Fractals, Vol. 10, No. 3 (2002) 291–296

BIRTH OF ANOMALOUS SCALING IN A MODEL OF HYDRODYNAMIC TURBULENCE WITH A TUNABLE PARAMETER*

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Abstract

We introduce a model of hydrodynamic turbulence with a tunable parameter ε, which represents the ratio between deterministic and random components in the coupling between N identical copies of the turbulent field. To compute the anomalous scaling exponents ζ_n (of the nth order structure functions) for chosen values of ε, we consider a systematic closure procedure for the hierarchy of equations for the n-order correlation functions, in the limit $N \to \infty$. The parameter ε regularizes the closure procedure, in the sense that discarded terms are of higher order in ε compared to those retained. It turns out that after the terms of $O(1)$, the first nonzero terms are $O(\varepsilon^4)$. Within this ε-controlled procedure, we have a *finite and closed* set of scale-invariant equations for the 2nd and 3rd order statistical objects of the theory. This set of equations retains all terms of $O(1)$ and $O(\varepsilon^4)$ and neglects terms of $O(\varepsilon^6)$. On this basis, we expect anomalous corrections $\delta\zeta_n$ in the scaling exponents ζ_n to increase with ε_n. This expectation is confirmed by extensive numerical simulations using up to 25 copies and 28 shells for various values of ε_n. The simulations demonstrate that in the limit $N \to \infty$, the scaling is normal for $\varepsilon < \varepsilon_{\rm cr}$ with $\varepsilon_{\rm cr} \approx 0.6$. We observed the birth of anomalous scaling at $\varepsilon = \varepsilon_{\rm cr}$ with $\delta\zeta_n \propto \varepsilon^4 - \varepsilon_{\rm cr}^4$ according to our expectation.

1. INTRODUCTION

The statistics of the small scale structure of turbulence is characterized by "anomalous scaling," which means that correlation functions and structure functions of velocity differences across a scale R exhibit a power law behavior with scaling exponents that are not correctly predicted by dimensional analysis. In the last few decades, there have been many attempts to compute the scaling

*This paper was presented at the 4th Nonlinear Variability in Geophysics and Astrophysics Conference in Roscoff, France, 12–17 July 1998.

exponents of turbulent fields from the equations of motion. In the context of simplified models of passive scalar advection, it was discovered that there exist natural small parameters that allow direct computations of anomalous scaling exponents.[1–3] In Navier Stokes turbulence and also in simplified models like shell models, the progress in computing scaling exponents was slowed down by the lack of a small parameter. It is thus worthwhile to consider models of turbulent velocity fields in which a tunable small parameter can be introduced and used to advantage. We achieve this by considering N copies of the original turbulent field, with proper couplings between the copies. When the copies are coupled with random coefficients, the model reduces to the Random Coupling Model and exhibits normal scaling. We construct a small parameter ε by allowing a small deterministic coupling of $O(\varepsilon)$. Furthermore, we arrange the coupling such that for $\varepsilon = 1$ the copies are decoupled, leading to the commonly studied anomalous scaling for each copy. This procedure can be adopted for Navier-Stokes turbulence as well as for shell models of turbulence. We use the small parameter to great advantage, developing closed form equations for the low order correlation functions which are valid to a stated order in ε. It turns out that the closed equations for the correlation functions are still so difficult to solve in the case of Navier-Stokes that it makes sense to study the method first in the context of shell models. Shell models have the additional obvious advantage that their numerical simulations are highly feasible, and as a consequence, we can back up the theory with simulations using a large number of coupled copies (up to $N = 25$). We stress however from the start that *in principle* all the steps provided here can be repeated in the context of Navier-Stokes turbulence.

