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Victor S. L'vov

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Chapter 1

INTRODUCTION TO NONLINEAR WAVE DYNAMICS

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Introduction to Nonlinear Wave Dynamics

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List of Main Symbols with Short Comments

All symbols are fully defined where they are first introduced in the text. As a convenience to the reader some of the most frequently used symbols are collected here.

$a(\boldsymbol{k},t)=a_{\boldsymbol{k}},\ b(\boldsymbol{k},t)=b_{\boldsymbol{k}},$ $c(\boldsymbol{k},t)=c_{\boldsymbol{k}},\ d(\boldsymbol{k},t)=d_{\boldsymbol{k}}$	canonical variables – complex amplitudes of $\boldsymbol{k}$–waves (having wave vector $\boldsymbol{k}$). These are classical analogues of the Bose operators, so-called c–numbers
$\mathcal{H}=\mathcal{H}_2+\mathcal{H}_{\rm int}$	Hamiltonian function or Hamiltonian which is the classical analogue the Hamiltonian operator in quantum mechanics. Here:
$\mathcal{H}_2=\sum_{\boldsymbol{k}}\omega_{\boldsymbol{k}}b^*_{\boldsymbol{k}}b_{\boldsymbol{k}}$	Hamiltonian of non-interacting $\boldsymbol{k}$–waves having frequency $\omega_{\boldsymbol{k}}$, quadratic in canonical variables, and
$\mathcal{H}_{\rm int}=\mathcal{H}_3+\mathcal{H}_4+\dots$	interaction Hamiltonian. Here:
$\mathcal{H}_3,\ \mathcal{H}_4$	the parts of $\mathcal{H}_{\rm int}$, of the third– and fourth–order in canonical amplitudes which describe the three– and four–wave processes of interaction
$V(\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)=V_{\mathbf{1},\mathbf{23}}$	amplitudes of three-wave interaction of type $\boldsymbol{k}_1\Leftrightarrow\boldsymbol{k}_2+\boldsymbol{k}_3$
$T(\boldsymbol{k},\boldsymbol{k}')=T_{\boldsymbol{k}\boldsymbol{k}'}$	amplitudes of four-wave interaction of type $\boldsymbol{k}+\boldsymbol{k}'\Leftrightarrow\boldsymbol{k}+\boldsymbol{k}'$, describing the nonlinear frequency shift
$S(\boldsymbol{k},\boldsymbol{k}')=S_{\boldsymbol{k}\boldsymbol{k}'}$	amplitudes of four-wave interaction of type $\boldsymbol{k}+(-\boldsymbol{k})\Leftrightarrow\boldsymbol{k}\prime+(-\boldsymbol{k}')$, describing the self-consistent interaction of pairs of wave $\pm\boldsymbol{k}$ and $\pm\boldsymbol{k}'$
$\gamma(\boldsymbol{k})=\gamma_{\boldsymbol{k}}$	damping of $\boldsymbol{k}$-wave
$n(\boldsymbol{k},t)=n_{\boldsymbol{k}}$	simultaneous double correlation function of $\boldsymbol{k}$–waves, "number" of $\boldsymbol{k}$–waves
$N=\sum_{\boldsymbol{k}}n_{\boldsymbol{k}}$	total number of waves
Θ,φ	polar and azimuthal angles
$\propto,\quad\approx$	proportional, approximately equal
$\simeq$	of the same order

1 Introduction to Nonlinear Wave Dynamics

Our aim is to describe the nonlinear properties of a system of interacting waves in various media from a common point of view. We would like to abstain from using the features of these media until it is absolutely necessary for studying a particular problem. On the other hand, the method of description must be sufficiently well-known and convenient. The classical Hamiltonian description method of nonlinear waves (including spin waves in magnetodielectrics) described in Sect. 1.1 seems to meet these requirements. By this method, the Hamiltonian equations for the canonical variables, i.e. complex wave amplitudes in continuous media, will be obtained, the general structure of the Hamiltonian function (*Hamiltonian*) for small-amplitude waves, irrespective of their physical nature, will be studied, and the dynamic perturbation theory will be formulated, so that nonresonant terms from the Hamiltonian of interaction should be excluded.

We should leave aside unwieldy calculations of the Hamiltonian function of the spin waves in magnetodielectrics until later. On the other hand, the general theory of nonlinear waves is better illustrated by simple specific examples. Therefore in Sect. 1.2, the dimensional analysis yields expressions for the frequency and interaction amplitudes of sound waves, of waves on a liquid surface, and of the simplest type of spin waves in ferromagnets. The following sections of the present chapter discuss self-action and interaction of almost monochromatic wave packets. They may be described by dynamic equations of motion for complex amplitudes. Phase relations in the equations are essential. Therefore these processes can naturally be called *dynamic*. On the other hand, interactions of wave packets broad in $\boldsymbol{k}$-space with almost random phases may be described by the kinetic equations for wave population number. Such processes can be called *kinetic*. These will be treated later, in Chap. 10.

In Section 1.3 of this chapter we shall add a phenomenological term to the Hamiltonian equations. This term describes a weak interaction of the dynamic wave system with the environment which serves as a thermal bath. This makes it possible to obtain from the very beginning much more realistic approximations in describing the wave dynamics than it is possible under the purely Hamiltonian approach (which is applicable to systems conserving their energy). This procedure will be rigorously proven by means of diagram technique. Its applicability domain was given, for example, in Chap. 7 of my Russian book [1.1].

Sections 1.4 and 1.5 of this chapter describe some dynamic processes (confluence of two waves into one; generation of the second harmonic; decay of one wave into two various four-wave processes, including self-focusing and collapse) irrespective of the wave type and the medium in which the waves propagate. Only in the third chapter shall we study the specific characteristics of these processes using the explicit form of dispersion laws and the interaction Hamiltonian of spin waves in ferromagnets and antiferromagnets.

1.1 Hamiltonian Method for Description of Waves in a Continuous Medium

The Hamiltonian method is applicable to a wide group of weakly interacting and weakly dissipative wave systems and is used to reveal the common properties of such system. Indeed, equations of motion for waves may vary considerately if these equations are written in terms of natural medium variables. For instance, the Bloch equations describing magnetic moment motion do not resemble the Maxwellian equations for a nonlinear dielectric. The latter differ drastically from the Eulerian equations for a compressible liquid. At the same time spin electromagnetic and sound waves are primarily waves, i.e. oscillations of the medium transferred from one of its points to another. If we are interested only in the propagation peculiarities of waves with small amplitudes, e.g. in the diffraction phenomenon, it is not important whether it is the it magnetic moment, electric field or medium density that oscillates. To study how non-interacting waves propagate in the medium it is quite sufficient to use information given by the dispersion law $\omega(\boldsymbol{k})$. Similarly, the other expansion coefficients of the Hamiltonian provide all necessary (and almost superfluous) information for investigating the nonlinear properties of the wave system. It is therefore clear that two wave systems with similar dispersion laws and $\boldsymbol{k}$-dependencies of "essential" Hamiltonian coefficients will show a similar nonlinear behavior. Their equations of motion in natural variables may at the same time look completely different.

The method of second quantization can also reveal the common properties of various wave systems. In this method, operators characterizing the natural variables of the medium are represented in terms of creation and annihilation Bose operators. This quantum-mechanical method is commonly used in the physics of nonlinear waves, in spite of the fact that the values of population numbers under which nonlinear effects are significant are usually great and the wave system is classical. Therefore when the powerful method of second quantization is used at the stage of problem statement, the Planck constant has to appear. In the resulting expressions it is afterwards cancelled. Such inconsistency must be due to the fact that modern physicists are taught quantum mechanics in much more detail than classical mechanics.

