. (2.18) to verify that the operator

$$\mathcal{G}_{12}^{-1}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t) \equiv \partial_t + \hat{\mathcal{D}}_1(\mathbf{r}_1-\mathbf{r}_0) + \hat{\mathcal{D}}_1(\mathbf{r}_2-\mathbf{r}_0) \quad \text{(A1)}$$

is the inverse of the product of Green's functions, that is,

$$\mathcal{G}_{12}^{-1}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t)\mathcal{G}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_1',t)\mathcal{G}(\mathbf{r}_0|\mathbf{r}_2,\mathbf{r}_2',t) = \delta(\mathbf{r}_1-\mathbf{r}_1')\delta(\mathbf{r}_2-\mathbf{r}_2')\delta(t). \quad \text{(A2)}$$

We have used here the fact that the Green's function when multiplied by $\delta(t)$ may be evaluated at $t=0$. Applying this operator and performing the spatial integrations, one obtains directly

$$\left[\hat{\mathcal{D}}_1(\mathbf{r}_1-\mathbf{r}_0) + \hat{\mathcal{D}}_1(\mathbf{r}_2-\mathbf{r}_0)\right]\mathcal{F}(\mathbf{r}_1,\mathbf{r}_2) = \Phi(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2) + \Phi_0(\mathbf{r}_1,\mathbf{r}_2). \quad \text{(A3)}$$

Now using the definition of $\Phi(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2)$ from Eq. (2.12) and dropping the $\mathbf{r}_0$ dependence, one may rewrite this as

$$-\left[\kappa(\nabla_1^2+\nabla_2^2) - h_{ij}(\mathbf{r}_1-\mathbf{r}_2)\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}} - \hat{\mathcal{H}}(\mathbf{r}_1,\mathbf{r}_2)\right] \times\mathcal{F}(\mathbf{r}_1,\mathbf{r}_2) = \Phi_0(\mathbf{r}_1,\mathbf{r}_2), \quad \text{(A4)}$$

where the operator $\hat{\mathcal{H}}$ is defined in (2.32). In the case that all quantities are only functions of $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$, $\hat{\mathcal{H}}$ vanishes. The remaining terms are equivalent to the definition of the operator $\hat{\mathcal{D}}_2(\mathbf{R})$ in Eq. (2.33) and one recovers Eq. (2.29).

APPENDIX B: EQUATION FOR THE CUMULANT OF THE 2N-POINT CORRELATOR

In this appendix we derive equations for the many-time correlators using the diagrammatic approach. The infinite diagrammatic series for the four-point correlator was presented in [4]. Let us consider the series for the correlator. A typical diagram for a $2n$-point correlator consists of n "tramways": strings of Green's functions, connected in pairs at one end by two-point correlators. These tramways may be interconnected by velocity correlators. Therefore, to build up an arbitrarily complex diagram, one successively adds connections between chosen pairs of tramways. Let us consider a given pair of end points, x_α and x_β. Group all diagrams together in which the Green's function beginning at x_α and that beginning at x_β are linked by a velocity correlator. The series of diagrams to the right of this first velocity correlator is again the series of diagrams for the full correlator.

The difficulty with writing a resummed equation for the full correlator is that the lowest order terms are Gaussian, and disconnected. In building upon the Gaussian terms by the addition of "rungs" as just described, one will generate, among other terms, a series in which the disconnected parts remain disconnected, and may be resummed again into the Gaussian decomposition. This means that these terms actually appear twice. In order to avoid this one may write an equation for the *cumulants.* For the cumulant the lowest order terms are those in which all tramways have one velocity correlator connection to another.

To simplify the appearance of this equation we introduce the operator $\widehat{C}_{\alpha\beta}$ which represents the addition of a rung, and here operates on $\mathcal{F}_{2n}(0|x_1,...,x_\alpha,x_\beta,...,x_{2n})$. The definition is

$$\widehat{C}_{\alpha\beta}\mathcal{F}_{2n}(0|x_1,...,x_\alpha,x_\beta,...,x_{2n}) \equiv \int d\mathbf{r}_1' d\mathbf{r}_2' \int_{t_m}^{\infty} dt\mathcal{G}_2^0(0|x_\alpha,x_\beta,\mathbf{r}_1',t,\mathbf{r}_2',t)H_{ij}(\mathbf{r}_1',\mathbf{r}_2')\frac{\partial}{\partial r_1'}\frac{\partial}{\partial r_2'} \times\mathcal{F}_{2n}(0|x_1,...,\mathbf{r}_1',t,\mathbf{r}_2',t,...,x_{2n}). \quad \text{(B1)}$$

Since $\mathcal{F}_{2n}(0|x_1,x_2,...,x_{2n})$ is symmetric with respect to all exchanges of coordinates this definition is sufficient for any pair of indices $1\le\alpha,\beta\le 2n$.

