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Dissipative scaling functions in Navier-Stokes turbulence: Experimental tests

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Abstract. – A recent theoretical development in the understanding of the small-scale structure of Navier-Stokes turbulence has been the proposition that the scales $\eta_n(R)$ that separate inertial from viscous behavior of many-point correlation functions depend on the order n and on the typical separations R of points in the correlation. This is of fundamental significance in itself but it also has implications for the scaling behaviour of various correlation functions. This dependence has never been observed directly in laboratory experiments. In order to observe it, turbulence data which both display a well-developed scaling range with clean scaling behaviour and are well-resolved in the small scales to well within the viscous range is required. The data of the experiments performed in the laboratory of P. Tabeling of Navier-Stokes turbulence in a helium cell with counter-rotating disks approach these criteria, and provide supporting evidence for the existence of the predicted scaling of the viscous scale.

The familiar approach to the statistical theory of Navier-Stokes turbulence [1] concentrates on the properties of two-point differences of the Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ and their moments, termed structure functions:

$$S_n(R) = \langle |\mathbf{u}(\mathbf{x}+\mathbf{R}) - \mathbf{u}(\mathbf{x})|^n \rangle. \tag{1}$$

In isotropic homogeneous turbulence, these structure functions are observed to behave as a power law in R, $S_n(R) \sim R^{\zeta_n}$, with scaling exponents ζ_n that may be universal. This scaling holds within a range of scales between the outer scale L determined by the system size or the forcing, and some inner scale η determined by the viscosity, below which the velocity field is essentially smooth. In this regime $S_n(R) \sim R^n$. The usual definition of the viscous scale was established by Kolmogorov from the balance of the viscosity ν and the mean energy flux $\bar{\epsilon}$, according to $\eta \sim L\mathrm{Re}^{-1/(2-\zeta_2)}$. The experimental observation of anomalous exponents ζ_n which depend *nonlinearly* on n is contrary to this straightforward picture of a single viscous

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length scale, and suggests that a more complex situation applies. To see this immediately, consider the equation for S_n derived from the Navier-Stokes equations:

$$\frac{\partial S_n}{\partial t} + D_n(R) = \nu J_n(R) \ . \tag{2}$$

In the event of temporal stationarity $\partial S_n/\partial t = 0$. The term D_n results from the nonlinear term $\mathcal{P}\boldsymbol{u} \cdot \boldsymbol{\nabla u}$, where $\mathcal{P}$ is the standard projection operator used to eliminate the pressure gradient term. The term J_n results from the viscous dissipation $\nu\nabla^2\boldsymbol{u}$ and it takes the simple form $\nabla^2 S_n(R)$. It has been previously shown [2] that D_n scales as $\mathrm{d}S_{n+1}/\mathrm{d}R$. If it were true that there is a unique viscous scale, we could estimate $J_n(R) \sim J_2 S_n(R)/S_2(R)$ (and see below for details). By power counting one would then predict $\zeta_{n+1} - 1 = \zeta_n - \zeta_2$, resulting in classical scaling. The fact that this is not observed means that dissipative scales introduce a scaling of their own, changing the evaluation of J_n to $J_n(R) \sim S_{n+1}(R)/R$, making power-counting useless. This important point has never been put to direct experimental test. One reason for this omission is that workers in the field tend to place priority on extending their inertial range rather than resolving the dissipative scales. The aims of this letter are to show that the best of the existing data indicates strongly towards the predictions of the theory that we will now outline, but also to stimulate further efforts in the direction of subdissipative resolution.

We review briefly the derivation of the scaling behaviour of the viscous scale (found in more detail in [3]). Cross-over scales are naturally discussed within the framework of simultaneous many-point correlation functions $\mathcal{F}_n$ of velocity differences rather than on two-point quantities only. These are defined in terms of two-point differences

$$\boldsymbol{w}(\mathbf{x}, \mathbf{x}', t) \equiv \mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t), \tag{3}$$