In our opinion, the problems of nonlinear wave dynamics are much easier to solve using the classical Hamiltonian method [1.1-5]. One can avoid unnecessary difficulties arising from the non-commutativity of operators a and a^+. The Hamiltonian method provides as simple an interpretation as the method of second quantization since the complex canonical variables a and a^* are classical analogues (c-numbers) of the Bose operators. That is why we shall use the classical term of *spin waves* and the quantum-mechanical term of *magnons* as near synonyms.

1.1.1 Hamiltonian Equations of Motion

The Hamiltonian equations for systems with a single degree of freedom, as is well known, have the form [1.2]:

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} . \tag{1.1.1}$$

Here the Hamiltonian function $\mathcal{H}$ (henceforth called *Hamiltonian*) depends on the canonical variables, i.e. the generalized coordinate q and generalized momentum p. Usually, $\mathcal{H}$ is the system energy expressed in terms of the canonical variables. Systems with n degrees of freedom are characterized by pairs of canonically conjugated variables $q_1, q_2, ...q_n$ and $p_1, p_2, ...p_n$. They satisfy the equations

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial P_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} , \tag{1.1.2}$$

where $\mathcal{H}$ is a function of all q_i and p_i. In the simplest case the continuous medium can be characterized by a pair of canonical variables $q(\boldsymbol{r}, t), p(\boldsymbol{r}, t)$ at each point $\boldsymbol{r}$. The equations of motion for $q(\boldsymbol{r}, t),\ p(\boldsymbol{r}, t)$ are obtained by generalizing (1.1.2):

$$\frac{\partial q(\boldsymbol{r}, t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta p(\boldsymbol{r}, t)}, \qquad \frac{\partial p(\boldsymbol{r}, t)}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q(\boldsymbol{r}, t)} . \tag{1.1.3}$$

The Hamiltonian $\mathcal{H}$ is dependent on $p(t)$ and $q(t)$ taken at all the points $\boldsymbol{r}$, i.e. it is a functional of $q(\boldsymbol{r}, t),\ p(\boldsymbol{r}, t)$. The symbols $\delta/\delta p$ and $\delta/\delta q$ designate variational derivatives which generalize notions of partial derivatives for the case of continuous degrees of freedom.

It must be noted that (1.1.3) specify the dynamics of only one wave type, e.g. the sound. To allow for several wave types or polarizations the medium must be characterized at each point $\boldsymbol{r}$ by several pairs of variables

$q_j(\boldsymbol{r},t)\,,\ p_j(\boldsymbol{r},t)\,,\ j=1...$. The respective generalization of the equations of motion (1.1.3) yields

$$\frac{\partial q_j(\boldsymbol{r},t)}{\partial t}=\frac{\delta\mathcal{H}}{\delta p(\boldsymbol{r},t)}\,,\qquad \frac{\partial p_j(\boldsymbol{r},t)}{\partial t}=-\frac{\delta\mathcal{H}}{\delta q_j(\boldsymbol{r},t)}\,. \tag{1.1.4}$$

1.1.2 Transfer to Complex Variables

A formal advantage of the Hamiltonian method is the symmetrical representation of the equation for the coordinate q and the momentum p. To this end, first change over to new canonical variables $Q(\boldsymbol{r})=\lambda q(\boldsymbol{r}), P(\boldsymbol{r})=p(\boldsymbol{r})/\lambda$, the dimension factor λ being chosen in such a way that P and Q have the same dimension. Then introduce complex variables:

$$a_j=(Q_j+iP_j)/\sqrt{2}\,,\qquad a_j^*=(Q_j-iP_j)/\sqrt{2} \tag{1.1.5}$$

with equation of motion

$$\sqrt{2}\frac{\partial a_j}{\partial t}=\frac{\delta\mathcal{H}}{\delta P_j}-i\frac{\delta\mathcal{H}}{\delta Q_j}\,,\qquad \sqrt{2}\frac{\partial a_j^*}{\partial t}=\frac{\delta\mathcal{H}}{\delta P_j}+i\frac{\delta\mathcal{H}}{\delta Q_j}\,.$$

Substituting in the above expression $\mathcal{H}(a,a^*)$ we obtain

$$i\frac{\partial a_j}{\partial t}=\frac{\delta\mathcal{H}}{\delta a_j^*}\,,\qquad -i\frac{\partial a_j^*}{\partial t}=\frac{\delta\mathcal{H}}{\delta a_j}\,, \tag{1.1.6}$$

The second equation may be obtained from the first one by complex conjugation and, consequently, instead of two real equations (1.1.6) we obtain one complex equation. In quantum mechanics there is a corresponding change from the coordinate-momentum representation to the representation of creation and annihilation Bose operators. Their classical analogues are the complex canonical variables.

Canonical variables (1.1.5) are by no means unique. There is a wide range of possible changes from the canonical variables (a,a^*) to other variables (b,b^*) where the equations of motion retain their canonical form:

$$i\frac{\partial b(\boldsymbol{r},t)}{\partial t}=\frac{\delta\mathcal{H}}{\delta b^*(\boldsymbol{r},t)}\,, \tag{1.1.7}$$

Such transformations are called *canonical* [1.2].

The possibility of choosing various canonical variables is an important advantage of the Hamiltonian method. Thus, the variables suitable for a given problem can be chosen.

1.1.3. Hamiltonian Structure Under Small Nonlinearity

For greatly varying problems of nonlinear wave dynamics the wave amplitude may be characterized by some natural dimensionless parameter x. For sound waves the parameter is the relation between the density variation in a sound wave and the mean medium density; for surface waves in liquids it is the relation between the vertical declination and the wavelength. For spin waves the angle of the precession of the magnetic moment serves as a dimensionless parameter x. If the parameter x is of the order unity it leads to phenomena specific to a given problem, e.g. sound is transformed into shock waves, "white horses" appear on the liquid surface, magnetization in ferromagnets is reversed, producing a domain wall which, generally speaking, is able to move. Obviously, it is not worth considering all these phenomena from a single viewpoint. However, if the parameter of wave nonlinearity is small, the specific features of the medium are no longer essential and the wave dynamics can be described in general terms (irrespective of these features) i.e. in terms of the dispersion law $\omega(\boldsymbol{k})$, group and phase velocity, probabilities of elementary processes of interaction involving three, four waves, etc. In this section it will be shown how these notions are introduced in the scope of the Hamiltonian formalism under small nonlinearity x.

The canonical variables $a\ a^*$ will be chosen so as to characterize the wave amplitude and become zero in the absence of a wave. The index $j = 1, ...n$ determines the wave polarization or type, since, generally speaking, various waves can be simultaneously excited in media: sound, light, etc. Assuming $a\ , a^*$ to be small, let us expand the functional $\mathcal{H}\{a(\boldsymbol{r},t), a^*(\boldsymbol{r},t)\}$ in a power series in $a\ , a^*$. We are not interested in the zero term in $\mathcal{H}\{0,0\}$, since it does not enter into the equations of motion: $\delta\mathcal{H}\{0,0\}/\delta a = 0$. The terms of the first order $\mathcal{H}$ are zero, since we take the medium to be in an equilibrium state in absence of waves, consequently, with a minimum at $a = a^* = 0$. Therefore, the expansion of $\mathcal{H}$ begins with the terms of the second order

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}}\ . \tag{1.1.8}$$

The most general form of $\mathcal{H}_2$ is

$$\begin{aligned}\mathcal{H}_2 = \sum_{i,j=1}^{n} \int \Big\{ & A_{ij}(\boldsymbol{r},\boldsymbol{r}')a_i(\boldsymbol{r},t)a_j^*(\boldsymbol{r}',t) \\ & + \frac{1}{2}[B_{ij}^*(\boldsymbol{r},\boldsymbol{r}')a_i(\boldsymbol{r},t)a_j(\boldsymbol{r}',t) + \text{c.c.}]\Big\}\, d\boldsymbol{r}d\boldsymbol{r}'\ .\end{aligned} \tag{1.1.9}$$

