Nonperturbative zero modes in the Kraichnan model for turbulent advection

Omri Gat, Victor S. L'vov, Evgenii Podivilov, and Itamar Procaccia
Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
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The anomalous scaling behavior of the nth order correlation functions $\mathcal{F}_n$ of the Kraichnan model of turbulent passive scalar advection is believed to be dominated by the homogeneous solutions (zero modes) of the Kraichnan equation $\hat{\mathcal{B}}_n\mathcal{F}_n=0$. Previous analysis found zero modes in perturbation theory with respect to a small parameter. We present a computer-assisted analysis of the simplest nontrivial case of $n=3$: we demonstrate nonperturbatively the existence of anomalous scaling, and compare the results with the perturbative predictions. [S1063-651X(97)51104-4]

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The Kraichnan model of turbulent passive scalar advection [1] has attracted enormous attention recently [2–6] as a nontrivial model of turbulent statistics in which the phenomenon of multiscaling appears to be analytically derivable. The model describes an advected field $T(\boldsymbol{r},t)$ satisfying the equation of motion

$$\frac{\partial T(\boldsymbol{r},t)}{\partial t}+\boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla T(\boldsymbol{r},t)=\kappa\nabla^2T(\boldsymbol{r},t)+\xi(\boldsymbol{r},t). \qquad (1)$$

Here $\xi(\boldsymbol{r},t)$ is a Gaussian white random force, κ is the diffusivity, and the driving field $\boldsymbol{u}(\boldsymbol{r},t)$ is chosen to have Gaussian statistics, and to be "fast varying" in the sense that its time correlation function is proportional to $\delta(t)$. The statistical quantities that one is interested in are the many point correlation functions

$$\mathcal{F}_{2n}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_{2n})\equiv\langle\langle T(\boldsymbol{r}_1,t)T(\boldsymbol{r}_2,t)\cdots T(\boldsymbol{r}_{2n},t)\rangle\rangle, \qquad (2)$$

where double angular brackets denote an ensemble average with respect to the stationary statistics of the forcing *and* the statistics of the velocity field. One of Kraichnan's major results [2] is an exact differential equation for this correlation function,

$$\left[-\kappa\sum_{\alpha}\nabla^2_{\alpha}+\hat{\mathcal{B}}_{2n}\right]\mathcal{F}_{2n}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_{2n})=(\text{right-hand side}), \qquad (3)$$

where the right-hand side is known explicitly, but does not need to be written down here for reasons to be stated momentarily. The operator $\hat{\mathcal{B}}_{2n}\equiv\Sigma_{\alpha>\beta}^{2n}\hat{\mathcal{B}}_{\alpha\beta}$, where $\hat{\mathcal{B}}_{\alpha\beta}$ is defined by

$$\hat{\mathcal{B}}_{\alpha\beta}\equiv\hat{\mathcal{B}}(\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta)=h_{ij}(\boldsymbol{r}_\alpha-\boldsymbol{r}_\beta)\partial^2/\partial r_{\alpha,i}\partial r_{\beta,j}; \qquad (4)$$

the "eddy-diffusivity" tensor $h_{ij}(\boldsymbol{R})$ is given by

$$h_{ij}(\boldsymbol{R})=h(R)[(\zeta_h+d-1)\delta_{ij}-\zeta_hR_iR_j/R^2],$$

and $h(R)=H(R/\mathcal{L})^{\zeta_h}$, $0\leqslant\zeta_h\leqslant2$. Here $\mathcal{L}$ is some characteristic outer scale of the driving velocity field. The scaling properties of the scalar depend sensitively on the scaling exponent ζ_h that characterizes the R dependence of $h_{ij}(\boldsymbol{R})$, and can take values in the interval [0,2].

