Fusion Rules in Turbulent Systems with Flux Equilibrium

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Fusion rules in turbulence specify the analytic structure of many-point correlation functisons of the turbulent field when a group of coordinates coalesce. We show that the existence of universal flux equilibrium in fully developed turbulent systems combined with a direct cascade induces universal fusion rules. In certain examples these fusion rules suffice to compute the multiscaling exponents exactly, and in other examples they give rise to an infinite number of scaling relations that constrain enormously the structure of the allowed theory.

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In a series of recent papers, elements of the analytic theory of Navier-Stokes turbulence [1–4] and passive-scalar turbulent advection [5–8] were presented. In this Letter we explain that the structure of the essential part of these theories is economically summarized by a set of "fusion rules" that determine the analytic structure of n-point correlation functions when a group of coordinates tend together. We show here that the fusion rules can be deduced from very few general assumptions about the nature of the universal flux equilibrium that exists in fully developed turbulent systems. Of course, the same fusion rules can also be established by direct calculations in specific examples. We first deduce the fusion rules, then we exemplify their utility in determining scaling exponents, and, lastly, we demonstrate how in one example the fusion rules follow from first principles.

Consider a turbulent field $\mathbf{u}(\mathbf{r},t)$ which is either a vector or a scalar and denote the difference $\mathbf{w}(\mathbf{r}_1|\mathbf{r}_2,t) \equiv \mathbf{u}(\mathbf{r}_2,t) - \mathbf{u}(\mathbf{r}_1,t)$. We discuss the statistical properties of the turbulent field in terms of simultaneous many-point correlation functions of differences with respect to one, two, or more references points:

$$S_n(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n) = \langle \mathbf{w}(\mathbf{r}_0|\mathbf{r}_1)\mathbf{w}(\mathbf{r}_0|\mathbf{r}_2)\cdots\mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)\rangle,$$
$$S_{n,m}(\mathbf{r}_0,\mathbf{r}_0'|\mathbf{r}_1,\ldots,\mathbf{r}_n;\mathbf{r}_{n+1},\ldots,\mathbf{r}_{n+m}) = \langle \mathbf{w}(\mathbf{r}_0|\mathbf{r}_1)$$
$$\times \mathbf{w}(\mathbf{r}_0|\mathbf{r}_2)\cdots\mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)\mathbf{w}(\mathbf{r}_0'|\mathbf{r}_{n+1})\cdots\mathbf{w}(\mathbf{r}_0'|\mathbf{r}_{n+m})\rangle, \quad (1)$$

etc. Note that when $\mathbf{u}$ is a vector the n-point correlation is an n-rank tensor. The class of systems that we discuss are driven on a characteristic scale referred to as the outer scale L. This driving can be achieved either by a time dependent low frequency "stirring force" or by specifying given values of $\mathbf{u}$ at a set of "boundary" points with a characteristic separation L away from our observation points $\mathbf{r}_0, \mathbf{r}_0', \mathbf{r}_1$, etc. The systems have dissipation (viscosity, diffusivity, etc.), and in the dissipationless limit there exists an integral of motion which we refer to as "energy." We consider systems with a "direct" cascade in which the intake of energy on the scale L is balanced by dissipation on a small scale $\eta \ll L$.

We invoke two assumptions of the Kolmogorov [9] type.

(1) Scale invariance: All correlation functions are homogeneous functions of their arguments in the core of the inertial interval $\eta \ll |\mathbf{r}_i - \mathbf{r}_0| \ll L$:

$$S_n(\lambda\mathbf{r}_0|\lambda\mathbf{r}_1,\lambda\mathbf{r}_2,\ldots,\lambda\mathbf{r}_n) = \lambda^{\zeta_n} S_n(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n), \quad (2)$$

where ζ_n are scaling exponents.

