Scaling Behavior in Turbulence is Doubly Anomalous

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It is shown that the description of anomalous scaling in turbulent systems requires the simultaneous use of two normalization scales. This phenomenon stems from the existence of two independent (infinite) sets of anomalous scaling exponents that appear in leading order, one set due to infrared anomalies and the other due to ultraviolet anomalies. To expose this clearly we introduce here a set of local fields whose correlation functions depend simultaneously on the two sets of exponents. Thus the Kolmogorov picture of "inertial range" scaling is shown to fail because of anomalies that are sensitive to the *two ends* of this range. [S0031-9007(96)00258-X]

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Anomalous multiscaling in turbulence is usually discussed [1,2] in terms of the simultaneous structure functions of velocity differences across a scale R,

$$\tilde{S}_n(R) \equiv \langle |\mathbf{u}(\mathbf{r} + \mathbf{R}) - \mathbf{u}(\mathbf{r})|^n \rangle \simeq (\bar{\epsilon}R)^{n/3}(L/R)^{\delta_n}, \quad (1)$$

where $\langle \cdots \rangle$ stands for a suitably defined ensemble average, $\bar{\epsilon}$ is the mean energy flux per unit time per unit mass, and δ_n is the deviation of the scaling exponent ζ_n of the structure function from the 1941 Kolmogorov (K41) prediction $\zeta_n \equiv n/3 - \delta_n$. Since K41 follows from dimensional analysis [3], deviations require a renormalization scale, and it is accepted [1,2,4–6] that in $\tilde{S}_n(R)$ it is the outer scale of turbulence L that serves this purpose. The same renormalization scale appears in the correlation function of the energy dissipation rate $\epsilon(\mathbf{r}, t)$ (which is roughly $\nu|\nabla\mathbf{u}(\mathbf{r}, t)|^2$ with ν the kinematic viscosity) [7]:

$$\tilde{K}_{\epsilon\epsilon}(R) = \langle (\epsilon(\mathbf{r} + \mathbf{R}) - \bar{\epsilon})(\epsilon(\mathbf{r}) - \bar{\epsilon}) \rangle \simeq \bar{\epsilon}^2(L/R)^{\mu}, \quad (2)$$

where μ is known as the "intermittency exponent" [7]. The appearance of the outer renormalization scale in Eqs. (1) and (2) has been correctly interpreted as a failure of the K41 basic assumption of inertial range scaling. The aim of the Letter is to discuss infinite sets of local turbulent fields whose correlation functions require *two* simultaneous renormalization scales, L and η where η is the viscous scale. One set of these local fields will be denoted below as $\mathbf{L}_l(\mathbf{r})$ where l is an index that takes on integer values. The central result of this Letter is that to leading order the correlation functions of these fields scale like

$$\langle \mathbf{L}_l(\mathbf{r} + \mathbf{R})\mathbf{L}_{l'}(\mathbf{r}) \rangle \sim \frac{(\bar{\epsilon}R)^{4/3}}{\eta^{l+l'}} \left(\frac{\eta}{R}\right)^{\beta_l + \beta_{l'}} \left(\frac{L}{R}\right)^{\delta_4}. \quad (3)$$

Other local fields exhibit other exponents from the family ζ_n. The point is that these correlation functions demonstrate that K41 fails doubly, once because of infrared and once due to ultraviolet anomalies. This double anomaly results, in addition to an infinite set of multiscaling exponents ζ_n, with a second infinite set of exponents that are denoted here as β_l. In fact, one can separately measure two types of exponents which govern the anomalous scaling related to the L and η renormalization scales [in Eq. (3) these are $\beta_l + \beta_{l'}$ and δ_4] by keeping the outer scale L constant and varying R and η. The latter can be done by varying the outer velocity and using the dependence of the ratio L/η on the Reynolds number. The scale η can be changed by varying the viscosity (for example, by temperature control in a helium gas near the critical point, e.g., [8]). Theoretically we can compute β_l exactly in simple models of scalar turbulent advection. The phenomenon of doubly anomalous scaling occurs, however, in a similar fashion in Navier-Stokes turbulence where we can estimate β_1 and β_2. Since the development of the ideas is simpler in the case of scalar fields, we will present them in the context of scalar advection and generalize later to turbulent vector fields.