Here "c.c" denotes complex conjugate. As $\mathcal{H}$ must be Hermitian

$$A_{ij}(\boldsymbol{r},\boldsymbol{r}') = A_{ij}^*(\boldsymbol{r}',\boldsymbol{r})\,, \quad B_{ij}(\boldsymbol{r},\boldsymbol{r}') = B_{ji}(\boldsymbol{r}',\boldsymbol{r})\ . \tag{1.1.10}$$

In a spatially homogeneous medium A and B depend only on the coordinate difference $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ and if there is inversion symmetry, they are even functions of the difference $\boldsymbol{R}$. Then it follows from (1.1.10) that

$$A_{ij}(\boldsymbol{R}) = A^*_{ij}(-\boldsymbol{R}) , \qquad B_{ij}(\boldsymbol{R}) = B_{ij}(-\boldsymbol{R}) . \tag{1.1.11}$$

The Hamiltonian can be significantly simplified by the Fourier transform

$$\begin{aligned} a(\boldsymbol{k}, t) &= a_{\boldsymbol{k}} = \frac{1}{V_s} \int a(\boldsymbol{r}, t) \exp(-i\boldsymbol{k}\boldsymbol{r})\, d\boldsymbol{r} \ , \\ a(\boldsymbol{k}, t) &= \sum_{\boldsymbol{k}} a(\boldsymbol{k}, t) \exp(i\boldsymbol{k}\boldsymbol{r}) \ , \end{aligned} \tag{1.1.12}$$

where V_s is the sample volume (of the medium in which the waves propagate). We consider the wave vector $\boldsymbol{k}$ to be a discrete variable. If necessary, one may pass from summation over $\boldsymbol{k}$ to integration

$$(2\pi)^3 \sum_{\boldsymbol{k}} = V_s \int d\boldsymbol{k} \tag{1.1.13}$$

in the conventional way. The Fourier transform (1.1.12) is canonical but not unimodal. This implies that the Hamiltonian equation retains its canonical form but the new Hamiltonian differs from the previous one by a factor: it is divided by the sample volume

$$i\partial a_{\boldsymbol{k}}/\partial t = \delta\mathcal{H}(a_{\boldsymbol{k}}, a^*_{\boldsymbol{k}})/\delta a^*_{\boldsymbol{k}} , \quad \mathcal{H}(a_{\boldsymbol{k}}, a^*_{\boldsymbol{k}}) = \mathcal{H}(a(\boldsymbol{r}), a^*(\boldsymbol{r}))/V_s \ . \tag{1.1.14}$$

It is significant that in the new variables $a(\boldsymbol{k}, t)$ the quadratic part of the Hamiltonian is a sum over $\boldsymbol{k}$ and contains no summation over $\boldsymbol{k}'$:

$$\begin{aligned} \mathcal{H}_2 = & \sum_{i,j} \sum_{\boldsymbol{k}} \Big\{ A_{ij}(\boldsymbol{k}) a_i(\boldsymbol{k}, t) a^*_j(\boldsymbol{k}, t) \\ & + \frac{1}{2}[B^*_{ij}(\boldsymbol{k}) a_i(\boldsymbol{k}, t) a_j(-\boldsymbol{k}, t) + \text{c.c.}] \Big\} \ , \end{aligned} \tag{1.1.15}$$

$$\begin{aligned} A_{\boldsymbol{k}} = & A(\boldsymbol{k}) = \int A(\boldsymbol{R}) \exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R} , \\ B_{\boldsymbol{k}} = & B(\boldsymbol{k}) = \int A(\boldsymbol{R}) \exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R} \ . \end{aligned} \tag{1.1.16}$$

Obviously, the spatial homogeneity of the medium is responsible for this fact.

In some cases the Hamiltonian (1.1.15) may be diagonalized in “wave types” by means of a linear transformation

$$b_i(\boldsymbol{k}) = \sum_j [u_{ij}(\boldsymbol{k}) a_j(\boldsymbol{k}) + v_{ij}(\boldsymbol{k}) a^*_j(-\boldsymbol{k})] \, . \tag{1.1.17}$$

The matrices $[u]$ and $[v]$ may be chosen so that the Hamiltonian $\mathcal{H}$ takes the form

$$\mathcal{H}_2 = \sum_j \sum_{\boldsymbol{k}} \omega_j(\boldsymbol{k}) b_j(\boldsymbol{k}) b_j^*(\boldsymbol{k}) \ . \tag{1.1.18}$$

Diagonal elements of the matrix $[u]$ will further be considered real. This can always be arranged through suitable choice of phases for complex variables $b(\boldsymbol{k}, t)$.

Let us consider in more detail the case when waves of a single type propagate in the medium. Taking in (1.1.15, 17, 18) $i = j = 1$, substitute (1.1.17) into (1.1.15). By condition of coincidence of (1.15) and (1.18) we obtain:

$$\omega(\boldsymbol{k})[u^2(\boldsymbol{k}) + |v(\boldsymbol{k})|^2] = A(\boldsymbol{k}) \, , \qquad 2\omega(\boldsymbol{k}) u(\boldsymbol{k}) v(\boldsymbol{k}) = B^*(\boldsymbol{k}) \, , \tag{1.1.19a,b}$$

$$2\omega(\boldsymbol{k}) u(\boldsymbol{k}) v^*(\boldsymbol{k}) = B(\boldsymbol{k}) \, , \qquad u^2(\boldsymbol{k}) - |v(\boldsymbol{k})|^2 = 1 \ . \tag{1.1.19c,d}$$

If the second relation is satisfied, the transformation is canonical. Comparison of (1.1.19b) and (1.1.19c) shows that obtaining a diagonalizing transformation is possible only if $\omega(\boldsymbol{k})$ is real. Note that in the variables $b_j(\boldsymbol{k})$ the equations of motion (1.1.7) with the Hamiltonian (1.1.18) take a trivial form

$$\partial b_j(\boldsymbol{k}, t)/\partial t + i\omega_j(\boldsymbol{k}, t) b(\boldsymbol{k}, t) = 0 \, , \tag{1.1.20a}$$

and have a solution

$$b_j(\boldsymbol{k}, t) = b(\boldsymbol{k}, 0) \exp[i\omega_j(\boldsymbol{k})t] \ . \tag{1.1.20b}$$

Therefore the condition of $\omega(\boldsymbol{k})$ being real simply implies the medium's stability with respect to an exponential increase of the wave amplitudes. $A(\boldsymbol{k})$, as follows from (1.1.19a) must in this case be real.

From (1.1.18) transformation coefficients can be easily obtained

$$u(\boldsymbol{k}) = \sqrt{\frac{A(\boldsymbol{k}) + \omega(\boldsymbol{k})}{2\omega(\boldsymbol{k})}} \, , \qquad v(\boldsymbol{k}) = -\frac{B(\boldsymbol{k})}{|B(\boldsymbol{k})|} \sqrt{\frac{A(\boldsymbol{k}) - \omega(\boldsymbol{k})}{2\omega(\boldsymbol{k})}} \ . \tag{1.1.21}$$

as well as two expressions of the opposite sign for the frequency

$$\omega(\boldsymbol{k}) = \pm \sqrt{A^2(\boldsymbol{k}) - |B(\boldsymbol{k})|^2} \ . \tag{1.1.22}$$

According to the relation (1.1.18a) we choose these expression of the same sign as $A(\boldsymbol{k})$ (*Zakharov* [1.3]).