(2) Universality of the fine scale structure of turbulence: In its strong version this means that the correlation functions of the type (1) have a universal functional dependence on the separation distances when they are all in the interior of the inertial interval (η, L). This means that we can fix an arbitrary set of velocity differences on the scale of L, and the correlation functions will depend on their precise choice only via an overall factor determined by the L-scale motions. Mathematically this is expressed as the following property of the conditional average:

$$\langle \mathbf{w}(\mathbf{r}_0|\mathbf{r}_1)\mathbf{w}(\mathbf{r}_0|\mathbf{r}_2)\cdots\mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)|\mathbf{w}(\mathbf{r}_0|\mathbf{r}_1')$$
$$\times \mathbf{w}(\mathbf{r}_0|\mathbf{r}_2')\cdots\mathbf{w}(\mathbf{r}_0|\mathbf{r}_N')\rangle$$
$$= \tilde{S}_n(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n)\Phi_{n,N}(\mathbf{r}_0|\mathbf{r}_1',\ldots,\mathbf{r}_N') \quad (3)$$

for $|\mathbf{r}_i' - \mathbf{r}_0| \sim L$ and $|\mathbf{r}_i - \mathbf{r}_0| \ll L$. The functions $\tilde{S}_n$ coincide with S_n in the inertial interval. They can differ in their crossover to viscous behavior, and their (different) crossover scales may depend on the large scale motions. A weaker version of the universality assumption concerns with the scaling exponent only. In this version the functions S_n and $\tilde{S}_n$ may differ, but their scaling exponents coincide. This weaker version is sufficient for most of our developments below. Note that in both versions ζ_n can be identified with the scaling exponent of the n-order structure function $\langle|\mathbf{w}(\mathbf{r}_0|\mathbf{r})|^n\rangle$. We are particularly interested in multiscaling systems in which ζ_n is a nonlinear function of n.

The derivation of these two properties from first principles differs from system to system. In this Letter we discuss the fusion rules and their consequences in systems for which these assumptions are valid. The first set of fusion rules that we derive concerns S_n when p points $(p < n)$ tend to $\mathbf{r}_0$ (so that the typical separation from $\mathbf{r}_0$ is r), and all the other separations remain much larger, of

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the order of R, $r \ll R \ll L$. Without loss of generality, we can choose these p coordinates as $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p$. We claim that

$$S_n(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \tilde{S}_p(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p)\Psi_{n,p}(\mathbf{r}_0|\mathbf{r}_{p+1}, \mathbf{r}_{p+2}, \ldots, \mathbf{r}_n), \tag{4}$$

where $\Psi_{n,p}(\mathbf{r}_0|\mathbf{r}_{p+1}, \mathbf{r}_{p+2}, \ldots, \mathbf{r}_n)$ is a homogenous function with a scaling exponent $\zeta_n - \zeta_p$. The derivation of the fusion rule (4) follows from Bayes' theorem and assumptions (1) and (2). We write

$$S_n(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \int d\mathbf{w}(\mathbf{r}_0|\mathbf{r}_{p+1}) \cdots d\mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)\mathbf{w}(\mathbf{r}_0|\mathbf{r}_{p+1}) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)\mathcal{P}[\mathbf{w}(\mathbf{r}_0|\mathbf{r}_{p+1}) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)]$$
$$\times \langle \mathbf{w}(\mathbf{r}_0|\mathbf{r}_1), \mathbf{w}(\mathbf{r}_0|\mathbf{r}_2) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_p)|\mathbf{w}(\mathbf{r}_0|\mathbf{r}_{p+1})\mathbf{w}(\mathbf{r}_0|\mathbf{r}_{p+2}) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)\rangle, \tag{5}$$

where $\mathcal{P}[\mathbf{w}(\mathbf{r}_0|\mathbf{r}_{p+1}) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)]$ is the probability to see the tensor $\mathbf{w}(\mathbf{r}_0|\mathbf{r}_{p+1}) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_{pn})$. Next, note the consequence of assumption (2): The scaling laws of the correlation functions at scale r are the same, independent of whether we force the system on the scale $L \gg r$ or on the scale $R \gg r$. The conditional average in (5) is proportional to $\tilde{S}_p$, and hence (4). This result seems rather obvious at this point, but we will see that it leads to a totally unconventional scaling structure of the theory. We should stress that for Navier-Stokes and passive-scalar advection these fusion rules for $p = 2$ were derived from first principles [4,8].