Now, an important point needs to be made. It was claimed that in the inertial interval one can neglect the Laplacian operators in Eq. (3). Then it has been shown [3,4,6] that the solutions of Eq. (3) for $n>1$ are dominated by the homogeneous solutions ("zero modes"). This means that deep in the inertial interval the inhomogeneous solutions are negligible compared to the homogeneous ones; we thus need to consider the simpler homogeneous equation $\hat{\mathcal{B}}_{2n}\mathcal{F}_{2n}=0$.

Having exact differential equations for $\mathcal{F}_{2n}$ allowed Kraichnan to announce a mechanism for anomalous scaling [2]. Scaling implies that the physical solutions are scale invariant, in which case one may define a scaling (or homogeneity) exponent ζ_{2n} of $\mathcal{F}_{2n}$ by $\mathcal{F}_{2n}(\lambda\boldsymbol{r}_1,\lambda\boldsymbol{r}_2\cdots\lambda\boldsymbol{r}_{2n})=\lambda^{\zeta_{2n}}\mathcal{F}_{2n}(\boldsymbol{r}_1,\boldsymbol{r}_2\cdots\boldsymbol{r}_{2n})$. One expects this to hold in the inertial range, i.e., over the range of scales, where the separations r_{ij} satisfy $\eta\ll r_{ij}\ll L$, where η and L are the inner and outer scales, respectively. It is known [1] that such a solution exists for $\mathcal{F}_2$ with $\zeta_2=2-\zeta_h$. If one can determine these exponents for $n>1$, one can understand, at least in this simple model, what are the mechanisms for deviations from the predictions of dimensional analysis. In searching for methods for computing these exponents, two basic strategies have emerged. One strategy considered the differential equation in the "fully unfused" regime in which all the separations between the coordinates are in the inertial range. Then even in the simplest case of $n=2$ the function $\mathcal{F}_4$ depends on six independent variables (for dimensions $d>2$), and one faces a formidable analytic difficulty to determine exact solutions. Accordingly, several groups have considered perturbative solutions in some small parameter, such as ζ_h [3] or the inverse dimensionality $1/d$ [4]. The rationale for this approach is that at $\zeta_h=0$ and $d\to\infty$ one expects "simple scaling" with $\zeta_{2n}=n\zeta_2$. The exponents ζ_4 ζ_{2n} have been computed as a function of ζ_h near these simple scaling limits. The second approach considered the differential equation in the "fully fused" regime, in which the correlation function degenerates to the structure function $S_{2n}(R)=\langle\langle[T(\boldsymbol{r}+\boldsymbol{R})-T(\boldsymbol{r})]^{2n}\rangle\rangle$. This method gives an enormous simplification in having only one variable, but one loses information in the process of fusion. The lost information was supplemented [2] by an as yet underived conjecture about the properties of conditional averages, leading to a closed-form calculation of the exponents ζ_{2n} for arbitrary dimension and values of ζ_h. The results of the two strategies are not in agreement. Although both numerical simulations [5] and experiments [7,8] lend support to the assumption used in the second strategy and to the resulting values of ζ_{2n}, an important mystery remains as to why the two approaches reach such different conclusions. The aim of this paper is to explore nonperturbative calculations of the zero modes and their exponents. We will offer the first nonperturbative demonstration of the existence of anomalous scaling in a homogenous equation for a correlation function.

Our strategy is to solve exactly, including eigenfunctions, the homogeneous equation satisfied by the third order correlation function $\mathcal{F}_3(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$ in the isotropic sector. Note that in Kraichnan's model all the odd-order correlation functions $\mathcal{F}_{2n+1}$ are zero due to symmetry under the transformation $T\to -T$. This symmetry disappears, for example [9], if the random force $\xi(\boldsymbol{r},t)$ is not Gaussian (but δ correlated in time), in particular if it has a nonzero third order correlation

$$\mathcal{D}_3(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)\equiv\int dt_1 dt_2\langle\xi(\boldsymbol{r}_1,t_1)\xi(\boldsymbol{r}_2,t_2)\xi(\boldsymbol{r}_3,0)\rangle. \quad (5)$$

