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Hamiltonian structure of the Sabra shell model of turbulence: Exact calculation of an anomalous scaling exponent

V. S. L'vov[1], E. Podivilov[2] and I. Procaccia[1]

[1] *Department of Chemical Physics, The Weizmann Institute of Science Rehovot 76100, Israel*
[2] *Institute of Automatization and Electrometry, Russian Academy of Sciences Novosibirsk 630090, Russia*

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Abstract. – We show that the Sabra shell model of turbulence, which was introduced recently, displays a Hamiltonian structure for given values of the parameters. The requirement of scale independence of the flux of this Hamiltonian allows us to compute exactly a one-parameter family of anomalous scaling exponents associated with 4th-order correlation functions.

The field of turbulence and turbulent statistics cannot pride itself on a large number of exact results. One of the best known exact results is the "fourth-fifth law" of Kolmogorov, which pertains to the third-order moment of the longitudinal velocity differences, fixing its scaling exponent ζ_3 to unity [1]. This follows from the conservation of energy, a quadratic invariant, by the Euler part of the Navier-Stokes equations. This famous result reappeared also in simplified toy models of turbulence, like shell models, since the conservation of a quadratic invariant was built into their definition [2]. Nevertheless, the high degree of simplification involved in the shells model did not lead so far to additional exact results.

In this letter we report a discovery of a Hamiltonian structure of the Sabra shell model of turbulence that was introduced recently [3]. The Hamiltonian structure exists for given values of the parameters of the model, and it *does not coincide* with the usual quadratic invariants; in fact it is cubic in the velocities. There is a Hamiltonian density $\mathcal{H}_n$ which is *local* in shell-space, and its existence implies a flux of a local conserved density. The consequence of the constancy of this flux in shell-space fixes the value of a fourth-order scaling exponent which is "anomalous" in the sense that it must involve the existence of a renormalization scale. This result is both exact and nontrivial, constituting a kind of "boundary condition" on theories of anomalous scaling in such models.

The Sabra model of turbulence, like all shell models of turbulence, describes truncated fluid mechanics in wave number space in which we keep $N+1$ "shells". Denoting the n-th wave number component of the velocity field by v_n, $0 \le n \le N$, the model may be written as

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} \equiv \dot{v}_n = -i\big(ak_{n+1}v_{n+1}v_{n+2} - bk_n v_{n-1}v^*_{n+1} - ck_{n-1}v^*_{n-1}v_{n-2}\big) - \nu k_n^2 v_n + f_n\,, \quad (1)$$

where the coefficients a, b, and c are real, ν is the "viscosity" and f_n is a forcing term, usually limited to acting on the first shells only. Equation (1) is equivalent to the Sabra model suggested in [3] after the change of variables: $v_n = u_n$ for n even and $v_n = -u^*_n$ for n odd. Like in all shell models one builds in the conservation of "energy"

$$E = \sum_{n=0}^{N} |v_n|^2 \quad (2)$$

in the inviscid limit $\nu \to 0$ by requiring that $a + b + c = 0$. The actual values of the wave numbers k_n are determined by k_0 and the "level spacing" parameter λ, $k_n = k_0\lambda^n$. By rescaling we can always choose $a = 1$, leaving us with only two free parameters in this model, λ and $\epsilon = -b$. The inviscid and unforced part of our model then reads

$$\dot{v}_n = -ik_{n+1}\left[v_{n+1}v_{n+2} + \frac{\epsilon}{\lambda}v_{n-1}v^*_{n+1} + \left(\frac{1-\epsilon}{\lambda^2}\right)v_{n-2}v^*_{n-1}\right]. \quad (3)$$