The next set of fusion rules is obtained for the structure function $S_{n,m}$ of (1) when two groups of $p \le n$ and $q \le m$ points tend to $\mathbf{r}_0$ and $\mathbf{r}'_0$, respectively. The separation between these groups of points is of the order of R. The derivation of the fusion rules is now obvious, with the result

$$S_{n,m}(\mathbf{r}_0, \mathbf{r}'_0|\mathbf{r}_1, \ldots, \mathbf{r}_n; \mathbf{r}_{n+1}, \ldots, \mathbf{r}_{n+m})$$
$$= \tilde{S}_p(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p)\tilde{S}_q(\mathbf{r}'_0|\mathbf{r}_{n+1}, \mathbf{r}_{n+2}, \ldots, \mathbf{r}_{n+q})$$
$$\times \Psi_{n,m,p,q}(\mathbf{r}_0, \mathbf{r}'_0|\mathbf{r}_{p+1}, \ldots, \mathbf{r}_n; \mathbf{r}_{n+q+1}, \ldots, \mathbf{r}_{n+m}). \tag{6}$$

The scaling exponent of $\Psi_{n,m,p,q}$ is $\zeta_{n+m} - \zeta_p - \zeta_q$. Note that the fusion rules (4) and (6) are *not* decompositions into products of lower order correlation functions, and the functions Ψ are not correlations of velocity differences across large separations. In fact, we will show that Ψ is much larger than the corresponding correlation functions in all situations with multiscaling. Evidently, one can derive similar fusion rules for three, four, or more groups of coalescing points with large separations between the groups. The structure of the resulting correlation function will be a product of the correlation function associated with each group times some function Ψ of big separations which carries the overall exponent.

Next, we discuss fusion rules for correlation functions that include gradient fields. These rules depend on the type of rotational invariant that one can define from the tensors that appear after taking gradients. We will consider only the lowest order invariant which is a scalar under rotation. For passive scalars T this is $|\nabla T \cdot \nabla T|^2$, and for a vector $\mathbf{u}$ the quantity $|\nabla \mathbf{u}|^2$ is the square of the strain tensor $s_{ij}s_{ij}$, where $s_{ij} \equiv (\partial u_i/\partial r_j + \partial u_j/\partial r_i)/2$. Consider the quantity

$$J_{2p,n}(\mathbf{r}_0|\mathbf{r}_{2p+1}, \ldots, \mathbf{r}_n) = \langle |\nabla \mathbf{u}(\mathbf{r}_0)|^{2p}$$
$$\times \mathbf{w}(\mathbf{r}_0|\mathbf{r}_{2p+1}) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)\rangle. \tag{7}$$

To evaluate $J_{2p,n}$ we consider a related object in which all the gradients are taken at different points:

$$\tilde{J}_{2p,n} = \nabla^{i_1}_{r_1}\nabla^{j_1}_{r'_1}\nabla^{i_2}_{r_2}\nabla^{j_2}_{r'_2} \cdots \nabla^{i_p}_{r_p}\nabla^{j_p}_{r'_p} C^{i_1,j_1,\ldots,i_p,j_p}_{k_1,l_1,\ldots,k_p,j_p}$$
$$\times \langle w^{k_1}(\mathbf{r}_0|\mathbf{r}_1)w^{l_1}(\mathbf{r}_0|\mathbf{r}'_1) \cdots w^{k_p}(\mathbf{r}_0|\mathbf{r}_p)w^{l_p}(\mathbf{r}_0|\mathbf{r}'_p)$$
$$\times \mathbf{w}(\mathbf{r}_0|\mathbf{r}_{2p+1}) \cdots \mathbf{w}(\mathbf{r}_0|\mathbf{r}_n)\rangle, \tag{8}$$

where the contraction tensor C ensures that the requires scalar is obtained. We will represent this quantity in a compact form without displaying all the tensor indices as $\tilde{J}_{2p,n}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \nabla_{r_1}\nabla_{r'_1} \cdots \nabla_{r'_p} C\, S_n(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}'_1, \ldots, \mathbf{r}_n)$. The quantity (8) gives us $J_{2p,n}$ when the first $2p$ points coalesce together with $\mathbf{r}_0$, whereas all the rest of the coordinates remain a typical distance R from $\mathbf{r}_0$. When R is in the inertial interval we expect scaling behavior in terms of R,

$$J_{2p,n} \propto R^{-\xi(2p,n)}. \tag{9}$$

Considering the distances between all the coalescing points to be in the inertial range, we apply (4) and find for $2p$ coalescing points

$$J_{2p,n}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \nabla_{r_1} \cdots \nabla_{r'_p} \tilde{S}_{2p}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}'_1, \ldots, \mathbf{r}'_p)$$
$$\times \Psi_{n,2p}(\mathbf{r}_0|\mathbf{r}_{2p+1}, \mathbf{r}_{2p+2}, \ldots, \mathbf{r}_n). \tag{10}$$