Shell models as well as Navier-Stokes turbulence pose an infinite hierarchy of equations for the time derivatives of the n-order correlation functions, in terms of correlation functions of orders n and $n+1$. This hierarchy is *linear* in the correlation functions. We have shown in Refs. 4 to 6 that when the Reynolds number Re is very large (Re $\to \infty$), the viscous contributions in this hierarchy are negligible. It was pointed out[4,5] that in this limit, the hierarchy obeys a scaling symmetry which stems from the rescaling symmetry of the Euler equation.[7] This rescaling symmetry foliates the space of solutions into slices of different scaling exponents h of the velocity; these are referred to as h-slices. On each h-slice, the solution exhibits "normal scaling" with the given value of h. The full solution is a linear combination of all the solutions on the h-slices with non-universal weights which are determined by the forcing on the integral scale of turbulence. Different orders of correlation functions are dominated by different h-slices, and accordingly the full solution has anomalous scaling. The anomalous exponents are expected to be universal.

In attempting to use these findings to compute the anomalous exponents from first principles, we proposed in Refs. 4 to 6 to truncate the hierarchy of equations, preserving the fundamental rescaling symmetry that gives rise to anomalous scaling. Truncation is problematic; in turbulence there is no natural small parameter, and therefore any closure of an infinite hierarchy is uncontrolled (and non-unique). These difficulties can be effectively surmounted using the parameter ε discussed above. Having no anomaly at $\varepsilon = 0$ and full anomaly at $\varepsilon = 1$, this family of models is expected to exhibit anomalous scaling for some values of ε in (0, 1). We stress that we are not particularly interested in $\varepsilon = 1$. Any value of ε which is tractable analytically and for which the anomalous scaling exponents may be determined by numerical simulations is of equal interest.

In Sec. 2, we discussed a family of model[9] for which the closure procedure is controlled by means of a small parameter ε. We review briefly the "Sabra" shell model of turbulence[8] and introduce equations for coupled N copies of the Sabra models with the appropriate coupling which includes the parameter ε. We refer to these equations the "$(N - \varepsilon)$-Sabra (shell) model" of hydrodynamic turbulence.

As we showed in Ref. 9 the lowest order closed equations (for the 2nd and 3rd order correlation functions) contain all the terms of $O(1)$ and $O(\varepsilon^4)$ (terms of smaller order in ε are zero), whereas the neglected terms are of $O(\varepsilon^6)$. We can improve the closure scheme systematically, solving for 2nd, 3rd and 4th order correlation functions, including all the terms of $O(\varepsilon^6)$, neglecting terms of $O(\varepsilon^8)$, etc. As said above, for $\varepsilon = 1$, we lose the coupling between the copies, and recover the initial anomalous problem for any value of N. For $\varepsilon = 0$, the model is nothing but the well-known Random Coupling Model which is believed to lead to K41 scaling for shell models.

Detailed numerical investigations of the $(N - \varepsilon)$-Sabra model are described in Sec. 3. We have simulated up to 25 coupled Sabra copies for ten different values of ε, computed the 2nd, 4th and 6th order structure functions and found the corresponding scaling exponents $\zeta_n(\varepsilon, N)$. Then we analyzed their dependence on $1/N$ and showed that for $\varepsilon = 0$, the anomalous corrections $\delta\zeta_n(\varepsilon, N) - \zeta_n^{\rm K41}$ to the K41 scaling exponents $\zeta_n^{\rm K41} = n/3$ tend to vanish for $N \to \infty$. For $N = 25$, these corrections are already small enough. As expected, for $\varepsilon = 1$, the corrections $\delta\zeta_n(\varepsilon, N)$ are N independent while for some intermediate values of ε (say $\varepsilon = 0.8$) the values of $\delta\zeta_n$ decrease with increasing N but go to a finite limit when $N \to \infty$.

The important question is how $\delta\zeta_n(\varepsilon, \infty)$ depends on ε for different values of n. We found that $\delta\zeta_n(\varepsilon, \infty) = 0$ at ε is smaller than a critical value $\varepsilon_{\rm cr} \approx 0.6$, and

$$\delta\zeta_n(\varepsilon, N) \approx C_n(\varepsilon^4 - \varepsilon_{\rm cr}^4) \quad \text{for } \varepsilon > \varepsilon_{\rm cr}\,. \tag{1}$$

This functional dependence on ε^4 was expected from the preliminary analysis of the closure equations.