Like the popular GOY model [2], the Sabra model also conserves the so-called "helicity"

$$H = \sum_{n=0}^{N}\left(\frac{1}{\epsilon - 1}\right)^n |v_n|^2\,. \quad (4)$$

For the special choice of ϵ equal to the golden mean $\epsilon_{\mathrm{g}} = (\sqrt{5} - 1)/2$, eq. (3) reads

$$\dot{v}_n = -ik_{n+1}\left[v_{n+1}v_{n+2} + \frac{\epsilon_{\mathrm{g}}}{\lambda}v_{n-1}v^*_{n+1} + \left(\frac{\epsilon_{\mathrm{g}}}{\lambda}\right)^2 v_{n-2}v^*_{n-1}\right]. \quad (5)$$

By a direct calculation one can verify that in this case the model exhibits a third integral of motion which we denote as W:

$$W = \sum_{m=1}^{N-1}\epsilon_{\mathrm{g}}k_0\left(\frac{\lambda}{\epsilon_{\mathrm{g}}}\right)^n\left[v^*_{n-1}v_n v_{n+1} + \mathrm{c.c.}\right]\,. \quad (6)$$

The main statement of this letter is that the Sabra model (5) with ϵ chosen at the golden mean realizes a Hamiltonian structure in the sense that the equations of motion can be written in a canonical form

$$\dot{a}_n = -i\frac{\partial\mathcal{H}\{a_j, a^*_j\}}{\partial a^*_n}\,, \quad (7)$$

with $\mathcal{H}\{a_j, a^*_j\}$ being the Hamiltonian function of the set of pairs of canonical variables $\{a_j, a^*_j\}$ representing all the degrees of freedom. The Hamiltonian $\mathcal{H}$ is precisely the cubic invariant W, expressed in terms of canonical variables a_n, a^*_n, which are related to the velocity as follows:

$$a_n \equiv v_n/\epsilon_{\mathrm{g}}^{n/2}\,. \quad (8)$$

According to (6) and (8) the Hamiltonian has the following form:

$$\mathcal{H} = \sum_{m=1}^{N-1}\mathcal{H}_m\,, \quad (9)$$

$$\mathcal{H}_m = \epsilon_{\mathrm{g}}k_0(\lambda\sqrt{\epsilon_{\mathrm{g}}})^m\left[a_{m+1}a_m a^*_{m-1} + \mathrm{c.c.}\right]\,. \quad (10)$$

The statement is proven by verification: compute $\dot{a}_n$ according to the canonical equations (7) with the Hamiltonian (9):

$$\dot{a}_n = -i\frac{\partial \mathcal{H}}{\partial a_n^*} = -ik_0\epsilon(\lambda\sqrt{\epsilon_{\rm g}})^n[\lambda\sqrt{\epsilon_{\rm g}}a_{n+2}a_{n+1} + a_{n+1}^* a_{n-1} + \frac{1}{\lambda\sqrt{\epsilon_{\rm g}}}a_{n-1}^* a_{n-2}]. \qquad (11)$$

Using the relationship (8) this equation coincides with eq. (5), *Q.e.D.*

One should note that in the majority of physical systems in the conservative limit the Hamiltonian, if it exists, coincides with the energy. Examples abound: electrodynamics, hydrodynamics, acoustics, waves on the surface of fluids, spin waves etc., see for example [4]). There are no regular procedures to find the canonical variables in terms of the "natural" variables of a particular physical system. This relationship may be very nonobvious, like the Clebsch representation of the velocity field in hydrodynamics. There are also examples of physical problems with a Hamiltonian structure in which the Hamiltonian *does not* coincide with the energy, like induced scattering of electrons on ions in strongly nonisothermal plasmas [5]. The present example belongs to this class.