We expect that $J_{2p,n}$ is independent of the first $2p$ coordinates when the distances between them are well in the viscous regime. We will denote by $\eta(2p, n, R)$ the characteristic viscous length at which $\tilde{S}_{2p}$ crosses smoothly from inertial range behavior to dissipative behavior. Of course, this is also the crossover scale of $J_{2p,n}$ with respect to the first $2p$ coordinates. This allows us to evaluate the coalescing gradients by taking the $2p$ separations to be $\eta(2p, n, R)$:

$$J_{2p,n}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \sim \eta(2p, n, R)^{\zeta_{2p}-2p}$$
$$\times \Psi_{n,2p}(\mathbf{r}_0|\mathbf{r}_{2p+1}, \mathbf{r}_{2p+2}, \ldots, \mathbf{r}_n)$$
$$(2p \text{ coalescing points}). \tag{11}$$

If there are two groups of coalescing points with gradients, with p points coalescing onto $\mathbf{r}_0$ and q points coalescing on $\mathbf{r}'_0$, respectively, we consider $J_{p,q,n,m}$ (where, as before, $n + m \ge p + q$ is the total number of points). The rule for p and q coalescing points is

$$J_{p,q,n,m}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \sim \eta(p, n, R)^{\zeta_p - p}\eta(q, n, R)^{\zeta_q - q}$$
$$\times \Psi_{n,m,p,q}(\mathbf{r}_0|\mathbf{r}_{p+1}, \mathbf{r}_{p+2}, \ldots, \mathbf{r}_n). \tag{12}$$

The generalization of this fusion rule for three or more groups of coalescing points with gradients is obvious.

This is as much as one can do in general. Now the crucial issue is how $\eta(2p,n,R)$ depends on its arguments. The simplest version of the theory comes about when the dissipative length is independent of R, $\eta(2p,n,R) = \eta(2p,n)$. This is realized, for example, in passive scalar convection as is shown below. In this case the fusion rules imply various sets of scaling relations. For example, the exponents $\xi(2p,n)$ of $J_{2p,n}$ are given by

$$\xi(2p,n) = \zeta_n - \zeta_{2p} \,. \tag{13}$$

Note that fusion rules equivalent to this simple version were proposed in [10] on the basis of formal assumptions used in field theory, namely, existence of an ultraviolet universal renormalization-group fixed point and "asymptotic completeness."

As another example of scaling relations, consider the correlation functions

$$K_{\epsilon\epsilon}^{(2s)}(R) \equiv \langle |\nabla \mathbf{u}(\mathbf{r})|^{2s} |\nabla \mathbf{u}(\mathbf{r}+\mathbf{r})|^{2s} \rangle \propto R^{-\mu(2s)}. \tag{14}$$

From (12) in the case $n = m = p = q = 2s$ we get a set of scaling relations

$$\mu(2s) = 2\zeta_{2s} - \zeta_{4s} \,. \tag{15}$$

Next, we can consider a correlation of l gradient fields with the same power (i.e., $|\nabla \mathbf{u}|^{2s}$) at l different points which are separated by a distance of the order of R. The corresponding exponent $\mu(l,2s)$ is

$$\mu(l,2s) = l\zeta_{2s} - \zeta_{2sl} \,. \tag{16}$$

This algebra can be generalized to any correlation of powers of $|\nabla \mathbf{u}|^2$. For example, the scaling exponent $\mu(p_1,p_2,\ldots,p_n)$ of a correlation of fields $\langle |\nabla \mathbf{u}(\mathbf{r}_1)|^{p_1} |\nabla \mathbf{u}(\mathbf{r}_2)|^{p_2} \cdots |\nabla \mathbf{u}(\mathbf{r}_n)|^{P_n} \rangle$ in which all the separations is of the order of R is

$$\mu(p_1,p_2,\ldots,p_n) = \sum_{j=1}^{n} \zeta_{p_j} - \zeta_{\overline{p}}, \quad \overline{p} = \sum p_j \,. \tag{17}$$

In usual operator algebras [11–14] every local field is associated with a leading exponent and the correlation function scales with the sum of these exponents. In this case the algebra is different. There is a global exponent $\zeta_{\overline{p}}$ from which one subtracts the sum of individual exponents ζ_{p_j}. In multiscaling situations the global exponent is a nonlinear function of $\overline{p} = \sum_{p_j}$. Accordingly, it is not a property of every individual field. We note here without demonstration that invariants of the gradient field tensors other than scalars are associated with other individual exponents.