First we generate local fields that originate from the fusion of two points. Consider for that a turbulent scalar field $T(\mathbf{r}, t)$ and the product of two such fields at two adjacent points

$$\Psi(\rho, \mathbf{r}) \equiv T(\mathbf{r} + \rho/2)T(\mathbf{r} - \rho/2). \quad (4)$$

It is advantageous to represent this field as a multipole expansion $\Psi(\rho, \mathbf{r}) = \sum_{l=0}^{\infty} \Psi_l(\rho, \mathbf{r})$, where

$$\Psi_l(\rho, \mathbf{r}) = \sum_{m=-l}^{l} Y_{lm}(\hat{\rho}) \int \Psi(\rho\hat{\xi}, \mathbf{r}) Y_{lm}(\hat{\xi})\, d\hat{\xi}. \quad (5)$$

Here $\hat{\rho} = \rho/\rho$ and $\hat{\xi}$ are unit vectors. The orthonormal spherical harmonics $Y_{lm}(\hat{\rho})$ are the eigenfunctions of the angular momentum operator $\hat{L} = -i\rho \times \nabla$ which depend only on the direction of ρ: $\hat{L}^2 Y_{lm}(\hat{\rho}) = l(l+1)Y_{lm}(\hat{\rho})$. Next we represent $\Psi_l(\rho, \mathbf{r})$ in terms of (infinitely many) local fields depending on $\mathbf{r}$ only. To this aim we expand $\Psi(\rho, \mathbf{r})$ in a Taylor series in ρ. This turns Eq. (5) into

$$\Psi_l(\rho, \mathbf{r}) \equiv \sum_{m=-l}^{l} Y_{lm}(\hat{\rho}) \int d\hat{\xi}\, Y_{lm}(\hat{\xi}) \times \sum_{n=0}^{\infty} \frac{\rho^{2n}}{(2n)!} (\hat{\xi} \cdot \nabla')^{2n} \Psi(\rho', \mathbf{r}) \Big|_{\rho'=0}. \quad (6)$$

Here and below we use the operator $\nabla'_\alpha = \partial/\partial\rho'_\alpha$. Note that we have only even n orders since our field $\Psi(\rho, \mathbf{r})$ is even in ρ. Performing the angular integrations and recollecting terms we end up with

$$\Psi_l(\rho, \mathbf{r}) = \rho_{\alpha_1}\rho_{\alpha_2}\cdots\rho_{\alpha_l} \sum_{p=0}^{\infty} \times \frac{a_{l+2p,2p}}{(l+2p)!}\rho^{2p} L_l^{\alpha_1\alpha_2\cdots\alpha_l}(\mathbf{r}).$$

All the coefficients $a_{m,n}$ here can be computed explicitly. For example, $a_{p,p} = 1$ for any p, $a_{2,0} = \frac{1}{3}$, $a_{4,0} = -\frac{3}{35}$, etc. We have introduced the tensorial local fields $\mathbf{L}_{l,p}$, where

$$L_{l,p}^{\alpha_1\cdots\alpha_l}(\mathbf{r}) \equiv \nabla^{2p} D_l^{\alpha_1\alpha_2\cdots\alpha_l} T\left(\mathbf{r} + \frac{\rho}{2}\right) T\left(\mathbf{r} - \frac{\rho}{2}\right)\Bigg|_{\rho=0}, \tag{7}$$

with $\hat{\mathbf{D}}_l(\mathbf{r})$ being local differential operators. For the first values of l these differential operators are

$$\hat{D}_0 = 1, \quad \hat{D}_2^{\alpha\beta} = \nabla_\alpha\nabla_\beta - \tfrac{1}{3}\nabla^2\delta_{\alpha\beta},$$
$$\hat{D}_4^{\alpha\beta\gamma\delta} = \nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\delta - \tfrac{1}{7}\nabla^2(\delta_{\alpha\beta}\nabla_\gamma\nabla_\delta + \delta_{\alpha\gamma}\nabla_\beta\nabla_\delta + \delta_{\alpha\beta}\nabla_\beta\nabla_\gamma + \delta_{\beta\gamma}\nabla_\alpha\nabla_\delta + \delta_{\beta\delta}\nabla_\alpha\nabla_\gamma + \delta_{\gamma\delta}\nabla_\alpha\nabla_\delta) + \tfrac{1}{35}\nabla^4(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{8}$$