The lowest order terms may be expressed in terms of the operator $\hat{C}$ and two-point correlators as

$$\mathcal{F}^c_{2n,0}(0|x_1,x_2,..,x_{2n}) = \prod_{j=2,4}^{2n} \hat{C}_{j,j+1} \prod_{k=1,3,...}^{2n-1} \mathcal{F}(x_k,x_{k+1}). \tag{B2}$$

Then the resummed equation has the form

$$\mathcal{F}^c_{2n}(0|x_1,x_2,...,x_{2n}) = \sum_{\rm perm} \mathcal{F}^c_{2n,0}(0|x_1,x_2,...,x_{2n}) + \sum_{\alpha>\beta} \hat{C}_{\alpha\beta}\mathcal{F}^c_{2n}(0|x_1,x_2,...,x_{2n}). \tag{B3}$$

Now operate on both sides of the equation with the product of the inverse Green's functions (2.26). Using the fact that

$$\left(\frac{\partial}{\partial t_\alpha}+\hat{\mathcal{D}}_1(\mathbf{r}_\alpha)\right)\left(\frac{\partial}{\partial t_\beta}+\hat{\mathcal{D}}_1(\mathbf{r}_\beta)\right)C_{\alpha\beta} = \delta(t_\alpha-t_\beta)\mathcal{B}_{\alpha\beta}, \tag{B4}$$

one finds

$$\prod_{k=1}^{2n}\left(\frac{\partial}{\partial t_k}+\hat{\mathcal{D}}_1(\mathbf{r}_k)\right)\mathcal{F}^c_{2n}(0|x_1,x_2,..,x_{2n})$$

$$= \sum_{\langle\alpha,\beta\rangle}\sum_{\substack{(\gamma_i)\\ \mathrm{perm}(\gamma_1\neq\alpha,\beta)}}\left[\prod_{i=1}^{2n-2}\delta(t_{\gamma_i}-t_{\gamma_{i+1}})\right]\left(\frac{\partial}{\partial t_\alpha}+\hat{\mathcal{D}}_1(\mathbf{r}_\alpha)\right)\left(\frac{\partial}{\partial t_\beta}+\hat{\mathcal{D}}_1(\mathbf{r}_\beta)\right)\mathcal{B}_{\gamma_1,\gamma_2}\mathcal{F}(x_\alpha,x_{\gamma_1})$$

$$\times\left\{\prod_{j=2,4..}^{2n-4}\mathcal{B}_{\gamma_{j+1},\gamma_{j+2}}\mathcal{F}(\mathbf{r}_{\gamma_j},\mathbf{r}_{\gamma_{j+1}})\right\}\mathcal{B}_{\gamma_{j+1},\gamma_{j+2}}\mathcal{F}(x_{\gamma_{2n-2}},x_\beta)$$

$$+\sum_{\alpha>\beta}\delta(t_\alpha-t_\beta)\prod_{\gamma\neq\alpha,\beta}\left[\partial_t+\hat{\mathcal{D}}_1(\mathbf{r}_\gamma)\right]\mathcal{B}_{\alpha\beta}\mathcal{F}^c_{2n}(0|x_1,x_2,...,x_{2n}). \tag{B5}$$

Therefore we have derived a closed equation for the time development of the many-time cumulants of the $2n$-point correlator in terms of the operator $\mathcal{B}$ and the two-point moments only. We will not solve this equation in this paper.