as

$$\begin{aligned} &\boldsymbol{\mathcal{F}}_n(\boldsymbol{x}_1, \boldsymbol{x}'_1; \boldsymbol{x}_2, \boldsymbol{x}'_2; \ldots; \boldsymbol{x}_n, \boldsymbol{x}'_n) = \\ &= \langle \boldsymbol{w}(\boldsymbol{x}_1, \boldsymbol{x}'_1) \boldsymbol{w}(\boldsymbol{x}_2, \boldsymbol{x}'_2) \ldots \boldsymbol{w}(\boldsymbol{x}_n, \boldsymbol{x}'_n) \rangle \ , \end{aligned} \tag{4}$$

where $\langle \cdot \rangle$ denotes averaging, and all coordinates are distinct. Time labels have been dropped as only simultaneous correlations will be considered. Homogeneous scaling here means that

$$\mathcal{F}_n(\lambda \mathbf{x}_1, \lambda \mathbf{x}'_1, \ldots, \lambda \mathbf{x}'_n) = \lambda^{\zeta_n} \boldsymbol{\mathcal{F}}_n(\mathbf{x}_1, \mathbf{x}'_1, \ldots, \mathbf{x}'_n), \tag{5}$$

with ζ_n the scaling exponent. Taking the time derivative of $\boldsymbol{\mathcal{F}}_n$, using the Navier-Stokes equations to evaluate each $\partial \mathbf{u}/\partial t$ and considering the stationary state where $\partial \mathcal{F}/\partial t = 0$, one derives the following statistical balance equation:

$$\mathcal{D}_n(\mathbf{x}_1, \mathbf{x}'_1; \ldots; \mathbf{x}_n, \mathbf{x}'_n) = \nu \mathcal{J}_n(\mathbf{x}_1, \mathbf{x}'_1; \ldots; \mathbf{x}_n, \mathbf{x}'_n). \tag{6}$$

Here the term $\mathcal{D}_n$ arises from the nonlinear interaction term and may be written as

$$\begin{aligned} \mathcal{D}_n^{\alpha_1 \alpha_2 \ldots \alpha_n}(\mathbf{x}_1, \mathbf{x}'_1; \ldots; \mathbf{x}_n, \mathbf{x}'_n) &= \int \mathrm{d}\mathbf{x} \sum_{j=1}^{n} P_{\alpha_j \beta}(\mathbf{x}) \times \\ &\times \langle w_{\alpha_1}(\mathbf{x}_1, \mathbf{x}'_1) \ldots L^\beta(\mathbf{x}_j, \mathbf{x}'_j, \mathbf{x}) \ldots w_{\alpha_n}(\mathbf{x}_n, \mathbf{x}'_n) \rangle, \end{aligned} \tag{7}$$

$$\begin{aligned} L^\beta(\mathbf{x}_j, \mathbf{x}'_j, \mathbf{x}) \equiv \frac{1}{n} \sum_{k=1}^{n} \big[& w_\gamma(\mathbf{x}_j - \mathbf{x}, \mathbf{x}_k) \nabla_j^\gamma + \\ & + w_\gamma(\mathbf{x}'_j - \mathbf{x}, \mathbf{x}'_k) \nabla_{j'}^\gamma \big] \cdot w_\beta(\mathbf{x}_j - \mathbf{x}, \mathbf{x}'_j - \mathbf{x}). \end{aligned} \tag{8}$$

In the above $P_{\alpha_j\beta}(\mathbf{x})$ is the projection operator. The RHS with coefficient ν, the kinematic viscosity, results from the viscous term and is defined as

$$\mathcal{J}_n(\mathbf{x}_1,\mathbf{x}'_1;\dots;\mathbf{x}_n,\mathbf{x}'_n)=\sum_{j=1}^{n}(\nabla_j^2+\nabla_{j'}^2)\times$$
$$\times\langle w_{\alpha_1}(\mathbf{x}_1,\mathbf{x}'_1)\dots w_{\alpha_j}(\mathbf{x}_j,\mathbf{x}'_j)\dots w_{\alpha_n}(\mathbf{x}_n,\mathbf{x}'_n)\rangle. \tag{9}$$