The existence of a Hamiltonian has significant consequences for the solutions of nonlinear problems. In this letter we demonstrate one such consequence, which is the ability to determine exactly an anomalous exponent. The locality of $\mathcal{H}_n$ in the space of shell numbers means that there exists a current $J_4(k_n)$ over shells of the mean value of the"density" $\langle\mathcal{H}_n\rangle$. Here the symbol $\langle\ldots\rangle$ stands for an appropriate ensemble average. In order to derive an expression for $J_4(k_n)$ compute the time derivative of $\mathcal{H}_n$ (10) with the help of the canonical equations (11). The right-hand side of this equation is a combination of 4th-order correlation functions. The conservation of the Hamiltonian means that $\sum_n \dot{\mathcal{H}}_n = 0$. This is possible only if $\langle\dot{\mathcal{H}}_n\rangle$ can be written as a *difference* of currents depending on two neighboring shells (substituting the divergence in the continuous limit):

$$\langle\dot{\mathcal{H}}_n\rangle = J_4(k_{n-1}) - J_4(k_n)\ . \qquad (12)$$

We find

$$J_4(k_n) = -\epsilon k_0^2\left(\frac{\lambda^2}{\epsilon_{\rm g}}\right)^n \widetilde{F}_4(k_n)\ , \qquad (13)$$

where $\widetilde{F}_4(k_n)$ is the following combination of 4th-order velocity correlation functions:

$$\widetilde{F}_4(k_n) \equiv {\rm Im}[\lambda^2\langle v_{n-1}^* v_n v_{n+2} v_{n+3}\rangle + \epsilon_{\rm g}\langle v_{n-2}^* v_{n-1} v_{n+1} v_{n+2}\rangle + \\ +\epsilon_{\rm g}\lambda\langle v_{n-1}^* v_n^2 v_{n+2}^*\rangle + \lambda\langle v_{n-1}^* v_n^2 v_{n+1}^2 v_{n+2}\rangle]\ . \qquad (14)$$

In the steady state $\langle\dot{\mathcal{H}}_n\rangle = 0$, implying that $J_4(k_n)$ must be independent of k_n: $J_4(k_n) = J_4(k_{n-1})$. According to (13) we must demand

$$\widetilde{F}_4(k_n) \propto \left(\frac{\epsilon_{\rm g}}{\lambda^2}\right)^n\ , \qquad (15)$$

or, looking for a scaling solution $\widetilde{F}_4(k_n) \propto k_n^{-\tilde{\zeta}_4}$, we compute

$$\tilde{\zeta}_4 = \log_\lambda\left(\frac{\lambda^2}{\epsilon_{\rm g}}\right) = 2 + \log_\lambda\left(\frac{\sqrt{5}+1}{2}\right)\ . \qquad (16)$$

This is an exact solution for a one-parameter family of scaling exponents as a function of the level spacing λ. We can write our result in a more traditional form in terms of the rate of energy dissipation $\dot{E}$:

$$\widetilde{F}_4(k_n) = C\left(\frac{\dot{E}}{k_n}\right)^{4/3}\left(\frac{k_0}{k_n}\right)^{\tilde{\zeta}_4 - 4/3} . \tag{17}$$

This form underlines the "anomaly" of the exponent, requiring the appearance of a renormalization scale k_0 in the scaling form. We note that the anomalous exponent $\tilde{\zeta}_4 \geq 2$. The Hölder inequalities imply that the scaling exponent ζ_4 characterizing any of the correlation functions constituting $\widetilde{F}_4(k_n)$ is smaller than or equal to $4/3$. Denote the scale at which the Hamiltonian invariant is pumped into the system by $k_{\mathcal{H}}$. If this scale is chosen to coincide with the first shells of the model, the anomalous exponent found here will always be subleading. Thus any of the individual 4th-order correlations that appear on the RHS of $\widetilde{F}_4$ will scale with the leading exponent ζ_4. However, these leading scaling contributions exactly cancel in the combination of correlation functions making up $\widetilde{F}_4(k_n)$, due to the conservation of the Hamiltonian. Note, however, that it is possible to make $\tilde{\zeta}_4$ a leading scaling exponent by choosing $k_{\mathcal{H}}$ in the bulk of the inertial interval. If the pumping of the Hamiltionian and has an effect on the region of smaller k, $k < k_{\mathcal{H}}$, then in that region the contribution of $\tilde{\zeta}_4$ will be leading. This and other consequences of the existence of the Hamiltonian structure will be investigated further in the near future.

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