With such a forcing the third order correlator is nonzero, and it satisfies the equation

$$\hat{\mathcal{B}}_3\mathcal{F}_3(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)=\mathcal{D}_3,\quad \hat{\mathcal{B}}_3\equiv\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{13}+\hat{\mathcal{B}}_{23}. \quad (6)$$

As this equation pertains to the inertial interval we have neglected the Laplacian operators. We also denoted $\mathcal{D}_3 =\lim_{\boldsymbol{r}_{\alpha\beta}\to 0}\mathcal{D}_3(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$. The solution of this equation is a sum of inhomogeneous and homogeneous contributions, and below we study the latter. We will focus on scale invariant homogeneous solutions that satisfy $\mathcal{F}_3(\lambda\boldsymbol{r}_1,\lambda\boldsymbol{r}_2,\lambda\boldsymbol{r}_3) =\lambda^{\zeta_3}\mathcal{F}_3(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$. We refer to these as the "zero modes in the scale invariant sector." We note that the scaling exponent of the *inhomogeneous* scale invariant contribution can be read directly from power counting in Eq. (6) (leading to $\zeta_3=\zeta_2$). Any other scaling exponent can arise only from homogeneous solutions that do not need to balance the constant right-hand side. In addition, note that scale-invariant zero modes arise not only due to the omission of the diffusive terms from Eq. (6), but also as a result of the omission of the boundary conditions for large separations (at the outer scale L). The smooth connection to either small or large scales must ruin scale invariance at least at these scales. The scale-invariant solutions of Eq. (6) live in a projective space whose dimension is lowered by unity compared to the most general form; these solutions do not depend on three separations but rather on two dimensionless variables that are identified below. It will be demonstrated how boundary conditions arise in this space for which the operator $\hat{\mathcal{B}}_3$ is neither positive nor self-adjoint.

Equation (6) is also invariant under the action of the d dimensional rotation group SO(d), and under permutations of the three coordinates. Here we seek solutions in the scalar representation of SO(d), where the solution depends only on the three separations r_{12}, r_{23} and r_{31}. We transform coordinates to the variables $x_1=|\boldsymbol{r}_2-\boldsymbol{r}_3|^2$, $x_2=|\boldsymbol{r}_3-\boldsymbol{r}_1|^2$, $x_3=|\boldsymbol{r}_1-\boldsymbol{r}_2|^2$. The triangle inequalities in the original space are equivalent to the condition

$$2(x_1x_2+x_2x_3+x_3x_1)\geqslant x_1^2+x_2^2+x_3^2. \quad (7)$$

The advantage of the new coordinates is that the inequality (7) describes a circular cone in the x_1, x_2, x_3 space whose axis is the line $x_1=x_2=x_3$ and whose circular cross section is tangent to the planes $x_1=0$, $x_2=0$, and $x_3=0$. This cone can be parametrized by three new coordinates s, ρ, ϕ:

$$x_n=s\{1-\rho\cos[\phi+(2\pi/3)n]\},$$
$$0\leqslant s<\infty,\quad 0\leqslant\rho\leqslant 1,\quad 0\leqslant\phi\leqslant 2\pi. \quad (8)$$

The s coordinate measures the overall scale of the triangle defined by the original $\boldsymbol{r}_i$ coordinates, and configurations of constant ρ and ϕ correspond to similar triangles. The ρ coordinate describes the deviation of the triangle from the equilateral configuration ($\rho=0$) up to the physical limit of three collinear points attained when $\rho=1$; ϕ does not have a simple geometric meaning.