2. THE (N, ε)-SABRA MODEL

Let us introduce a small parameter via the coupling between N copies of a field u_n, which satisfies the dynamics of the Sabra shell model of turbulence suggested in Ref. 8:

$$\frac{du_n(t)}{dt} = i[ak_{n+1}u_{n+1}^*u_{n+2} + bk_n u_{n-1}^*u_{n+1} - ck_{n-1}u_{n-2}u_{n-1}] - \nu k_n^2 u_n + f)n(t)\,. \tag{2}$$

Shell models are simplified dynamical systems constructed such that the complex number u_n represents the amplitude associated with the Fourier transform of the velocity field $\mathbf{u}(\mathbf{r})$ with "wave vector" k_n. Rather than considering the full $\mathbf{k}$ space and all the nonlinear interactions, one allows for only one-dimensional k vectors on shells spaced such that $k_n \equiv k_0\lambda^n$, with λ being the spacing parameter, and local interactions. In Eq. (2), ν is the "viscosity" and $f_n(t)$ a random Gaussian force restricted to the lowest shells. The parameters a, b and c are restricted by the requirement $a+b+c = 0$ which guarantees the conservation of the "energy" $E = \sum_{n=0}^N |u_n(t)|^2$ in the in-viscid, unforced limit.

We now want to generalize this model to one which consist of N suitably coupled copies of it. In order to do that, we need to consider separately the equations for the real and imaginary parts of u_n. This procedure guarantees that the obtained model converges to the original Sabra model in the limit $N \to 1$. The copies are indexed by i, j or ℓ, and these indices take on values $-J, \ldots, +J$, $2J+1 = N$. The ith copy of the velocity field is denoted as $u_{n,\sigma}^{[i]}$. In this notation, $\sigma = \pm 1$ refers to the real and imaginary parts of u_n respectively. Let $D^{[ij\ell]}$ be the coupling between copies, which will be chosen later. Equation (2) for a collection of copies are

$$\begin{aligned}\frac{du_{n,\sigma}^{[i]}}{dt} = \sum_{j\ell} D^{[ij\ell]}[A_{\sigma'\sigma''}^{(\sigma)}(\gamma_{a,n+1}u_{n+1,\sigma'}^{[j]}u_{n+2,\sigma''}^{[\ell]} &+ \gamma_{b,n}u_{n-1,\sigma'}^{[j]}u_{n+1,\sigma''}^{[\ell]}) \\ &+ C_{\sigma'\sigma''}^{(\sigma)}\gamma_{c,n-1}u_{n-2,\sigma'}^{[j]}u_{n-1,\sigma''}^{[\ell]} \\ &- \nu k_n^2 u_{n,\sigma}^{[i]} + f_{n,\sigma}^{[i]}\end{aligned} \tag{3}$$

$$\gamma_{a,n} \equiv ak_n\,, \quad \gamma_{b,n} \equiv bk_n\,, \quad \gamma_{c,n} \equiv ck_n \tag{4}$$

$$\begin{aligned}\mathbf{A}^{(+1)} &\equiv \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathbf{A}^{(-1)} \equiv \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\\ \mathbf{C}^{(+1)} &\equiv \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathbf{C}^{(-1)} \equiv \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.\end{aligned} \tag{5}$$

Note that $A_{\sigma'\sigma''}^{(\sigma)} = A_{\sigma\sigma''}^{(\sigma')}$, $A_{\sigma'\sigma''}^{(\sigma)} = C_{\sigma''\sigma}^{(\sigma')}$. Recall that the index ℓ is defined modulo N, therefore it is convenient to introduce a Fourier transform in the "copy" space, defining the *collective* variables:

$$u_{n,\sigma}^{\alpha} = \frac{1}{\sqrt{N}}\sum_{\ell=-J}^{J} u_{n,\sigma}^{[\ell]} \exp\left(\frac{2i\pi\alpha\ell}{N}\right). \tag{6}$$