This equation provides the means to determine the scale of the viscous range. The balance equation expresses the competition between the small-scale viscous effects and the interesting nonlinear dynamics, and intuitively, the viscous scale should be the scale at which the two effects become comparable. This raises a rather subtle question. As there is a balance of the two terms, it appears that the scaling properties of the correlators must be determined by the viscous term. However, one believes that the properties of the inertial-range quantities are *independent* of the details of the viscous range. This apparent paradox can be understood if one considers the separations of the coordinates of the correlation functions. If all separations are in the inertial range, there are no small-scale quantities, and the contribution of $\mathcal{J}_n$ will be negligible, leaving a homogeneous equation $\mathcal{D}_n=0$. Nontrivial scaling can arise from special solutions for the terms in the sum in $\mathcal{D}_n$ that exactly cancel one another. Now as some coordinates in the correlation functions approach one another, the gradients in $\mathcal{J}_n$ will begin to show their effect: these pick up the smallest separation $r_{\min}$, introducing a factor of $1/r^2_{\min}$. As $r_{\min}\to 0$, this term is no longer negligible; the scaling solution of the homogeneous inertial-range equation will no longer be valid and one obtains the smooth viscous result.

Thus one wishes to estimate the terms of the balance equation both in the inertial range and in the limit where some coordinates approach one another, in order to observe this crossover and estimate its scale length.

It can be shown [3] that in the case where all separations are of order R one may evaluate $\mathcal{D}$ simply as

$$\mathcal{D}_n\sim S_{n+1}(R)/R. \tag{10}$$

This can be demonstrated by proving that the integral in (7) converges in both limits, so that the typical evaluation at R is correct. As points in the correlation approach one another, this evaluation remains valid; but cancellations between terms will no longer occur.

The second term $\mathcal{J}_n$ can be estimated directly as

$$\mathcal{J}_n(R)\sim S_n(R)/R^2. \tag{11}$$

In the limit when some separation $r_{ij}\to 0$, this evaluation is replaced by

$$\mathcal{J}_n(r_{ij};R)\sim \mathcal{F}_n(r_{ij};R)/r^2_{ij}, \tag{12}$$

where $\mathcal{F}_n(r_{ij};R)$ is shorthand notation for $\mathcal{F}_n$ with an overall typical separation R and some pair of coalescing points of smaller separation r_{ij}. Now the balance equation gives

$$\mathcal{F}_n(r_{ij};R)\sim r^2_{ij}S_{n+1}(R)/\nu R. \tag{13}$$

This gives the evaluation of $\mathcal{F}_n(r_{ij};R)$ for a small separation in the viscous regime.

Now we wish to compare this with an evaluation for $\mathcal{F}_n(r_{ij};R)$ when the small distance is still in the inertial range. To do so we invoke the fusion rules derived in [4, 5]. These rules predict the behaviour of multipoint correlation functions as some pairs of coordinates approach one another, or "fuse". The essential result concerns a correlation of n pairs of points $\boldsymbol{\mathcal{F}}_n$ where p pairs of coordinates $\mathbf{x}_1,\mathbf{x}'_1\dots\mathbf{x}_p,\mathbf{x}'_p$, $(p<n)$ of p velocity differences

coalesce, with typical separations between the coordinates $|\mathbf{x}_i - \mathbf{x}'_i| \sim r$ for $i \leq p$, and where all other separations are of the order of R, $r \ll R \ll L$. Let us denote such a correlation as $\boldsymbol{\mathcal{F}}_n^{(p)}$. In a homogeneous isotropic scaling system, the fusion rules predict

$$\boldsymbol{\mathcal{F}}_n^{(p)}(\mathbf{x}_1, \mathbf{x}'_1; \ldots ; \mathbf{x}_n, \mathbf{x}'_n) = \\ = \tilde{\boldsymbol{\mathcal{F}}}_p(\mathbf{x}_1, \mathbf{x}'_1; \ldots ; \mathbf{x}_p, \mathbf{x}'_p) \boldsymbol{\Psi}_{n,p}(\mathbf{x}_{p+1}, \mathbf{x}'_{p+1}; \ldots ; \mathbf{x}_n, \mathbf{x}'_n) \ ,$$