The transformation of the linear operator $\hat{\mathcal{B}}_3$ to the new coordinates is straightforward, and produces a second order linear partial differential operator in the s, ρ, ϕ variables (the full form of the operator is long and will not be given here). It suffices to note that the scale-invariant solution takes the form $s^{\zeta_3/2}f(\rho,\phi)$, and the transformed operator applied to this form gives an equation for $f(\rho,\phi)$:

$$\begin{aligned}\hat{B}_3(\zeta_3)f(\rho,\phi)=[&a(\rho,\phi)\partial_\rho^2+b(\rho,\phi)\partial_\phi^2+c(\rho,\phi)\partial_\rho\partial_\phi\\&+u(\rho,\phi,\zeta_3)\partial_\rho+v(\rho,\phi,\zeta_3)\partial_\phi\\&+w(\rho,\phi,\zeta_3)]f(\rho,\phi)=0.\end{aligned} \quad (9)$$

The new operator $\hat{B}_3$ depends on ζ_3 as a parameter and it acts on the unit circle described by the polar ρ,ϕ coordinates. The circle represents the projective space of the physical cone described above.

The discrete permutation symmetry of the original Eq. (6) leads to a symmetry of Eq. (9) with respect to the six element group generated by the transformation $\phi\to\phi+2\pi/3$ (cyclic permutation of the coordinates in physical space) and $\phi\to-\phi$ (exchange of coordinates). This symmetry extends to a full U(1) symmetry in the two marginal cases of $\zeta_h=0$ and $\zeta_h=2$ for which all the coefficients in Eq. (9) become ϕ independent. The coefficients in Eq. (9) all have a similar structure, and, for example, $a(\rho,\phi)$ reads

$$a(\rho,\phi)=\sum_n [1-\rho\cos(\phi+\tfrac{2}{3}\pi n)]^{(\zeta_h-2)/2}\tilde{a}(\rho,\phi+\tfrac{2}{3}\pi n),$$

where $\tilde{a}(\rho,\phi)$ is a low order polynomial in ρ, $\cos\phi$, and $\sin\phi$, which vanishes at $\rho=1,\phi=0$. We see that the coefficients are analytic everywhere on the circle except at the three points $\rho=1$, $\phi=2\pi n/3$ where $n=0,1,2$. These points correspond to the fusion of one pair of coordinates, and the coefficients exhibit a branch point singularity there. This singularity leads to a nontrivial asymptotic behavior of the solutions that had been described before in terms of the fusion rules [6,11]. Note that for $\zeta_h=2$ the singularity disappears trivially. For $\zeta_h=0$ there is also no singularity since $\widetilde{a}$ exactly compensates for the inverse power.

The boundary conditions follow naturally when one realizes that $\hat{B}_3$ is elliptic for points strictly inside the physical circle. On the other hand $\hat{B}_3$ becomes singular on the boundary $\rho=1$, where the coefficients $a(\rho,\phi)$ and $c(\rho,\phi)$ vanish. This singularity reflects the fact that this is the boundary of the physical region. It follows that $\hat{B}_3$ restricted to the boundary becomes a relation between the function $f(\rho=1,\phi)\equiv g(\phi)$ and its normal derivative $\partial_\rho f(\rho,\phi)|_{\rho=1}\equiv h(\phi)$. The relation is $bg''+uh+vg'+wg=0$. Solutions of Eq. (9) that do not satisfy this boundary condition are singular, with infinite ρ derivatives at $\rho=1$. Such solutions are not physical since they involve infinite correlations between the dissipation (second derivative of the field) and the field itself when the geometry becomes collinear, but without fusion.

Given this homogeneous equation with homogeneous boundary conditions we realize that nontrivial solutions are available only when $\det(\hat{B}_3)=0$. This determinant depends parametrically on ζ_3. Since the operator is defined on a compact domain we expect the determinant to vanish only at discrete values of ζ_3 for any given value of ζ_h and dimensionality d. There always exists a trivial constant solution associated with $\zeta_3=0$. Our aim is to find the lowest lying positive real values ζ_3 for which the determinant vanishes.