Note that the index α is also defined modulo $N = 2J+1$. It is convenient to consider values α within "the first Brillouin zone," i.e. from $-J$ to J. We will refer to it as the α-momentum. Since $u_{n,\sigma}^{[i]}$ is real, $u_{n,\sigma}^{-\alpha} = u_{n,\sigma}^{\alpha *}$. In "$\alpha$-Fourier space," Eq. (3) reads

$$\begin{aligned}\frac{du_{n,\sigma}^{\alpha}}{dt} = \sum_{\beta,\gamma}\Phi^{\alpha,\beta,\gamma}[\Delta_{\alpha,\beta+\gamma} + \Delta_{\alpha+N,\beta+\gamma} + \Delta_{\alpha,\beta+\gamma+N}] &\times\{A_{\sigma'\sigma''}^{\sigma}[\gamma_{a,n+1}u_{n+1,\sigma'}^{\beta}u_{n+2,\sigma''}^{\gamma} \\ &+ \gamma_{b,n}u_{n-1,\sigma'}^{\beta}u_{n+1,\sigma''}^{\gamma}] \\ &+ C_{\sigma'\sigma''}^{\sigma}\gamma_{c,n-1}u_{n-2,\sigma'}^{\beta}u_{n-1,\sigma''}^{\gamma}\} \\ &- \nu k_n^2 u_{n,\sigma}^{\alpha} + f_{n,\sigma}^{\alpha}\end{aligned} \tag{7}$$

where $\Delta_{\alpha,\beta}$ is the Kronecker symbol. Observe that we use Greek indices for components in α-Fourier space, and Latin indices for copies in the copy space. As a consequence of the discrete translation symmetry of the copy index $[i]$, Eq. (7) conserves α-momentum modulo N at the nonlinear vertex, as one can see explicitly in the above equation. The coupling amplitudes $\Phi^{\alpha,\beta,\gamma}$ in these equations are the Fourier transforms of the amplitudes $D^{[ij\ell]}$.

We choose the coupling amplitudes according to

$$\Phi^{\alpha,\beta,\gamma} = \frac{1}{\sqrt{N}}\left[\varepsilon + \sqrt{1-\varepsilon^2}\Psi^{\alpha,\beta,\gamma}\right] \qquad (8)$$

where $\Psi^{\alpha,\beta,\gamma}$ are quenched random phases, uniformly and independently distributed with zero average (cf. the "Random Coupling Model" (RCM) for the Navier-Stokes statistics[10] and the identical symmetry conditions there). Consequently for $\varepsilon = 0$, the model reduces to the RCM and exhibits normal scaling (K41) for $N \to \infty$. It was in fact understood[10,11] that in the limit $N \to \infty$ the direct interaction approximation (DIA) becomes the exact solution of the RCM. Moreover, a proper analysis of the DIA approximation leads to normal scaling for those systems in which sweeping effects are removed or absent by construction like in shell models.[11,12]

On the other hand for $\varepsilon = 1$, the coupling coefficients in the α-Fourier space (8) are index-independent. This corresponds to uncoupled Eq. (3) in the copy space, because in this case $D^{[ij\ell]} = \delta_{i,j}\delta_{i,\ell}$. Thus for $\ell = 1$, we recover the original Sabra model with anomalous scaling.[8] Our choice of couplings (8) allows an interpolation between the normal K41 scaling for $\varepsilon = 0$ (at $N \to \infty$) and the full anomalous scaling of Sabra model for $\varepsilon = 1$. A model of this type was proposed in the context of Navier-Stokes statistics by Kraichnan[13] and analyzed by Eyink[11] in terms of perturbative expansions.

3. NUMERICAL INVESTIGATION

Our aim here is to present intensive numerical simulation of the model with the variable parameter ε and to discover how anomalous corrections to the normal K41 scaling appear and develop with increasing ε.