Here $\nabla_\alpha = \partial/\partial\rho_\alpha$. The field $L_{0,1}$ is the dissipation field. Readers familiar with the representations of Lie groups recognize immediately that our local fields $\mathbf{L}_{l,0}$ are nothing but the $2l + 1$ rank irreducible representations of the SO(3) group [9]. This explains the meaning of the index l: Tensorial fields $\mathbf{L}_{l,p}$ (for any p) have the same transformation properties under rotation of the coordinate system as the spherical harmonics Y_{lm}. The procedure described above is a regular way to find such differential fields. The fact that fields $\mathbf{L}_{l,0}$ give irreducible representation of symmetry groups of the problem is the mathematical reason why these fields demonstrate "clean" scaling behavior. Note also that according to (7) and (8) fields $\mathbf{L}_{l,p}$ have $(l+2p)$-order differential operator, ∇^{l+2p}. The tensor fields thus obtained are symmetric to any pairwise exchange of indices. We will propose now that these gradient fields have η-related anomalous scaling which is governed by a set of anomalous exponents β_l. Autocorrelation functions of these fields, and correlation functions of these fields together with field differences across a scale R depend also on R/L with exponents determined by the set ζ_n.

To study the correlation of the newly defined local fields with the fundamental T field consider the following correlation function of the tensorial field $\mathbf{L}_{l,p}$ with $2n - 2$ scalar T fields:

$$\mathbf{C}_{2n,l,p}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) \equiv \langle \mathbf{L}_{l,p}(\mathbf{r}) T(\mathbf{r}_3)\cdots T(\mathbf{r}_{2n})\rangle. \tag{9}$$

Note that in this correlation function ρ does not appear. However, it is related to the standard $2n$-point correlation function in which two coordinates (say $\mathbf{r}_1$ and $\mathbf{r}_2$) are separated by a small distance ρ. By definition

$$\mathcal{F}_{2n}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) = \langle \Psi(\rho, \mathbf{r}) T(\mathbf{r}_3)\cdots T(\mathbf{r}_{2n})\rangle.$$

To connect these functions and (9) we represent $\mathcal{F}_{2n}$ as a multipole decomposition $\mathcal{F}_{2n} = \sum_{l=0}^{\infty}\mathcal{F}_{2n,l}$. Using (5) we have

$$\mathcal{F}_{2n,l}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) = \langle \Psi_l(\rho, \mathbf{r}) T(\mathbf{r}_3)\cdots T(\mathbf{r}_{2n})\rangle. \tag{10}$$

We are interested in the scaling properties of this quantity in the regime in which all the separations between all the coordinates $\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}$ are of the order of R. For $\rho \ll R$ we can write

$$\mathcal{F}_{2n,l}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) \sim \left(\frac{\rho}{R}\right)^{x_l} S_{2n}(R), \tag{11}$$

where x_l is a yet unknown exponent which in general may also depend on n. This exponent will be found below in a particular model and will be shown to be n independent. For ρ very small we can use (7) and (9) to write

$$\mathcal{F}_{2n,l}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) = \rho_{\alpha_1}\rho_{\alpha_2}\ldots\rho_{\alpha_l} \times \sum_{p=0}^{\infty}\frac{a_{l+2p,2p}}{(l+2p)!}\rho^{2p} C_{2n,l,p}^{\alpha_1\ldots\alpha_l}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}). \tag{12}$$

Finally, in the limit $\rho \ll \eta$ we use the fact that $\mathcal{F}_{2n}$ is smooth in ρ up to $\rho \sim \eta$ to evaluate the differential operator as divisions by η: $\rho^{2p}\nabla^{2p} \sim (\rho/\eta)^{2p}$. Accordingly, we have in the limit

$$\lim_{\rho\to 0}\mathcal{F}_{2n,l}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) = \rho_{\alpha_1}\rho_{\alpha_2}\cdots\rho_{\alpha_l}\frac{a_{l,0}}{l!} C_{2n,l,0}^{\alpha_1\ldots\alpha_l}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) \propto \rho^l. \tag{13}$$