where $\tilde{\boldsymbol{\mathcal{F}}}_p$ is a tensor of rank p associated with the first p tensor indices of $\boldsymbol{\mathcal{F}}_n$, and it has a homogeneity exponent ζ_p. The $(n-p)$-rank tensor $\boldsymbol{\Psi}_{n,p}(\mathbf{x}_{p+1}, \mathbf{x}'_{p+1}; \ldots ; \mathbf{x}_n, \mathbf{x}'_n)$ is a homogeneous function with a scaling exponent $\zeta_n - \zeta_p$, and is associated with the remaining $n - p$ indices of $\boldsymbol{\mathcal{F}}_n$. In terms of the scaling of structure functions, this can be expressed for p points coalescing to a distance r and all other points with typical separation R as (abbreviating the coordinate dependence of $\boldsymbol{\mathcal{F}}_n^{(p)}$)

$$\boldsymbol{\mathcal{F}}_n^{(p)}(r; R) \sim S_p(r) S_n(R)/S_p(R). \tag{14}$$

In the special case that $p = 1$, due to the vanishing of the average of a single difference in isotropic turbulence, the leading order result is

$$\boldsymbol{\mathcal{F}}_n^{(1)}(r; R) \sim S_2(r) S_n(R)/S_2(R). \tag{15}$$

Applying this result to the correlation we previously denoted $\mathcal{F}_n(r_{ij}; R)$ one obtains

$$\mathcal{F}_n(r_{ij}; R) \sim S_2(r_{ij}) S_n(R)/S_2(R). \tag{16}$$

Thus we have an inertial-range and a viscous-range evaluation of $\mathcal{F}_n(r_{ij}; R)$. Let us take the viscous scale η_n to be that at which the two evaluations coincide. Balancing in the two-point case one recovers the Kolmogorov estimate, $\eta_2 \sim L\mathrm{Re}^{-1/(2-\zeta_2)}$. For other values of n one finds

$$\eta_n(R) = \eta_2 \left(\frac{R}{L}\right)^{x_n}, \qquad x_n = \frac{\zeta_n + \zeta_3 - \zeta_{n+1} - \zeta_2}{2 - \zeta_2}. \tag{17}$$

In order to test this proposition experimentally, we will consider direct measurements of the function $\mathcal{J}_n(R)$. To make a comparison with the one-dimensional data obtained from experiments, we take a form defined by

$$J_n(\rho; R) = \left\langle \tilde{\nabla}_\rho^2 \mathbf{u}(\mathbf{x}) \left[\boldsymbol{w}(\mathbf{x}, \mathbf{x} + \mathbf{R})\right]^{n-1} \right\rangle \cdot \mathbf{R}/R. \tag{18}$$

For discrete data, the Laplacian operator $\tilde{\nabla}_\rho^2$ in (18) is taken to be a second-order finite difference of longitudinal components of the velocity,

$$\tilde{\nabla}_\rho^2 \mathbf{u}(\mathbf{x}) = \left[\boldsymbol{w}(\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}) - \boldsymbol{w}(\mathbf{x}, \mathbf{x} - \boldsymbol{\rho})\right] \cdot \boldsymbol{\rho}/\rho^3. \tag{19}$$

From the discussion above, one expects a different scaling for ρ above and below the dissipative scale. For ρ in the inertial range, the estimation of (14) is applicable and one predicts

$$J_n(\rho; R) = C_n J_2(\rho) S_n(R)/2S_2(R) \ , \quad \rho \gg \eta \, , \tag{20}$$

where C_n is an R-independent dimensionless constant which may have n-dependence. However, for ρ in the viscous regime,

$$J_n(\rho; R) = \tilde{C}_n J_2(\rho) S_{n+1}(R)/S_3(R) \ , \quad \rho \ll \eta \ , \tag{21}$$

where $\tilde{C}_n$ is some other coefficient. One can show that J_2 is equal to the mean dissipation $\langle |\nabla u(x)|^2 \rangle$, and is thus expected to be R-independent. The explicit prefactor containing ρ is included in J_2; we will consider only the R-dependence resulting from the scaling of (17).