We measured the scaling exponents of the structure functions:

$$S_{2p}(k_n) \equiv \frac{1}{N}\left\langle \sum_{i=-J}^{J} \sum_{\sigma} |u_{n,\sigma}^{[j]}|^{2p} \right\rangle \sim k_n^{-\zeta_{2p}}\,. \qquad (9)$$

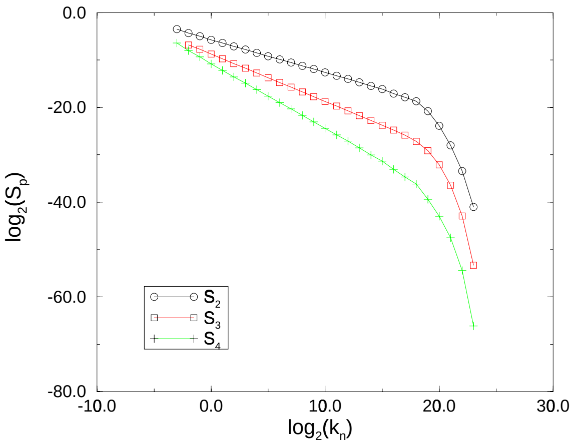

Fig. 1 Log-log plot of the structure functions $S_p(k_n)$ vs. k_n for $p = 2, 3, 4$, $\varepsilon = 0.8$ and $N = 25$.

The exponents ζ_{2p} have been calculated by a linear fit in the two decades inertial range (see Fig. 1). The equations of motion (7) with 28 shells, $a = 1$, $b = c = -0.5$, were integrated with the slaved Adams-Bashforth algorithm, viscosity $\nu = 4\times10^{-9}$, a time-step $\Delta t = 10^{-5}$. The forcing was applied on the first two shells, chosen random Gaussian with zero average and with variances such that $\sigma_2/\sigma_1 = 0.7$ (in order to minimize the input of helicity[8] which leads to period two oscillation in the structure functions). Averages were taken for a time equal to 250 eddy turnover times for the case $N = 1$. The averaging times were decreased when the number of copies increased, taking into account the faster convergence times in these cases. The quality of the scaling behavior and of the fits is demonstrated in Fig. 1.

To substantiate the birth of anomalous scaling at $\varepsilon > 0$, we simulated the model for different values of ε and of N. We are interested in the values of the scaling exponents for very large values of N (∞ in theory). In Fig. 2, one can see the plot of the value of the anomalous corrections to Kolmogorov scaling, $\delta\zeta_2 = \zeta_2 - 2/3$, as a function of $1/N$ for $\varepsilon = 0.8$ together with the same curve for $\varepsilon = 0$ and for $\varepsilon = 1$, for N ranging from 5 to 25. While for $\varepsilon = 0$, the corrections to Kolmogorov scaling go to zero, for $\varepsilon = 0.8$ and for $\varepsilon = 1$, the corrections converge to a finite value which increases with ε.

The random phases in the couplings were chosen with respect to a uniform probability with zero-mean at the beginning of each simulation. The rigorous procedure for quenched disorder would call for taking averages over different runs with different choices of the couplings. We did not do that,

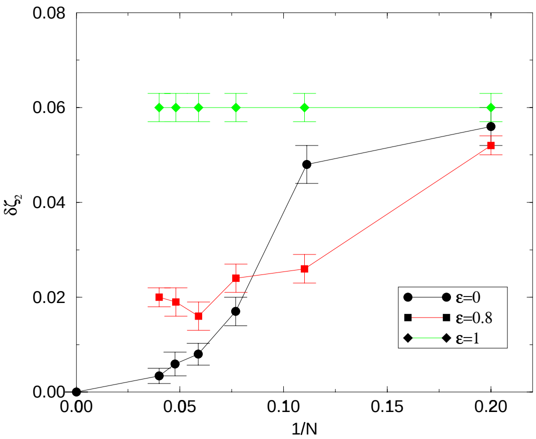

Fig. 2 $\delta\zeta_2 = \zeta_2 - 2/3$ vs. $1/N$ for $\varepsilon = 0$ (circles), $\varepsilon = 0.8$ (squares), and $\varepsilon = 1$ (diamonds) for N from 5 to 25. The point at $1/N = 0$ for the $\varepsilon = 0$ curve is the theoretical prediction of the RCM for $N \to \infty$.