The range of applicability of these fusion rules should be understood on the basis of the equations of motion for any given system. As an example, we explain here briefly why they are applicable for Kraichnan's model [15] of passive-scalar convection with a driving velocity field that is δ correlated in time. It was shown [15,16] that the cumulant part F_{2n}^c of the $2n$-order correlation function $F_{2n} = \langle T(\mathbf{r}_1,t)T(\mathbf{r}_2,t)\cdots T(\mathbf{r}_{2n},t)\rangle$ satisfies for $n > 1$ the exact homogeneous differential equation

$$\left[-\kappa \sum_{\alpha=1}^{2n} \nabla_\alpha^2 + \hat{\mathcal{B}}_{2n} \right] F_{2n}^c(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2n}) = 0 \,, \tag{18}$$

where κ is the molecular diffusivity and ∇_α^2 is the Laplacian operator acting on $\mathbf{r}_\alpha$. The operator $\hat{\mathcal{B}}_{2n}$ is the sum of the binary operators $\hat{\mathcal{B}}_{\alpha\beta}$:

$$\hat{\mathcal{B}}_{\alpha\beta} \equiv h_{i,j}(\mathbf{r}_\alpha - \mathbf{r}_\beta) \frac{\partial^2}{\partial r_{\alpha,i} \partial r_{\beta,j}}, \quad \hat{\mathcal{B}}_{2n} = \sum_{\alpha>\beta=1}^{2n} \hat{\mathcal{B}}_{\alpha\beta} \,. \tag{19}$$

Here $h_{i,j}(\mathbf{r})$ is the eddy diffusivity that behaves like HR^{ζ_h} with $0 < \zeta_h < 2$ and $\zeta_2 = 2 - \zeta_h$.

In the inertial range of scales we can disregard the Laplacian operators in Eq. (19). For deriving the fusion rules (4) we consider the p coalescing points with characteristic separation r and denote their coordinates by the index α or α'. The remaining $2n - p$ coordinates have characteristic separations R and are denoted by β or β'. We assemble [8] the $\hat{\mathcal{B}}$ operators in three groups: $\hat{\mathcal{B}}_p = \sum_{\alpha>\alpha'} \hat{\mathcal{B}}_{\alpha\alpha'}$, $\hat{\mathcal{B}}_{2n-p} = \sum_{\beta>\beta'} \hat{\mathcal{B}}_{\beta\beta'}$, and $\hat{\mathcal{B}}^R = \sum_{\alpha\beta} \hat{\mathcal{B}}_{\alpha\beta}$. The evaluation of the action of the operators in the first and second groups is H/r^{ζ_2} and H/R^{ζ_2}, respectively. The evaluation of the action of each term in the third group is HR/rR^{ζ_2}. However, space homogeneity results in a cancellation of this evaluation in the sum of the terms in this group [8]. The next order surviving evaluation is again H/R^{ζ_2}. We thus combine the second and third groups into an effective operator $\hat{\mathcal{B}}$. The equation to consider is $\hat{\mathcal{B}}_p + \hat{\mathcal{B}} F_{2n} = 0$. When $p = 2$ we can find the solution of this equation as the following expansion in powers of the small difference r_{12}: $A_2\{R\} + r_{12}^{\zeta_2} C_2\{R\} + r_{12}^{2\zeta_2} D_2\{R\} + r_{12}^2 E_2\{R\} + \cdots$, where A_2, C_2, D_2, etc. are some functions of the large separations of the order of R. When we use this solution to compute S_{2n} the leading contribution $A\{R\}$ drops, and we find (4) for $p = 2$. For $p > 2$ we need to distinguish between even and odd p's because of the special property of passive advection in which $S_{2n+1} = 0$. The next even p is $p = 4$. For this case we find a solution in the form

$$\begin{aligned} F_4^c = {} & A_4\{R\} + C_4\{R\} \left[\sum_{\alpha\alpha'=1}^{4} r_{\alpha\alpha'}^{\zeta_2} \right] \\ & + D_4\{R\} [(r_{12}r_{13})^{\zeta_2} + (r_{12}r_{14})^{\zeta_2} + (r_{12}r_{23})^{\zeta_2} + \cdots] \\ & + F_4^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \Psi_{2n,4}\{R\} + \cdots \,, \end{aligned} \tag{20}$$