If there are several oscillation branches, finding a diagonalizing representation is not easy, though the frequencies $\omega(\boldsymbol{k})$ can be determined without it. The equations of motion (1.1.14) with the Hamiltonian (1.1.15) must be written for the variables $a_j(\boldsymbol{k}, t)$:

$$\frac{\partial a_j(\boldsymbol{k},t)}{\partial t} + i\sum_i [A_{ij}(\boldsymbol{k})a_i(\boldsymbol{k},t) + B_{ij}(\boldsymbol{k})a_i^*(-\boldsymbol{k},t)] = 0 ,$$
$$\frac{\partial a_j^*(-\boldsymbol{k},t)}{\partial t} - i\sum_i [A_{ij}^*(\boldsymbol{k})a_i^*(-\boldsymbol{k},t) + B_{ij}^*(-\boldsymbol{k})a_i(\boldsymbol{k},t)] = 0 .$$

Substituting a, $a^* \propto \exp(-i\omega t)$ into this expression yields an algebraic "secular" equation for obtaining the frequencies:

$$\begin{bmatrix} A_{ij} - \omega\delta_{ij}, & B_{ij} \\ B_{ij}^*, & A_{ij}^* - \omega\delta_{ij} \end{bmatrix} = 0 . \tag{1.1.23}$$

Note that the diagonalization of the Hamiltonian (1.1.17) is possible if all the roots of (1.1.23), i.e. the wave frequencies $\omega_j(\boldsymbol{k})$, are real.

Variables $b_j(\boldsymbol{k},t)$ are normal variables of linear theory and therefore are especially suitable for solving nonlinear problems. All the "linear" difficulties of the medium model are tackled in the single step of obtained the transformation to the b_j variables. In these variables the linearized equations of motion become trivial (see (1.1.20a)). They describe the propagation of free waves obeying the dispersion laws $\omega_j(\boldsymbol{k})$. All the "linear" information required for the investigation of nonlinear problems is contained in the functions $\omega_j(\boldsymbol{k})$. All necessary data on the interaction of waves can be found in the other coefficients of the expansion of the Hamiltonian $\mathcal{H}$ in powers of b_j:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}} , \qquad \mathcal{H}_{\text{int}} = \mathcal{H}_3 + \mathcal{H}_4 + ... \tag{1.1.24}$$

The physical meaning of $\mathcal{H}_3$, $\mathcal{H}_4$ can be easily explained by analogy with quantum mechanics. The Hamiltonian $\mathcal{H}_3$ describes three-wave processes. In a simple case when all the waves are of a single type

$$\begin{aligned} \mathcal{H}_3 =& \frac{1}{2}\sum_{\mathbf{123}} (V_q b_{\mathbf{1}} b_{\mathbf{2}} b_{\mathbf{3}}^* + \text{c.c.})\delta(\mathbf{1}+\mathbf{2}-\mathbf{3}) \\ &+ \frac{1}{3}\sum_{\mathbf{123}} (U_q^* b_{\mathbf{1}}^* b_{\mathbf{2}}^* b_{\mathbf{3}}^* + c.c.)\delta(\mathbf{1}+\mathbf{2}+\mathbf{3}) . \end{aligned} \tag{1.1.25}$$

Hereafter we shall use the following short notation $b_{\mathbf{1}}, b_{\mathbf{2}}, ...$ to denote $b(\boldsymbol{k}_1,t), b(\boldsymbol{k}_2,t)$; $\delta(...)$ is the Kronecker symbol, $\delta(\mathbf{1}+\mathbf{2}+\mathbf{3})$ designates $\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)$ which represents the law of conservation of momentum. The multi index q denotes $(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$, therefore $V_q = V(q) = V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$, and finally

$$\sum_{\mathbf{123}} = \sum_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} , \qquad \sum_{\mathbf{1+2=3}} = \sum_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} \delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3) .$$

The Hamiltonian $\mathcal{H}_4$ describes the four-wave processes

$$\mathcal{H}_4 = \frac{1}{4}\sum_{1+2=3+4} W_p b_1^* b_2^* b_3 b_4 + \frac{1}{6}\sum_{1=2+3+4}(G_p b_1 b_2^* b_3^* b_4^* + \text{c.c.}) + \frac{1}{6}\sum_{1+2+3+4=0}(R_p b_1 b_2 b_3 b_4 + \text{c.c.})\ , \quad p = (\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)\ . \tag{1.1.26}$$

The following question may arise: to which order in b, b^* must be expansion of the Hamiltonian $\mathcal{H}_{\text{int}}$ be taken? This question is rather general and can be answered with similar generality: $\mathcal{H}_{\text{int}}$ and the terms of higher order, generally speaking, need not be included. This can be supported as follows: Since expansion into a series is performed in terms of a small parameter, each subsequent term is less that its antecedent and the dynamics of the wave system will be determined by the very first expansion term in $\mathcal{H}_{\text{int}}$, i.e. $\mathcal{H}_3$. However, three-wave processes may be *nonresonant* or, equivalently, *forbidden.* This means that the condition of spatio-temporal synchronism (or, in terms of quasi-particles, the law of energy-momentum conservation):

$$\omega(\boldsymbol{k} + \boldsymbol{k}_1) = \omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) \tag{1.1.27}$$

may be impossible to satisfy. Let d denote the dimension of the medium under study, $\boldsymbol{k}$ be a vector of the d-dimensional space ($d > 1$). Equation (1.1.27) determines a $2d - 1$ hypersurface in the $2d$-dimensional space of the vectors $\boldsymbol{k}$, $\boldsymbol{k}_1$. If this surface actually exists (i.e. $\omega(\boldsymbol{k})$ is real), then the law of dispersion $\omega(\boldsymbol{k})$ is called a *decay* law and three-wave processes are allowed. If (1.1.27) has no real solutions, three-wave processes are forbidden and the law of dispersion is *non-decaying.*

In isotropic media $\omega(\boldsymbol{k})$ is a function only of the amplitude k. In this case for Goldstone modes (for which $\omega(0) = 0$) a simple criterion can be formulated: the dispersion law is decaying if $\omega''(k) > 0$ and is non-decaying if $\omega''(k) < 0$ [1.3]. In particular, if $\omega(k) \propto k^z$, dispersion laws where $z > 1$ are decaying. If the dispersion law is quadratic with a gap

$$\omega(k) = \omega_0 + (sk)^2\ , \tag{1.1.28}$$

then for small k three-wave decaying processes (1.1.27) are forbidden since the wave of minimum energy can no longer decaying. More detailed analysis of (1.1.27,28) shows that three-wave processes of decay are forbidden only for the waves where $(sk)^2 < 2\omega_0$.

It is important that 4-wave processes described by the Hamiltonian are always allowed. It is obvious from the conservation law for scattering processes

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}') = \omega(\boldsymbol{k} + \boldsymbol{\kappa}) + \omega(\boldsymbol{k}' - \boldsymbol{\kappa}), \tag{1.1.29}$$

which are allowed at $\kappa \to 0$ under any law of dispersion. Consequently, $\mathcal{H}_4$ will specify the dynamics of the wave system with non-decay spectrum and the expansion terms $\mathcal{H}_5$, $\mathcal{H}_6$, ... will describe small and, as a rule, insignificant corrections.

Obviously, this formal scheme is oversimplified. Reality again proves to be much more complex. For example, for spin waves in ferromagnets even in the decay part of the spectrum not only $\mathcal{H}_3$ but also $\mathcal{H}_4$ must be allowed for. The hamiltonian $\mathcal{H}_3$ can only be due to the magnetic dipole-dipole interaction and is relatively small as compared to the Hamiltonian $\mathcal{H}_4$ describing the exchange interaction. In some problems of wave dynamics on a liquid surface $\mathcal{H}_5$ has to be taken into consideration. Nevertheless, it is generally sufficient to allow only for three-wave processes $\mathcal{H}_3$ in the decay part of the spectrum and for four-wave processes $\mathcal{H}_4$ in the non-decay spectrum.