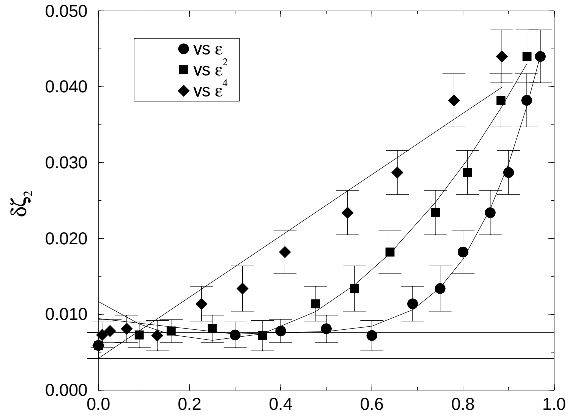

Fig. 3 $\delta\zeta_2 = \zeta_2 - 2/3$ vs. ε^4 (circles), vs. ε^2 (squares), and vs. ε (diamonds) together with linear, quadratic and quartic fits, respectively for $N = 25$.

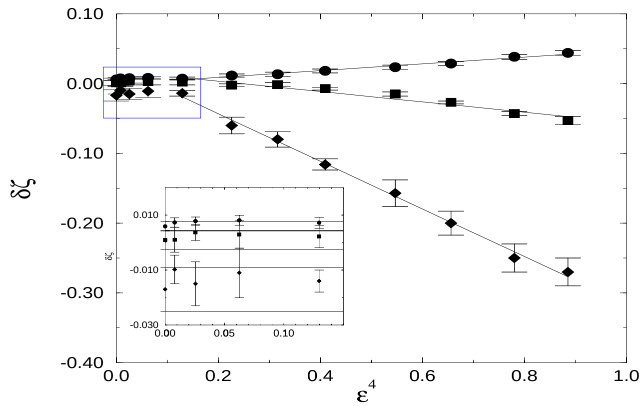

Fig. 4 $\delta\zeta_2 = \zeta_2 - 2/3$ (circles), $\delta\zeta_4$ (squares), and $\delta\zeta_6$ (diamonds) vs. ε^4 for $N = 25$.

but rather checked that self-averaging is already valid for $N = 5$, for $\varepsilon = 0.8$ (small random component in the couplings) and within our numerical precision. For $\varepsilon = 0$, self-averaging occurs only for large numbers of copies.

The most interesting aspects of the numerical findings are the dependence of the anomalies on ε and the question whether anomaly appears for any $\varepsilon > 0$ or only above a critical value ε_c. The first issue is settled with sufficient clarity in Fig. 3 in which we show the behavior of $\delta\zeta_2$ (for $N = 25$) as a function of ε, ε^2 and ε^4 with the respective linear, quadratic and quartic fits. One can see clearly that the anomalous corrections $\delta\zeta_2$ go to zero like ε^4. The same behavior is exhibited by $\delta\zeta_4$ and $\delta\zeta_6$ as one can see in Fig. 4.

The existence of a critical value ε_c is harder to settle. In the following, we show arguments in favor of the existence of a finite ε_c. First, one should notice in the magnification in Fig. 4 that at $\varepsilon = 0$, we do not get the K41 values $\delta\zeta_n = 0$ as theoretically predicted. This is a result of the finiteness of the number of copies ($N = 25$ in Fig. 4).

To establish the existence of a finite ε_c we proceeded as follows: (i) to take into account the finite size effects, we subtracted the $\varepsilon = 0$ value of $\delta\zeta_n$ from all the values of ζ_n; (ii) we calculated the maximal and minimal slope line, that is the best fit slope plus and minus the standard deviation, for $\delta\zeta_n$ versus ε^4 in two different ranges: $\varepsilon^4 \in [0.13, 0.88]$ and $\varepsilon^4 \in [0.23, 0.88]$, i.e. we fitted with the two-parameter function $f(\varepsilon^4) = a_n + b_n\varepsilon^4$; (iii) we fitted the plot of $\delta\zeta_n$ versus ε with the four-parameter function $g(\varepsilon) = a_n\varepsilon + b_n\varepsilon^2 + c_n\varepsilon^3 + d_n\varepsilon^4$ in the range $\varepsilon \in [0, 0.97]$. Note that with the "recalibration" of zero that we performed, $\delta\zeta_n = 0$ for $\varepsilon = 0$. The χ^2 test is much better for the first procedure than for the second, where the minimal χ^2 is tripled, although in principle it should be easier to fit a function with a larger number of parameters.