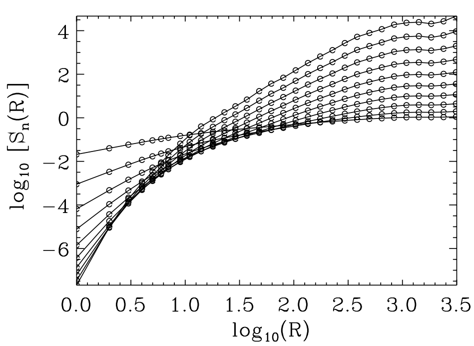

Fig. 1

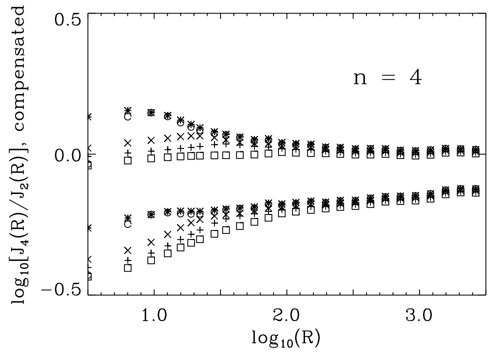

Fig. 2

Fig. 1. – Log-log plot of the structure functions $S_n(R)$ as a function of R for $n = 1$–10.

Fig. 2. – Log-log plot of the normalised function $J_4(R)/J_2(R)$ as a function of R for $\rho = 1, 3, 7, 11$ and 29, represented by □, +, ×, ∗ and ○, respectively. The data are compensated in the upper plots by the inertial-range fusion rule prediction $S_4(R)/S_2(R)$ and in the lower ones by the result in the viscous regime $S_5(R)/S_3(R)$.

These predictions are tested in data obtained by F. Belin and H. Williame in the laboratory of P. Tabeling at Ecole Normale Supérieure; see for example [6, 7]. The data are time signals of the velocity field taken from a low-temperature cell of helium gas enclosed in a cylinder and driven by counter-rotating disks. The helium is maintained at a constant temperature around 5 K and at a controlled pressure. The recordings were made using a hot-wire probe consisting of a 7 μm carbon fibre coated with evaporated gold apart from an active area of size 7 μm. The frequency response of the probe can range between 10 and 50 kHz.

The data had very long acquisition times, containing up to 30 million samples. The resulting statistics are well-resolved and stationary. The Taylor microscale Reynolds number and the Kolmogorov microscale η were determined through the usual procedure of surrogating time for space (by Taylor's hypothesis), and data is available for a range both of R_λ and of minimum resolved length scale r/η. We selected data sets according to the small-scale resolution, and although the R_λ was not extremely high, a distinct inertial range is evident. The data presented here has a minimum resolved distance r/η of 1.18 and R_λ of 418.

In fig. 1 we present the structure functions $S_n(R)$ as a function of R. In all figures, spatial separations have units of sampling times, and the velocity is normalised by the RMS velocity. This figure shows that we have one and a half decades of "inertial range" (between, say 10 and 500 sampling units) and that the length scales below 10 units are smooth and well-resolved. The initial logarithmic slope of the n-th structure function at 1 unit is close to n (deviating successively more for higher-order n, as would be expected). There is reasonably well-defined scaling behaviour up to order 10.

Our aim is to try to expose the postulated crossover in scaling behaviour in R of $J_n(\rho; R)$ as a function of ρ. We have calculated the correlation functions $J_n(\rho; R)$ for several values of ρ from the minimum distance of 1 unit up to a value well into the inertial range. Note that the difference in scaling that is expected is rather small; one expects the scaling exponent of $J_n(\rho; R)$ to cross over from $\zeta_n - \zeta_2$ to $\zeta_{n+1} - \zeta_3$, which for the usual values of scaling exponents obtained in turbulent experiments gives a difference of the order of 0.15 for $n = 4$ to 0.2 for $n = 8$. Thus we do not present the results in terms of calculated exponents, as one cannot justifiably separate values of this closeness on the basis of exponents calculated on a limited