1.1.4 Dynamic Perturbation Theory. Elimination of "Non-resonant" terms from Hamiltonian

Consider a non-decay dispersion law when three-wave processes are forbidden by conservation laws. In this case the Hamiltonian of interaction $\mathcal{H}_3$ describing these processes has to be a certain extent non-essential. Let us show that in this case we can pass to new canonical variables $c(\boldsymbol{k},t)$, $c^*(\boldsymbol{k},t)$ such that $\mathcal{H}_3\{c(\boldsymbol{k}),c^*(\boldsymbol{k})\}=0$. The quadratic part of the Hamiltonian therewith retains its previous form

$$\mathcal{H}_2=\sum_{\boldsymbol{k}}\omega(\boldsymbol{k},t)c^*(\boldsymbol{k})c(\boldsymbol{k},t)\ , \tag{1.1.30}$$

and in the four-wave Hamiltonian of interaction there appear some additional terms quadratic in the amplitudes $\mathcal{H}\{b(\boldsymbol{k}),b^*(\boldsymbol{k})\}$ in the old variables. To this end, a quasi-linear transformation is performed

$$\begin{aligned} b_{\boldsymbol{k}} =& c_{\boldsymbol{k}}-\sum_{\boldsymbol{1,2}}\left[\frac{U_{\boldsymbol{k},\boldsymbol{1},\boldsymbol{2}}c_{\boldsymbol{1}}^*c_{\boldsymbol{2}}^*\delta(\boldsymbol{k}+\boldsymbol{1}+\boldsymbol{2})}{\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{1}}+\omega_{\boldsymbol{2}}}\right. \\ &-\left.\frac{(V_{\boldsymbol{k};\boldsymbol{1},\boldsymbol{2}}c_{\boldsymbol{1}}c_{\boldsymbol{2}}-2V_{\boldsymbol{2};\boldsymbol{k},-\boldsymbol{1}}c_{-\boldsymbol{1}}c_{\boldsymbol{2}}^*)\delta(\boldsymbol{k}-\boldsymbol{1}-\boldsymbol{2})}{\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{1}}-\omega_{\boldsymbol{2}}}\right]+O(c^3)\ . \end{aligned} \tag{1.1.31}$$

The necessity of cubic terms $O(c^3)$ in the transformation to obtain the correct value of the four-wave interaction amplitudes was first pointed out by *Krasitskii* [1.6]. The transformation (1.1.31) (with cubic terms!) is approximately canonical, with sufficient accuracy [1.6].

Substitution of (1.1.31) into the Hamiltonian (1.1.24) shows that $\mathcal{H}\{c^*,c\}$ retains the form (1.1.30), $\mathcal{H}_3=0$ and additional terms appear in matrix elements T_p. Such terms will be written down only for the scattering processes of the type $2\Rightarrow 2$ which will later be most important for us

$$\mathcal{H}_4=\frac{1}{4}\sum_{\boldsymbol{1}+\boldsymbol{2}=\boldsymbol{3}+\boldsymbol{4}}T_p c_{\boldsymbol{1}}^*c_{\boldsymbol{2}}^*c_{\boldsymbol{3}}c_{\boldsymbol{4}}\ ,\quad p=(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)\,, \tag{1.1.32}$$

$$T_p = W_p + \tilde{W}_p$$

$$\tilde{W}_p - \frac{U_{(\bar{1}+\bar{2}),1,2}U^*_{(\bar{3}+\bar{4}),3,4}}{\omega_{(3+4)} + \omega_3 + \omega_4} - \frac{V^*_{(1+2),1,2}V_{3+4;3,4}}{\omega_{(1+2)} - \omega_1 - \omega_2} - \frac{V^*_{1;3,(1-3)}V_{4;2,(4-2)}}{\omega_{(4-2)} + \omega_2 - \omega_4}$$
$$- \frac{V^*_{2;4,(2-4)}V_{3;1,(3-1)}}{\omega_{(3-1)} + \omega_1 - \omega_3} - \frac{V^*_{2;3,(3-2)}V_{4;1,(4-1)}}{\omega_{(4-1)} + \omega_1 - \omega_4} - \frac{V^*_{1;4,(1-4)}V_{3;2,(2-4)}}{\omega_{(3-2)} + \omega_2 - \omega_3} \, .$$

Here $(\boldsymbol{j} \pm \boldsymbol{i}) = \boldsymbol{k}_j \pm \boldsymbol{k}_i$. Additional terms in T_p describe the processes of scattering that occur in the second order of perturbation theory for three-wave processes. In this case a "virtual" induced wave appears at the intermediate stage; for this wave the resonance condition is not satisfied.

There is an important property of scattering processes: they do not alter the total number of waves. Therefore the equations of motion corresponding to the Hamiltonian (1.1.32) retain not only the energy integral but also the following integral

$$N = \int c^*(\boldsymbol{k}, t)c(\boldsymbol{k}, t)\, d\boldsymbol{k} \ , \qquad (1.1.33)$$

which has a meaning of the total number of quasi-particles or the integral of the wave action. Within the scope of the total system allowing also for small effects caused by higher order processes the value N is an adiabatic invariant.

In conclusion it must be recalled that the described procedure is fundamentally based on the fact that the transformation (1.1.31) is almost linear, which is possible if the frequency denominator does not become zero. In this case the wave spectrum must be non-decay. In other words, canonical transformations enable us to eliminate only such nonresonant terms of the Hamiltonian of interactions for which the conservation laws of energy-momentum are not valid. Thus the best quasi-linear canonical variables are those for which the Hamiltonian includes no terms corresponding to the forbidden processes. Note that the above-described transformation is analogous to transformation of the Hamiltonians to their normal forms, i.e., in the vicinity of fixed points in classical analytical mechanics [1.2].

1.2 Dimensional Estimation of Hamiltonian Coefficients

It was shown above that when the coefficient of nonlinearity is small, the Hamiltonian of a system of interacting waves has a standard form (1.1.18, 25 and 26). The question is whether the functions ω, V_q, T_p can be estimated without going carefully into each specific problem and whether we can understand their dependence on wave vectors. The answer: It can be done using dimensional considerations unless the characteristic parameters of these waves can be combined in such a way that the resulting product has the dimension of length. It is said in such a case that the problem is *completely self-similar* (has *self-similarity of the first type*).