We interpret this result as an evidence for the existence of a finite ε_c. In order to have a better estimate of the "zero" level of $\delta\zeta_n$'s, and so a better estimate of ε_c, instead of subtracting the value of the anomalous correction at $\varepsilon = 0$, we subtracted the average value of $\delta\zeta_n$ calculated by using the first five points belonging to the flat region. The value of ε_c with its error has been found by looking at the intersections of the minimal and maximal slope lines with the zero line: the line of the value of $\delta\zeta_n$ for $\varepsilon = 0$ with its error (see Fig. 5). Table 1 exhibits the resulting values for the various exponents and their linear fits. Note that all the values of ε_c coincide within the error bars.

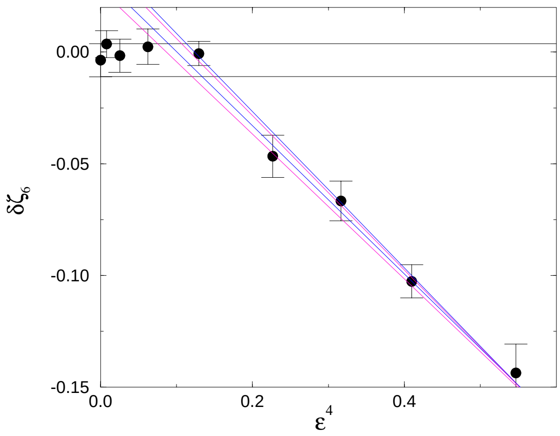

Fig. 5 $\delta\zeta_6 = \zeta_6 - 2$ vs. ε^4 for $N = 25$ with the maximal and minimal slope lines obtained for the linear fits in the two ranges $\varepsilon^4 \in [0.13, 0.88]$ and $\varepsilon^4 \in [0.23, 0.88]$. All the values of the anomalous corrections have been shifted by the average of the first five points.

Table 1 Results of the linear fits on different ranges for different scaling exponents. The values of ε_c have been calculated as explained in the body of the text.

$\delta\zeta_n$	Range of ε^4	Slope $\times$ 10^3	ε_c
$\delta\zeta_2$	[0.13, 0.88]	48 ± 3	0.60 ± 0.07
$\delta\zeta_2$	[0.23, 0.88]	50 ± 3	0.62 ± 0.06
$\delta\zeta_4$	[0.13, 0.88]	-74 ± 8	0.69 ± 0.07
$\delta\zeta_4$	[0.13, 0.88]	-82 ± 9	0.71 ± 0.06
$\delta\zeta_6$	[0.13, 0.88]	-341 ± 8	0.59 ± 0.04
$\delta\zeta_6$	[0.23, 0.88]	-335 ± 6	0.57 ± 0.05

4. DISCUSSION

In Ref. 9, we discussed the hierarchy of equation for the nth order correlation functions F_n and the Green's (response) functions G_n, and showed that in the limit $N \to \infty$, the closure procedure suggested in Ref. 5 is controlled by the small parameter ε. In other words, it is possible to express, say, F_4 in terms of the lowest order objects F_2, F_3 and G_2, G_3. In doing so, one surely leaves out some information about F_4 that cannot possibly be represented in these terms. Yet, the main result of the analysis of Ref. 9 is that the neglected terms in this procedure are of $O(\varepsilon^6)$ whereas the retained terms are of $O(1)$ and of $O(\varepsilon^4)$! Since we understand that for $\varepsilon = 0$, the anomaly must vanish, we expect the anomalies to be proportional to ε^4. We interpret therefore the numerical results shown in Figs. 3 and 4 as an excellent confirmation of this theoretical expectation.

ACKNOWLEDGMENTS

This work has been supported in part by the European Commission under the Training and Mobility of Researchers program, the German-Israeli Foundation, the Israel Science Foundation administered by the Israel Academy of Sciences, and the Naftali and Anna Backenroth-Bronicki Fund for Research in Chaos and Complexity.

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