Next we want to explore the scaling behavior of $\mathcal{F}_{2n,l}$ for values of ρ in the inertial range $\eta \ll \rho \ll L$. This we cannot do in general. We now need to specialize to a particular dynamical model. We choose Kraichnan's model of passive advection of a scalar field $T(\mathbf{r}, t)$ by a random velocity field whose statistics are Gaussian, and whose correlation functions are scale invariant in space and δ correlated in time [10,11]. The relevance of the results to Navier-Stokes turbulence will be discussed later. For a scalar diffusivity κ the dissipation field is $\epsilon(\mathbf{r}) \equiv \kappa|\nabla T|^2$ and the quantities (1) and (2) are replaced by

$$S_{2n}(R) \equiv \langle |T(\mathbf{R}) - T(0)|^{2n}\rangle \simeq [S_2(R)]^n (L/R)^{\delta_n}, \tag{14}$$

$$K_{\epsilon\epsilon}(R) = \langle(\epsilon(\mathbf{R}) - \bar{\epsilon})(\epsilon(0) - \bar{\epsilon})\rangle \simeq \bar{\epsilon}^2(L/R)^{\mu}. \tag{15}$$

In the present case the scaling exponent of S_{2n} is $\zeta_{2n} = n\zeta_2 - \delta_n$.

It was shown in [12] that the correlation function $\mathcal{F}_{2n}$ solves a particularly simple equation when two of its coordinates (say $\mathbf{r}_1$ and $\mathbf{r}2$) are much closer to one another than all the rest. Explicitly, for ρ small the ρ dependence of this function is governed by the equation

$$\hat{\mathcal{B}}(\rho)\mathcal{F}_{2n}(\mathbf{r} + \rho/2, \mathbf{r}-\rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) = \Phi_{2n-2}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}). \quad (16)$$

Here $\Phi_{2n-2}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n})$ is a homogeneous function with scaling exponent $\zeta_{2n} - \zeta_2$. In three dimensions the operator $\hat{\mathcal{B}}(\rho)$ is given by [10,12]

$$\hat{\mathcal{B}}(\rho) \equiv H\left[\frac{\partial}{\rho^2 \partial\rho}\rho^{4-\zeta_2}\frac{\partial}{\partial\rho} - \frac{(4-\zeta_2)}{2\rho^{\zeta_2}}\hat{L}^2\right]. \quad (17)$$

Here H is a constant. It has been shown [12,13] that the leading scaling solution for the ρ dependence of function $\mathcal{F}_{2n}$ is an eigenfunction of the operator $\hat{\mathcal{B}}(\rho)$ with eigenvalue 0 and thus can be expanded in spherical harmonics by

$$\mathcal{F}_{2n}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}^{(2n)}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_n)\rho^{\beta_l}Y_{l,m}(\hat{\rho}), \quad (18)$$

where $A_{lm}^{(2n)}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_n)$ is a homogeneous function whose scaling exponent is $\zeta_{2n} - \beta_l$. To compute the exponents β_l we note that the (right-hand side) of (16) is ρ independent, and therefore contributes only when $l = 0$. In this case we can compute β_0 by power counting with the result $\beta_0 = \zeta_2$. For $l \neq 0$ we need to find a solution of the homogeneous part of (16). By a direct substitution of (18) into the (left-hand side) of (16) one finds $\beta_l(\beta_l + 3 - \zeta_2) = (4 - \zeta_2)l(l + 1)/2$. Note that the lhs of this relation originates from the radial part of the operator $\hat{\mathcal{B}}$, whereas the rhs results from the angular part that is proportional to $\hat{L}^2$. Solving the quadratic equation for β_l we find in three dimensions [12,13]

$$\beta_l = \tfrac{1}{2}\left[\zeta_2 - 3 + \sqrt{(3-\zeta_2)^2 + 2l(l+1)(4-\zeta_2)}\right]. \quad (19)$$

The multipole decomposition of (18), similarly to (5), leads to

$$\mathcal{F}_{2n,l}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) = \rho^{\beta_l}\sum_{m=-l}^{l} Y_{lm}(\hat{\rho})A_{lm}^{(2n)}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_n). \quad (20)$$

In the situations in which all the separations between the coordinates $\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_n$ are of the same order of magnitude R, and $R \gg \rho \gg \eta$ we can write

$$\mathcal{F}_{2n,l}(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) \propto \rho^{\beta_l}R^{\zeta_{2n}-\beta_l}. \quad (21)$$