First of all, let us find the dimensions of the canonical variables $b(\boldsymbol{k},t)$ and matrix elements of the Hamiltonian of interaction V_q, T_p. The dimension of $b(\boldsymbol{k},t)$ is obtained from (1.1.18), taking into account that $\mathcal{H}$ has the dimension of energy density, and ω that of frequency:

$$[\mathcal{H}] = \mathrm{g\,cm}^{2-d}\,\mathrm{s}^{-2},\ [\omega(\boldsymbol{k})] = \mathrm{s}^{-1}\ ,\ [b(\boldsymbol{k})] = \mathrm{g}^{1/2}\mathrm{cm}^{1-d/2}\mathrm{s}^{-1/2}, \tag{1.2.1}$$

where d is the dimension of the medium. Taking into consideration that $[\omega(\boldsymbol{k})] = [V_q b] = [T_p b^2]$ we readily obtain

$$[b(\boldsymbol{k})] = \mathrm{g}^{-1/2}\mathrm{cm}^{d/2-1}\mathrm{s}^{-1/2}\ , \qquad [T_{\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4}}] = \mathrm{g}^{-1}\mathrm{cm}^{d-2}\ . \tag{1.2.2}$$

As should be expected, the dimension $[V_s b^2]$ (here V_s is system volume) is equal to that of Planck's constant $\hbar$. Evidently, our classical approach is valid when quantum mechanical population numbers $N(k) = Vb^2/\hbar$ are much greater than unity. On the other hand, the amplitudes of the waves $b(\boldsymbol{k},t)$ must not be too large for the Hamiltonian of interaction $\mathcal{H}_{\mathrm{int}}$ to remain small compared to $\mathcal{H}_2$. This yields an upper bound on $b(\boldsymbol{k},t)$, which schematically can be written as

$$\omega(\boldsymbol{k}) < V_{\boldsymbol{k},\boldsymbol{k},\boldsymbol{k}} \sum_{\boldsymbol{k}'} b(\boldsymbol{k}',t)\ . \tag{1.2.3}$$

Introducing the dimensionless wave amplitude

$$x(\boldsymbol{k},t) = b(\boldsymbol{k},t)/B(\boldsymbol{k})\ , \qquad B(\boldsymbol{k}) = |\omega(\boldsymbol{k})/V(\boldsymbol{k},\boldsymbol{k},\boldsymbol{k})|\ , \tag{1.2.4}$$

the condition of small nonlinearity can be written as

$$x(\boldsymbol{k}) \ll 1\ . \tag{1.2.5}$$

Now we can consider some examples.

Sound in continuous medium. Only the medium density ρ and elasticity coefficient κ (with dimensions $[\rho] = \mathrm{g\,cm}^{-3}$, $[\kappa] = \mathrm{g\,cm}^{-1}\,\mathrm{s}^{-2}$) can enter as parameters into the equations of motion for this problem. These values and the wave vector $\boldsymbol{k}$ can unambiguously enter into the combination with the dimension of frequency $[\omega(\boldsymbol{k})] = \mathrm{s}^{-1} = [\rho^x\kappa^y k^z] = \mathrm{g}^{x+y}\,\mathrm{cm}^{-(3x+y+z)}\,\mathrm{s}^{-2y}$. Equating exponents of g, cm, s, we obtain three equations $x + y = 0$, $3x + y + z = 0$, $2y = 1$. Hence $x = -1/2$, $y = 1/2$, $z = 1$. Therefore, dimensional considerations result in the linear dispersion law

$$\omega(\boldsymbol{k}) = c_s k\ , \qquad c_s = a\sqrt{\kappa/\rho}\ . \tag{1.2.6}$$

Here c_s is the sound velocity, and a is a dimensionless parameter of order unity. Parameters of our problem similarly can produce the combination $B(\boldsymbol{k})$ with dimension equal to that of the canonical variable $b(\boldsymbol{k}, t)$

$$B(\boldsymbol{k}) = \sqrt{\rho c_s / k} \tag{1.2.7}$$

and three-wave interaction amplitudes

$$V_{\mathbf{1,2,3}} = \sqrt{k_1 k_2 k_3 c_s/\rho} f(\boldsymbol{k}_1/k_1, \boldsymbol{k}_2/k_1, \boldsymbol{k}_3/k_1) \; . \tag{1.2.8}$$

The dimensionless function f depends in this case on eight dimensionless arguments, i.e. 2 ratios k_2/k, k_3/k and six angular variables giving the directions of the three vectors. Actually, there are only three essential angular variables, i.e. $\cos\theta_{12}$, $\cos\theta_{23}$ and $\cos\theta_{31}$ ($\cos\theta_{ij} = (\boldsymbol{k}_i \cdot \boldsymbol{k}_j)/(k_i k_j)$) since in our problem there is no preferential direction.

In the Hamiltonian description the wave amplitude is proportional to $b(\boldsymbol{k})$. In the field of the sound wave, density and velocity of the medium oscillate. Writing the density as a sum of the constant level ρ_0 and the oscillating component $\rho_1(\boldsymbol{r}, t)$, one has

$$\begin{aligned} \rho_1(\boldsymbol{r}, t) &= \mathrm{Re}\{\rho(\boldsymbol{k}) \exp[i\boldsymbol{k}\boldsymbol{r} - i\omega(\boldsymbol{k})t]\} \; , \\ \boldsymbol{v}(\boldsymbol{r}, t) &= \mathrm{Re}\{\boldsymbol{v}(\boldsymbol{k}) \exp[i\boldsymbol{k}r - i\omega(\boldsymbol{k})t]\} \; , \end{aligned} \tag{1.2.9}$$

where $\rho(\boldsymbol{k})$, $\boldsymbol{v}(\boldsymbol{k})$ denotes the wave amplitude in the natural variables. The relation between natural and normal canonical variables can be easily obtained in linear approximation using dimensional considerations

$$\rho(\boldsymbol{k}, t) \simeq \sqrt{k\rho_0/c_s} b(\boldsymbol{k}, t) \; , \qquad v(\boldsymbol{k}, t) \simeq \sqrt{k c_s/\rho_0} b(\boldsymbol{k}, t) \; . \tag{1.2.10}$$

The condition of small nonlinearity in terms of canonical variables can be rewritten as follows:

$$x(\boldsymbol{k}) \simeq \rho(\boldsymbol{k})/\rho_0 \simeq v(\boldsymbol{k})/c_s \ll 1 \; . \tag{1.2.11}$$

Gravitational waves on liquid surface. Gravitational waves are sufficiently long waves for which the surface tension is not essential and the restoring force tending to make the surface plane is due to gravitation. Clearly, essential parameters must include besides the liquid density ρ also the free fall acceleration g, $[g]$ = cm s^{-2}. Using a procedure similar to that employed in the previous example and taking into account that the problem is plane ($d = 2$) we have

$$\omega(\boldsymbol{k}) = \sqrt{gk} \; , \qquad B(\boldsymbol{k}) = (\rho^2 g k^{-5})^{1/4} \; . \tag{1.2.12}$$

Clearly in this case the dispersion law is non-decaying: $\omega(\boldsymbol{k}) \simeq k^z$, $z = 1/2 < 1$. Therefore, the principal interaction is the four-wave interaction with the following amplitude element:

$$T(\boldsymbol{k},\mathbf{2};\mathbf{3},\mathbf{4}) = \frac{k^3}{\rho} f(\frac{\boldsymbol{k}_1}{k}, \frac{\boldsymbol{k}_2}{k}, \frac{\boldsymbol{k}_3}{k}, \cos\theta_{\boldsymbol{k}\mathbf{1}}, \cos\theta_{\boldsymbol{k}\mathbf{2}}, \cos\theta_{\boldsymbol{k}\mathbf{3}}) \ . \tag{1.2.13}$$

The natural variable describing waves on the surface of water is the deviation of the liquid surface from the plane $\mu(\boldsymbol{r},t)$. The dimensionless wave amplitude is $x(\boldsymbol{k},t) = \mu(\boldsymbol{k},k)k = b(\boldsymbol{k},t)/B(\boldsymbol{k})$, hence the relation of $\mu(\boldsymbol{k},t)$ and $b(\boldsymbol{k},t)$ is obtained

$$\mu(\boldsymbol{k},t) = (k/\rho^2 g)^{1/4} b(\boldsymbol{k},t) \ . \tag{1.2.14}$$

Capillary waves. For sufficiently short waves the restoring force must be fully determined by the surface tension. The essential parameters in this case will include instead of g the coefficient of the surface tension σ with dimension equal to that surface energy density $[\sigma] = \mathrm{g\,s}^{-2}$. Therefore,

$$\begin{gathered} \omega(\boldsymbol{k}) = \sqrt{\sigma k^3/\rho} \ , \quad B(k) = (\rho\sigma/k^3)^{1/4} \ , \\ \mu(\boldsymbol{k},t) = (\rho\sigma k)^{-1/4} b(\boldsymbol{k},t) \ . \end{gathered} \tag{1.2.15}$$