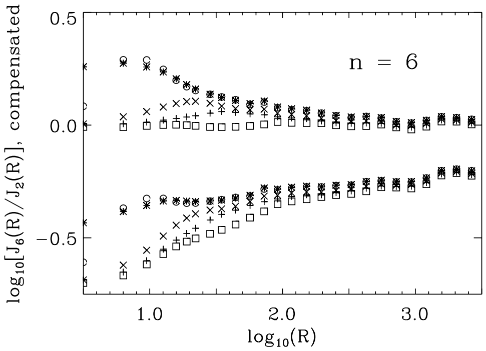

Fig. 3

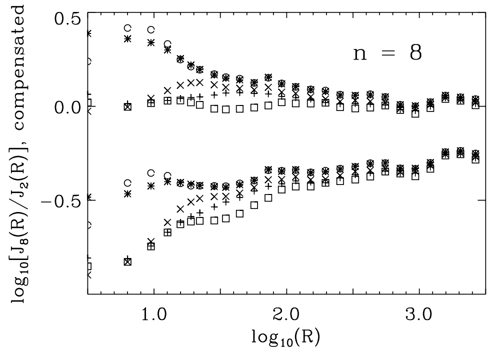

Fig. 4

Fig. 3. – Log-log plot of the normalised function $J_6(R)/J_2(R)$ as a function of R for $\rho = 1, 3, 7, 11$ and 29, represented by □, +, ×, ∗ and ○, respectively, compensated in the upper plots by $S_6(R)/S_2(R)$ and in the lower ones by $S_7(R)/S_3(R)$.

Fig. 4. – Log-log plot of the normalised function $J_8(R)/J_2(R)$ as a function of R for $\rho = 1, 3, 7, 11$ and 29, represented by □, +, ×, ∗ and ○, respectively, compensated in the upper plots by $S_8(R)/S_2(R)$ and in the lower ones by $S_9(R)/S_3(R)$.

inertial range. Instead we will examine the function as a whole.

In figs. 2-4 we display the results. The three figures show $J_n(\rho; R)$ for a single value of n, for $n = 4$, 6 and 8. For each n there are results for five values of ρ, $\rho = 1, 3, 7, 11$ and 29. The figures each show two sets of data, one in which the calculated J_n's have been compensated by the inertial-range prediction $J_2(\rho)S_n(R)/S_2(R)$ (the upper set of functions), and the second showing the same data compensated by the dissipative-range result $J_2(\rho)S_{n+1}(R)/S_3(R)$. Hence in the upper set we expect to see that for inertial-range values of ρ, the resulting plots are constant in R in the inertial range. In principle as there is no knowledge of the coefficient C_n, the value of the constant C_n can be different for different n. (It is trivially 1 for $n = 2$.) We hope to see that the dissipative-range scaling is a better fit as $\rho \to 0$.

The upper sets of plots in each figure show that for the inertial-range values of ρ, the inertial scaling (20) is very well realised. This scaling has been previously observed in turbulence data [8] but the agreement in this data is more impressive: it has smaller fluctuations, and the agreement continues into the viscous scales, which has not previously been seen to be the case. Comparing between the figures, for different values of n, one finds that all coefficients C_n are very near to 1. Comparing different values of ρ, there is a continuous dependence on ρ in the functional behaviour of $J_n(\rho; R)$. There is a clear deviation from the inertial-range scaling as ρ decreases, and the smallest value of ρ shows a small but definite slope. The plots corrected by the dissipative-range scaling show a distinct indication that there is a tendency toward this slope. The effect becomes more apparent for larger n as the difference between the two scaling laws becomes larger. These functions of course also show larger statistical fluctuations.

One should note that the plots for $\rho = 1$ and $\rho = 3$ are almost identical. This may indicate that the data is only resolved well to $\rho = 3$, and no further information is gained in the subsequent refinements of scale. It would therefore not be surprising that a clean scaling of (21) is not precisely observed. Nonetheless the trend in that direction is clearly visible.

It should be remarked that a true experimental verification of these predictions is very difficult at the current time. To date experimental effort has focused on the extension of the inertial range rather than the resolution of the dissipative scales. The data examined here is currently the best available to us and nonetheless clearly has its limitations. Despite this, we see an indication of the predicted behaviour. We hope that this investigation will serve as an impetus for further experimental work in small-scale resolution that will allow these issues to be more fully understood.

We have been able to find supporting experimental evidence that the viscous scale of n-point multipoint correlation functions is an anomalous scaling function. We have verified the inertial-range fusion rules and given evidence that the small-scale structure behaves according to the theoretical predictions.

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