Comparing with Eq. (11) we identify the exponent x_l as β_l and write the final form,

$$\mathcal{F}_{2n,l}(\mathbf{r} + \frac{\rho}{2}, \mathbf{r} - \frac{\rho}{2}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) \sim \left(\frac{\rho}{R}\right)^{\beta_l} S_{2n}(R). \quad (22)$$

At this point we want to match solution (22), which is valid for $\rho \gg \eta$, with solution (13) which is valid for $\rho \ll \eta$. This can be done if the solution is varying continuously across η without any non-monotonic behavior. The rigorous proof of this property is beyond the scope of this Letter. It can be demonstrated numerically by solving the ordinary differential equation (16). Equating (21) and (13) for $\rho = \eta$ (up to an unknown R-independent coefficient) we find

$$\mathbf{C}_{2n,l,0}(\mathbf{r}, \mathbf{r}_3, \ldots, \mathbf{r}_{2n}) \sim \frac{1}{\eta^l}\left(\frac{\eta}{R}\right)^{\beta_l} S_{2n}(R), \quad (23)$$

where we remind the reader that R stands for the order of magnitude of all the separations between the coordinates of $\mathbf{C}_{2n,l,0}$. Comparing with Eq. (14) we conclude that the correlation function of $\mathbf{L}_{l,0}$ with any even number of T fields separated by distances of the order of R depends simultaneously on two renormalization scales, η and L, and on the two sets of anomalous exponents β_l and ζ_n.

Next examine a cross correlation of two (generally different) local fields. Repeating the analysis one finds Eq. (3). We see that, in general, such correlations depend on the two renormalization scales and on two sets of exponents. It is therefore interesting to ask why this phenomenon is absent in $K_{\epsilon\epsilon}$ which is closely related to such correlation functions. We note that in our terms the correlation (15) is given by $K_{\epsilon\epsilon}(R) = \kappa^2\langle\mathbf{L}_{0,1}(\mathbf{r} + \mathbf{R})\mathbf{L}_{0,1}(\mathbf{r})\rangle$ as can be checked by substituting the definition of the local fields. This is a very special case among the correlations of the local fields. Using the fact that $\beta_0 = \zeta_2$, and taking into account that for $p = 1$ in Eq. (7) we have two derivatives, it follows that in this case [11–13]

$$K_{\epsilon\epsilon}(R) \sim \frac{\kappa^2}{\eta^4}\left(\frac{\eta}{R}\right)^{2\zeta_2} S_4(R) \sim \bar{\epsilon}^2\left(\frac{L}{R}\right)^{2\zeta_2-\zeta_4}. \quad (24)$$

In the last step we used the fact that by definition $\bar{\epsilon} = -\kappa \lim_{|\mathbf{r}_1-\mathbf{r}_1|\to\eta}\nabla_1\nabla_2\mathcal{F}(\mathbf{r}_1, \mathbf{r}_2)$. Since $\mathcal{F}(\mathbf{r}_1, \mathbf{r}_2) \sim |\mathbf{r}_1 - \mathbf{r}_2|^{\zeta_2}$ we get $\bar{\epsilon} \propto \kappa\eta^{\zeta_2-2}$. This leads directly to the final step in (24), in which the renormalization scale η disappears from the correlator. The deep reason for this is that this is the rate of dissipation of the integral of motion in the passive scalar problem, and therefore it is independent of the value of the diffusivity. Only such a combination of η and κ can appear that cancels in favor of the constant $\bar{\epsilon}$. In this sense $K_{\epsilon\epsilon}$ is unusual, and all the generic correlations (3) are simultaneously dependent on two renormalization scales.

One can generate more local fields that will have scaling properties which may depend on new exponents. Instead of starting with the fusion of two points we can fuse three, four, or more points [14]. Instead of (4) we can introduce

$$\Psi_3(\rho_1, \rho_2, \mathbf{r}) \equiv T(\mathbf{r} + \rho_1)T(\mathbf{r} + \rho_2)T(\mathbf{r} - \rho_1 - \rho_2), \quad (25)$$

$\Psi_4 \propto T^4$ etc. Expanding these fields in Taylor series with respect to ρ_1, ρ_2, etc., we can generate new sets of local fields that contain derivatives of three, four, etc., T fields.