The dispersion law of capillary waves is decay, i.e. $z = 3/2 > 1$. Therefore the principal interaction is the 3-wave interaction

$$V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = (\sigma k^9/\rho^3)^{1/4} f(\frac{k_1}{k}, \frac{k_2}{k}, \cos\theta_{\boldsymbol{k}1}, \cos\theta_{\boldsymbol{k}2}) \ . \tag{1.2.16}$$

Comparing dispersion laws of capillary waves (1.2.15) and gravitational waves (1.2.12) the boundary value of the wave vector under which those frequencies are equal can be easily found

$$k_* = \sqrt{\rho g/\sigma} \ . \tag{1.2.17}$$

At $k \ll k_*$ the gravitational energy of the wave is greater than the surface tension energy, and the latter can be neglected. Thus, waves of long wavelength on the liquid surface will be gravitational. Correspondingly, with $k \ll k_*$ surface waves will be capillary ones with the law of dispersion (1.2.15). It can be shown that at arbitrary k the dispersion law on the surface of a deep liquid has the form [1.4]:

$$\omega(\boldsymbol{k}) = \sqrt{gk + \sigma k^3/\rho} \ . \tag{1.2.18}$$

In spite of the fact that dimensional estimates yield results accurate up to a dimensionless factor of order unity, the dispersion laws (1.2.12, 15) are correct for long or short waves respectively.

For the waves on the water surface at room temperature we have $k_* = 4$ cm (ρ =1 g/cm, $\sigma = 70$ g/s). The respective wavelength $\lambda = 2\pi/k_* \simeq 1.6$ cm and the frequency $f_* = \omega/2\pi \simeq 0.2$ Hz.

Spin waves in the Heisenberg ferromagnet. We shall proceed from the following expression for the energy of the Heisenberg ferromagnet (see, e.g., [1.1] or (2.3.6))

$$\mathcal{H}_{\rm ex} = \alpha \int (\partial M_i/\partial x_j)^2 \, d\boldsymbol{r} \ . \tag{1.2.19}$$

Here α is the parameter of nonhomogeneous exchange, $\boldsymbol{M}$ denotes magnetization. The physical relevance of these values will be discussed in the next chapter. Now we shall only give their dimensions $[\alpha] = \mathrm{cm}^2$, $[M] = \mathrm{g}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{s}^{-1}$. There can be no combination of those values and the wave vector with dimension equal to that of the frequency. Therefore one more dimensional parameter must enter into the problem of the spin waves. This parameter determines the dynamics of the electron spin system in the magnetic field and is the ratio of the magnetic moment of the electron $\mu_{\rm B}/2$ to its mechanical moment $\hbar/2$:

$$g = \mu_{\rm B}/\hbar \simeq 2\pi \, 2.8 \ \mathrm{Hz/Oe} \ . \tag{1.2.20}$$

It should not be confused with the dimensionless g-factor of the electron which approximates two.

The values α, M, g and k can be combined to build up frequency. This construction is ambiguous since the combination αk^2 is dimensionless and dimensional considerations do not determine with what exponent the combination αk^2 must enter into various expressions. If we take into account that $\mathcal{H}_{\rm ex} \propto \alpha$ and impose a condition that $\omega(\boldsymbol{k}) \simeq \alpha$ we obtain

$$\omega(k) \simeq gM\alpha k^2 \ . \tag{1.2.21}$$

Later the dynamics of spin waves in the Heisenberg (exchange) approximation will be shown to be determined by four-wave scattering processes $2 \Rightarrow 2$. The coefficients of the Hamiltonian of interaction $T(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4})$ as well as the frequencies $\omega(\boldsymbol{k})$ must be proportional to α. The other factors are found from the analysis of dimensions

$$T(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4}) \simeq g^2 \alpha k^2 \ . \tag{1.2.22}$$

Spin waves are oscillation of magnetization. Consequently, the canonical amplitude of the spin wave $b(\boldsymbol{k},t)$ in the linear approximation must be proportional to $\boldsymbol{m}(\boldsymbol{k},t)$, i.e. to the space Fourier transform of the variable part of the magnetization. On the other hand, α does not have to enter into the expression for the $b(\boldsymbol{k},t)$ and $\boldsymbol{m}(\boldsymbol{k},t)$ relation, since this relation does not depend on the specific form of the expression for energy. Assuming $\boldsymbol{m}(\boldsymbol{k},t) = b(\boldsymbol{k},t) M^x g^y k^z$ we obtain from dimensional analysis

$$m(\boldsymbol{k},t) \simeq \sqrt{gM} b(\boldsymbol{k},t) \ . \tag{1.2.23}$$

The condition of small nonlinearity in terms of canonical variables for the spin waves is rewritten as follows

$$x(\boldsymbol{k}) \simeq m(\boldsymbol{k})/M \simeq \sqrt{g/M} b(\boldsymbol{k}) \ll 1 \ . \tag{1.2.24}$$

The dimensionless parameter $x(\boldsymbol{k})$ (1.2.24) describes the precession angle of the magnetic moment.

1.3 Dynamic Equations of Motion for Weakly Non-conservative Wave Systems

In Sect. 1.2. we considered the simplest case of a Hamiltonian equation of motion for the wave amplitudes when the medium of the wave propagation is spatially homogeneous, and at the same time conservative; then the interaction of the waves with the medium may be neglected. These equations may be represented as:

$$\partial b(\boldsymbol{k}, t)/\partial t + i\omega(\boldsymbol{k}) b(\boldsymbol{k}, t) = -i\delta \mathcal{H}_{\text{int}}/\delta b^*(\boldsymbol{k}, t) \ . \tag{1.3.1}$$

There are, of course, no completely isolated systems and even their weak interaction with the medium may sometimes prove significant. To understand the role of such an interaction and describe it properly the problem must be formalized and reduced.

Let us subdivide the physical system under consideration (ferromagnet, liquid, plasma, etc.) into two parts: the dynamic subsystem of nonlinear waves and the remaining medium. The state of the nonlinear wave system may be strongly excited. These waves will be described in detail by means of dynamic equations of motion for the wave amplitudes $b(\boldsymbol{k}, t)$. Let us assume that the "medium" weakly interacts with nonlinear waves and is in thermodynamic equilibrium. Then it can act as a *thermal bath* for the nonlinear wave systems.

The physical characteristics of the remaining part of the medium that we assume to be a thermal bath may be absolutely different for specific cases. If our aim is a detailed description of long-wave length motions of the continuous medium (liquid or plasma) by means of dynamical equations, small-scale non-cooperative motions of the individual particles (liquid molecules, electrons and ions in plasma) will act as a thermal bath. In the dynamic description of long spin waves in ferromagnets the thermal bath can be formed by short spin waves in thermodynamic equilibrium. If the excitation level of the spin waves is so great that the thermodynamic equilibrium of the short spin waves is significantly disturbed then the subsystem of phonons which are weakly connected with the subsystem of spin waves starts to act as a thermal bath. The spin system can evidently be affected so that the equilibrium of phonons will also be drastically disturbed. Then the medium in which the sample is immersed (air, liquid nitrogen, helium , etc.) can be treated as a thermal bath. The above examples show that the subdivision of the considered physical system into two parts, the nonlinear dynamic subsystem and the thermal bath, is fairly arbitrary. How a particular physical system will be subdivided will depend on its specific features, the way and level of its excitation and the intended accuracy of the description.