We approach the problem numerically by discretizing the operator $\hat{B}_3$ including the boundary conditions, and solving the analogous problem for the discretized operator. Using the symmetry of the problem the domain was restricted to one-sixth of the circle, and a nine-point finite difference scheme defined for the evaluation of the second order derivatives. The discretized boundary conditions at $\rho=1$ were implemented with the same scheme. The symmetry implies that the new boundary conditions on the lines $\phi=0,\pi/3$ are simple Neuman boundary conditions $\partial_\phi f(\rho,\phi)=0$. After discretization the problem transforms to a matrix eigenvalue problem $B_3\mathbf{\Psi}=\mathbf{0}$, where B_3 is a large sparse matrix, whose rank depends on the mesh of the discretization, and $\mathbf{\Psi}$ is the discretized f. We used NAG's sparse Gaussian elimination routines to find the zeros of $\det(B_3)$, and determined the values of ζ_3 for these zeros as a function of ζ_h. The results of this procedure for space dimensions $d=2,3,4$ are presented in Figs. 1, 2, and 3.

The various branches shown in Figs. 1-3 can be organized on the basis of the perturbation theory of the type proposed into [3] near $\zeta_h=0$. We performed that type of analysis and found that at $\zeta_h=0$ the allowed values of ζ_3 are organized into two sets,

$$\zeta_3^+(m,n)=2(3m+2n),$$
$$\zeta_3^-(m,n)=-2(d-1+3m+2n), \tag{10}$$

where n and m are any non-negative integer. The lowest lying positive values are 4,6,8, etc, whereas for $d=2$ the highest negative value is -2. We see that the nonperturbative solution displays in all dimensions a branch (dashed line) that begins at $\zeta_h=0,\zeta_3=4$ and ends at $\zeta_h=2,\zeta_3=0$. This branch is identical to the lowest lying positive branch predicted by the perturbation theory. We computed the slope of this branch near $\zeta_h=0$ in perturbation theory, and found that it is $2(2-d)/(d-1)$, in agreement with the numerics. Also the slopes of the other branches that begin at $\zeta_h=0$ were obtained perturbatively and found to agree with the numerics. The negative branch (shown only for $d=2$) never rises above its perturbative limit and is not relevant for the scaling behavior at any value of ζ_h. Note also that the point $\zeta_h=2,\zeta_3=0$ appears to be an accumulation point of many branches, and we are not confident that all the branches there were identified by our finite discretization scheme. This raises a worry about the availability of a smooth perturbative theory around $\zeta_h=2$. At least we expect such a perturbation theory to be very singular. Preliminary analytical work indicates that all branches meet the point $\zeta_h=2,\zeta_3=0$ with an infinite slope.

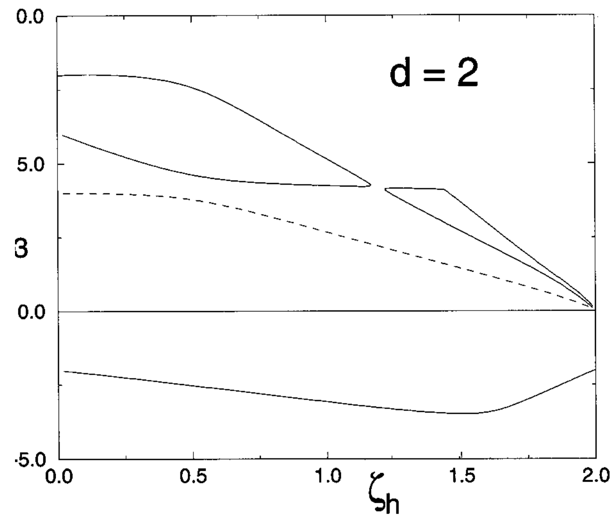

FIG. 1. The scaling exponent ζ_3 as a functions of ζ_h found as the loci of zeros of the determinant of the matrix B_3, for $d=2$.