Their correlation functions will depend on the ultraviolet exponents which appear due to three-point, four-point, etc., coalescing clusters, and on the infrared scaling exponents of six, eight, and more, point correlation functions. Of course, the actual values of the exponents depend on the dynamical model, but the structure of the theory is general. To stress this generality we now make a few comments about the Navier-Stokes problem. In dealing with Navier-Stokes turbulence we need to worry from the beginning about Galilean invariance in addition to the SO(3) symmetry group. To this aim we will consider local fields that originate from the fusion of gradient fields. The simplest object is

$$\Psi_2^{\alpha\beta\gamma\delta}(\rho,\mathbf{r}) \equiv \frac{\partial u_\alpha(\mathbf{r}+\rho/2)}{\partial\rho_\beta}\frac{\partial u_\gamma(\mathbf{r}-\rho/2)}{\partial\rho_\delta}. \quad (26)$$

From this point on we can proceed following the route sketched above for the scalar case. Representing this field as a multipole decomposition with respect to the direction of ρ, and considering the Taylor expansion in ρ, we can generate infinitely many local fields. These fields have two $\mathbf{u}$ fields and as many gradients as we want to consider, starting from two. It is interesting to notice that in the present case we have two different vectors, i.e., $\boldsymbol{\nabla}$ and $\mathbf{u}$, from which we can form antisymmetric combinations, like the vorticity $\omega_\alpha = \epsilon_{\alpha\beta\gamma}\partial u_\beta/\partial r_\gamma$ (where $\epsilon_{\alpha\beta\gamma}$ is the fully antisymmetric tensor). Consequently we will have odd as well as even l components in this scheme. In addition we have symmetric combinations of velocity derivatives like the strain tensor $s_{\alpha\beta} = [\partial u_\alpha/\partial r_\beta + \partial u_\beta/\partial r_\alpha]/2$. In general the tensor (26) has 36 independent components, serving as a basis for a 36-dimensional reducible representation of the O(3) group [SO(3) + inversion]. This basis may be decomposed into a set of irreducible bases of lower dimensions. There are two scalar fields, $\omega_\alpha\omega_\alpha$ and $s^2 = s_{\alpha\beta}s_{\beta\alpha}$, each of which is a basis for one-dimensional irreducible representation with $l = 0$. The pseudovector $s_{\alpha\beta}\omega_\beta$ is a three-dimensional basis for an irreducible representation with $l = 1$. There exist three traceless tensor fields, each of which is a five-dimensional basis belonging to $l = 2$ and taking care of $3 \times 5 = 15$ components. An example is

$$O_2^{\alpha\beta}(\mathbf{r}) = \omega_\alpha(\mathbf{r})\omega_\beta(\mathbf{r}) - \delta_{\alpha\beta}\omega^2(\mathbf{r})/3. \quad (27)$$

In addition, we have one 3-rank pseudotensor corresponding to $l = 3$ and one 4-rank tensor corresponding to $l = 4$. The last two fields exhaust the remaining $7 + 9$ components. As in the scalar case there are fields with all values of l which are obtained when more gradients act on our field (26). Finally, we can also start with a higher number of fusing gradient fields $\partial u_\alpha/\partial\rho_\beta$ to generate new sets of local fields having three, four, and more velocity fields. The exploration and utilization of this rich structure is beyond the scope of this Letter. It will suffice here to state that these local fields will have correlation function with anomalous scaling properties that generally depend on two renormalization scales and on two sets of anomalous scaling exponents. The correlator (2) will again be special and η independent since it involves the rate of dissipation of the integral of motion (energy). Correlations of fields with $l \neq 0$ will be generic. For example, the correlation of $\nu\mathbf{O}_2$ with $\epsilon = 2\nu s^2$ is

$$\nu^2\langle O_2^{\alpha\beta}(\mathbf{r}+\mathbf{R})s^2(\mathbf{r})\rangle \sim \bar{\epsilon}^2\left(\frac{L}{R}\right)^x\left(\frac{\eta}{R}\right)^y. \quad (28)$$

Our guess is that x is numerically close to μ and that y is numerically close to $\frac{2}{3}$, with an accuracy which is of the order of the difference between ζ_2 and its K41 estimate of $\frac{2}{3}$. We stress, however, that the main point of this Letter is *not* the numerical value of this or that exponent, but that normal scaling, which is based on dimensional analysis (like K41 for Navier-Stokes turbulence), fails doubly due to the explicit appearance of two physically important scales, the inner *and* the outer renormalization scales.

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