where F_4^c is a contribution of a new type, as it solves the homogeneous equation (18). Computing S_{2n} the first two terms disappear, and in a multiscaling situation the leading contribution becomes the last. In fact, this is the general rule for any even order, and is the explicit mechanism for the fusion rules in this particular case. It arises here from the possibility to split the total operator $\hat{\mathcal{B}}_{2n}$ into the two groups $\hat{\mathcal{B}}_p$ and $\hat{\mathcal{B}}$, such that for p coalescing points $\hat{\mathcal{B}}_p$ carries the leading contribution. Since the sum of Laplacians in (18) is also dominated by the sum up to

p, the crossover scale $\eta(p, 2n, R)$ from inertial range to dissipative behavior is determined in this case by a balance between $-\kappa \sum_{\alpha=1}^{p} \nabla_\alpha^2$ and $\hat{\mathcal{B}}_p$. It therefore cannot depend on n or on R: $\eta(p, 2n, R) = \eta(p)$. The fusion rules (14)–(16) which were based on the independence of η on R follow.

In fact, these result and, in particular, the scaling relations (13) seem sufficient to determine the exponents ζ_n in their entirety. The necessary steps were detailed in [8] and will not be repeated here.

The case of Navier-Stokes turbulence calls for additional considerations. The fusion rules (4) and (6) were found from first principles for $p = 2$ [4], and we believe that similar techniques can be used to establish them for any p. Equations (11) and (12) follow, but in the Navier-Stokes case it is possible that the dissipative scale $\eta(p, n, R)$ is R dependent. If we assume that this is not the case, the scaling relations obtained above should apply also to the Navier-Stokes case. To explore another possibility we follow Kolmogorov's refined similarity hypothesis [17] in assuming that the conditional average $\nu\langle|\nabla \mathbf{u}|^2|\mathbf{w}(0|\mathbf{r})\rangle \sim w(0|\mathbf{r})^3/R$. This assumption means

$$\nu J_{2,n}\{R\} \sim S_{n+1}(R)/R\,. \qquad (21)$$

Comparing with (11) this can be consistent only if

$$\eta(2, n, R)/\eta(2)^{2-\zeta_2} \sim (R/L)^{\zeta_n - \zeta_{n-1} + \zeta_3 - \zeta_2}, \qquad (22)$$

where $\eta(2)$ is by definition the viscous cutoff of the second order structure function, $\nu S_2(\eta(2)) \sim \eta(2)^2 \bar{\epsilon}$. The Hölder inequalities guarantee that $\zeta_n - \zeta_{n-1}$ is a decreasing function of n in a multiscaling situation. Accordingly, the effective dissipative scale of $J_{2,n}$ for two coalescing points $\eta(2, n, R)$ is much smaller than the viscous cutoff for S_2, $\eta(2)$. The consequences of the above two assumptions were discussed in detail in [4].

Needless to say, with assumption (21) all our scaling relations change. For example, consider $\xi(2, n)$ of (9). Instead of (13) we now have

$$\xi(2, n) = \zeta_{n+1} - 1\,. \qquad (23)$$

Another example is $K^{(2)}_{\epsilon\epsilon}$. We now find

$$\mu(1) = 2 - \zeta_6\,. \qquad (24)$$

This result is known as "the bridge" in the phenomenonlogical theory of multiscaling turbulence, cf. [18,19]. Notwithstanding the different scaling relation, the operator algebra that is induced has "global" and individual scaling exponents as discussed above. The values of these exponents may be changed due to the R dependence of the dissipative cutoff as it appears in different models.

In summary, we proposed fusion rules for multiscaling turbulent statistics that induce an unusual operator algebra. These rules are of two classes. The first does not involve gradients and is universal for all turbulent systems with a direct cascade of energy in which there exists a universal flux equilibrium. The second class involves gradients, and these bring in an explicit dependence on a viscous scale that, in general, is not universal. We explained why in the case of passive-scalar advection this problem may be solved. Accordingly, one can derive an infinite set of nontrivial scaling relations that allow the expression of the scaling exponents of the correlation functions of gradient fields with the exponents ζ_n of the structure functions. For hydrodynamic turbulence the fusion rules that involve gradients must be supplemented with a Navier-Stokes based theory for the R dependence of the viscous cutoff, which is our next project.

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