The results of our nonperturbative approach lend support to the validity of the perturbative calculations of the zero

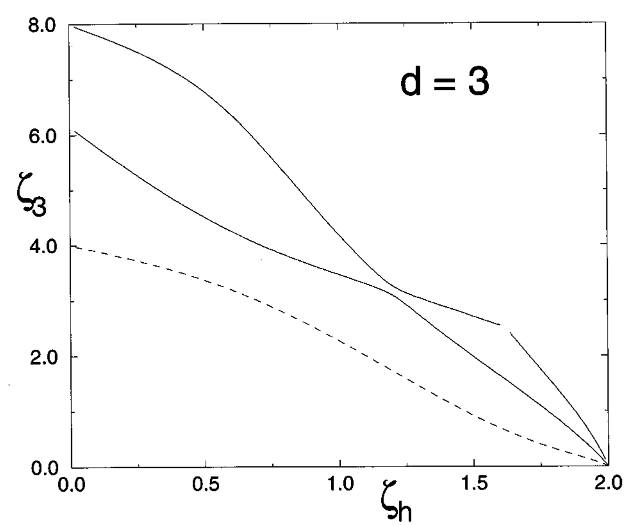

FIG. 2. Same as Fig. 1, but for $d=3$.

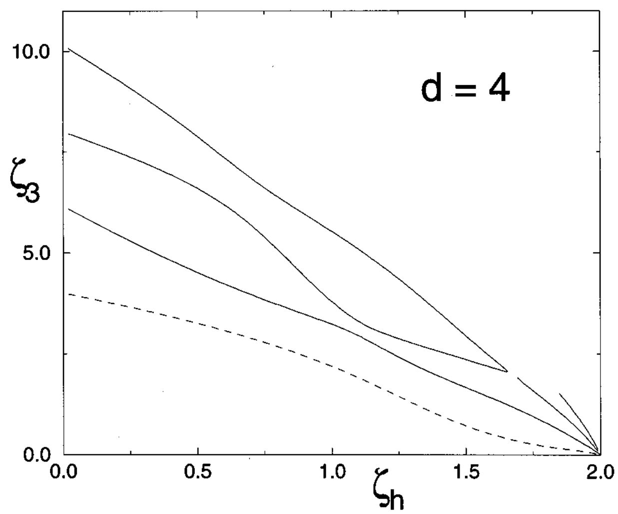

FIG. 3. Same as Fig. 1, but for $d=4$.

modes of $\hat{B}_4$. The disagreement between the scaling exponents ζ_4 and the higher order exponents ζ_n computed via the perturbative approach and the predictions of the other approach based on the fully fused theory cannot be ascribed to a formal failure of the perturbation theory. There are therefore a few possibilities that have to be sorted out by further research: (i) The crucial assumption in the fully fused approach, the linearity of the conditional average of the Laplacian of the scalar, is wrong. (ii) The perturbative approach fails for $\hat{B}_4$ even though it succeeds for $\hat{B}_3$. (iii) The computation of the zero modes, which is achieved by discarding the diffusive terms in $\hat{B}_n$, is irrelevant for the physical solution. It is not impossible that the diffusive term acts as a singular perturbation on some of the scale invariant modes. (iv) Lastly, it is possible that the physical solution is not scale invariant [12]. In other words, it is possible that $\mathcal{F}_3(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$ is not a homogeneous function with a fixed homogeneity exponent ζ_3, but rather (for example) that ζ_3 depends on the ratios of the separations (or, in other words, the geometry of the triangle defined by the coordinates). If this were also the case for even correlation functions $\mathcal{F}_{2n}$, this would open an exciting route for further research to understand how non-scale-invariant correlation functions under fusion become scale-invariant structure functions.

In light of the numerical results of Ref. [5] and the experimental results displayed in [7,8] we tend to doubt option (i). More work is needed to clarify the important riddle of the contradiction between the two current approaches in this field.

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