APPENDIX C: EVALUATION OF THE OPERATORS

Here we evaluate the sum of operators $\hat{\mathcal{B}}_{13}+\hat{\mathcal{B}}_{14}+\hat{\mathcal{B}}_{23}+\hat{\mathcal{B}}_{24}$ which will be denoted as Σ. We begin by transforming to the coordinate system described in Sec. IV A, where due to translational and rotational invariance we may move to a system described by a sufficient set of variables related to the relative positions, six in three dimensions. It is convenient to instead work with seven, the vector distance $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{r}_{34}=\mathbf{r}_3-\mathbf{r}_4$, and the vector $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4)/2$, with the coordinate frame fixed such that $\mathbf{R}$ lies along the $\mathbf{z}$ axis and $\mathbf{r}_{12}$ lies in the x-z plane. We will consider also the limit of the separations $r_{12},r_{34}\ll R$. The transformation of derivatives in this coordinate system reduces simply to the following:

$$\begin{aligned}\frac{\partial}{\partial\mathbf{r}_1}&\to\frac{\partial}{\partial\mathbf{r}_{12}}+\frac{1}{2}\frac{\partial}{\partial\mathbf{R}},\\ \frac{\partial}{\partial\mathbf{r}_2}&\to-\frac{\partial}{\partial\mathbf{r}_{12}}+\frac{1}{2}\frac{\partial}{\partial\mathbf{R}},\\ \frac{\partial}{\partial\mathbf{r}_3}&\to\frac{\partial}{\partial\mathbf{r}_{34}}-\frac{1}{2}\frac{\partial}{\partial\mathbf{R}},\\ \frac{\partial}{\partial\mathbf{r}_4}&\to-\frac{\partial}{\partial\mathbf{r}_{34}}-\frac{1}{2}\frac{\partial}{\partial\mathbf{R}},\end{aligned} \tag{C1}$$

from which we may define

$$\mathbf{D}_{12}\equiv\frac{\partial}{\partial\mathbf{r}_{12}}+\frac{1}{2}\frac{\partial}{\partial\mathbf{R}},\quad \mathbf{D}_{34}\equiv\frac{\partial}{\partial\mathbf{r}_{34}}-\frac{1}{2}\frac{\partial}{\partial\mathbf{R}}. \tag{C2}$$

The operators acting on the small distances, $\hat{\mathcal{B}}_{12}$ and $\hat{\mathcal{B}}_{34}$, then give, for example,

$$\frac{\partial}{\partial\mathbf{r}_1}\cdot h(\mathbf{r}_2-\mathbf{r}_1)\cdot\frac{\partial}{\partial\mathbf{r}_2}\to\mathbf{D}_{12}\cdot h(\mathbf{r}_{12})\cdot\mathbf{D}_{12}\sim r^{-\zeta_2}, \tag{C3}$$

neglecting the contribution from the derivatives on $\mathbf{R}$.

To evaluate Σ, we take into account the limit $R\gg r_{12},r_{34}$ to expand the operators around $\mathbf{R}$. The term of zeroth order in the small distance is given by

$$\Sigma^{(0)}(\mathbf{R},\mathbf{r}) = \sum_{(\alpha,\beta)\in A} (-1)^{\alpha+\beta} R^{\zeta_h} \times \left[c_1 \mathbf{D}_{12}\cdot\mathbf{D}_{34} + c_2 \frac{R_i R_j}{R^2}\mathbf{D}_{12i}\mathbf{D}_{34j}\right], \quad \text{(C4)}$$

where A denotes the set $\{(1,3),(1,4),(2,3),(2,4)\}$ and c_1 and c_2 are numerical constants depending on ζ_h. This term clearly vanishes in the summation over α,β. In the first order term again the sum vanishes due to pairwise cancellations. The first significant term is therefore

$$\Sigma^{(2)}(\mathbf{R},\mathbf{r}) = R^{-\zeta_2} \sum_{(\alpha,\beta)} (-1)^{\alpha+\beta} r_{\alpha\beta,k} r_{\alpha\beta,l} [A\hat{R}_i\hat{R}_j\hat{R}_k\hat{R}_l + B\delta_{ij}\hat{R}_k\hat{R}_l + C(\delta_{ik}\hat{R}_j\hat{R}_l + \delta_{il}\hat{R}_j\hat{R}_k + \delta_{jk}\hat{R}_i\hat{R}_l + \delta_{jl}\hat{R}_i\hat{R}_k + \delta_{kl}\hat{R}_i\hat{R}_j) + D\delta_{ij}\delta_{kl} + E\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\mathbf{D}_{12i}\mathbf{D}_{34j}, \quad \text{(C5)}$$

where $A, B, ..., E$ are constants depending on ζ_h and d. The leading order term therefore involves $(\partial/\partial\mathbf{r}_{12})(\partial/\partial\mathbf{r}_{34})$ but as the singularity is cancelled in the numerator it is of order $R^{-\zeta_2}$.

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