1.3.1 Taking into Account Linear Wave Damping

The interaction of the waves with the thermal bath leads to damping. For sound in fluid this is viscous damping, which converts wave energy into heat. For long spin waves in ferromagnets damping is mostly caused by their interaction with the "magnon thermal bath" - the thermally excited reservoir of the other magnons. An important role, especially in antiferromagnets, can be played by the interaction of spin waves with phonons (i.e. with sound). Spin waves may also be damped as a result of their interactions with different impurities, crystal defects, pores and other inhomogeneities.

Damping of small amplitude waves is known in most cases to be exponential

$$|b(\boldsymbol{k},t)| = |b(\boldsymbol{k},0)| \exp(-\gamma(\boldsymbol{k})t) \ . \tag{1.3.2}$$

The decrement $\gamma(\boldsymbol{k})$ depends on the wave vector $\boldsymbol{k}$, medium temperature T, magnetic field $\boldsymbol{H}$ and other experimental conditions. In good samples of magnetodielectrics (YIG, $MnCO_3$, $CsMnF_3$, etc.) the spin wave damping rate is small compared to their frequency: the relation $\omega(\boldsymbol{k})/\gamma(\boldsymbol{k}) \simeq 10^3 \div 10^4$. Therefore the Hamiltonian equations (1.3.1) can be used in the zeroth approximation not allowing for the damping. In the first approximation the small damping can be taken into account by adding to each equation the term $\gamma(\boldsymbol{k})b(\boldsymbol{k},t)$ which results in an exponentially decreasing amplitude (1.3.2)

$$\partial b(\boldsymbol{k},t)/\partial t + [i\omega(\boldsymbol{k}) + \gamma(\boldsymbol{k})]b(\boldsymbol{k},t) = -i\delta\mathcal{H}_{\rm int}/\delta b^*(\boldsymbol{k},t) \ . \tag{1.3.3}$$

This phenomenological procedure allowing for the small wave damping as an imaginary addition to the frequency is commonly used in theoretical physics and normally causes no objection. For (1.3.3) to be valid not only the damping and interaction must be small (these conditions may be written as $\gamma(\boldsymbol{k}) \ll \omega(\boldsymbol{k})$, $\mathcal{H}_{\rm int} \ll \mathcal{H}_2$) but the wave amplitudes must be bounded below and above. We actually think the whole world consists of two parts: the system of waves (with the amplitudes $b(\boldsymbol{k},t)$ and Hamiltonian $\mathcal{H}$, which we are going to describe in detail dynamically using (1.3.3)) and thermal bath, which we shall assume to be in thermodynamic equilibrium with the temperature T. The term $\gamma(\boldsymbol{k})b(\boldsymbol{k},t)$ describes "the wave friction on the thermal bath", i.e. energy flux to the thermal bath, and results in the wave amplitude b gradually vanishing. These amplitudes, however, must relax not to zero, but to the thermodynamical equilibrium value which is specified by the Rayleigh-Jeans distribution

$$|b_0(\boldsymbol{k})|^2 = n_0(\boldsymbol{k}) = T/\omega(\boldsymbol{k}), \tag{1.3.4}$$

which holds true in the classical limit $T \ll \hbar\omega(\boldsymbol{k})$. Therefore (1.3.3) is true when $|b(\boldsymbol{k})| \ll |b_0(\boldsymbol{k})| = \sqrt{T/\omega(\boldsymbol{k})}$.

1.3.2 Allowing for Thermal Noise

If the inequality $|b| \gg |b_0|$ is not satisfied the random Langevin force $f(\boldsymbol{k}, t)$ must be added to (1.3.3)

$$\frac{\partial b(\boldsymbol{k},t)}{\partial t} + [i\omega(\boldsymbol{k}) + \gamma(\boldsymbol{k})]b(\boldsymbol{k},t) = -i\frac{\delta\mathcal{H}_{\text{int}}}{\delta b^*(\boldsymbol{k},t)} + f(\boldsymbol{k},t)\ . \tag{1.3.5}$$

This force imitates thermal noise - chaotic shocks on the part of the thermal bath which usually cause an increase in wave energy $\propto b^*(\boldsymbol{k},t)b(\boldsymbol{k},t)$. As is known from statistical physics, random forcing is Gaussian forcing with the correlator

$$\langle f(\boldsymbol{k},t)f^*(\boldsymbol{k}',t')\rangle = 2\delta(\boldsymbol{k}-\boldsymbol{k}')\delta(t-t')\gamma(\boldsymbol{k})n_0(\boldsymbol{k})\ . \tag{1.3.6}$$

The properties of the thermal noise (1.3.6) are sufficiently clear: it is not correlated at different moments of time and for waves with different $\boldsymbol{k}$. The factor $2\gamma(\boldsymbol{k})n_0(\boldsymbol{k})$ has been selected so that the numbers of waves $n(\boldsymbol{k})$ should relax to the equilibrium value $n_0(\boldsymbol{k})$. Equations (1.3.5, 6) at $\mathcal{H}_{\text{int}} = 0$ quite readily yield

$$\frac{1}{2}\frac{\partial n(\boldsymbol{k},t)}{\partial t} = -\gamma(\boldsymbol{k})[n(\boldsymbol{k},t) - n_0(\boldsymbol{k})]\ , \tag{1.3.7}$$

$$\langle b(\boldsymbol{k})b^*(\boldsymbol{k}')\rangle = n(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{1.3.8}$$

describing this phenomenon. The term "number of waves" is used here by analogy with quantum mechanics. It must, however, be borne in mind that in our classical approach the value $n(\boldsymbol{k})$ has dimensions of the action (erg s), while quantum-mechanical occupation numbers $n_{\text{qm}}(\boldsymbol{k})$ are dimensionless. At $n_{\text{qm}}(\boldsymbol{k}) \gg 1$ they are correlated:

$$n(\boldsymbol{k},t) = \hbar n_{\text{qm}}(\boldsymbol{k},t)\ . \tag{1.3.9}$$

1.3.3 Nonlinearity of Wave Damping

One more restriction on the validity regime of (1.3.3) is due to the assumption that $\gamma(\boldsymbol{k})$ is completely independent of all the other numbers of waves $n(\boldsymbol{k}')$. Generally speaking, this is not the case. But under small $n(\boldsymbol{k}')$, $\gamma(\boldsymbol{k})$ may be expanded in terms of $n(\boldsymbol{k}')$ and we may restrict ourselves to the first terms

$$\gamma(\boldsymbol{k}, n(\boldsymbol{k}')) = \gamma_0(\boldsymbol{k}) + \sum_{\boldsymbol{k}'} \mu(\boldsymbol{k}, \boldsymbol{k}') n(\boldsymbol{k}') \ . \tag{1.3.10}$$

If $\mu(\boldsymbol{k}, \boldsymbol{k}') > 0$, the nonlinear damping is commonly called positive; for under $\mu(\boldsymbol{k}, \boldsymbol{k}') < 0$ it is called negative. The nature of the damping nonlinearity will be discussed in Chap. 10. At the moment note only that the function $\mu(\boldsymbol{k}, \boldsymbol{k}')$ may be of any sign and be either symmetrical under transposition $\boldsymbol{k} \Leftrightarrow \boldsymbol{k}'$ or antisymmetric. It may be also characterized by no symmetry at all. At the first stage (and throughout Chap. 1) we shall assume that the numbers $n(\boldsymbol{k}')$ are sufficiently small for the nonlinearity of wave damping to be neglected.

In conclusion it must be emphasized that (1.3.3, 5) can be employed to describe various nonlinear wave phenomena. To this end, we must only put the original equations of medium motion into the Hamiltonian form. Unfortunately, there is as yet no sufficiently effective general method of obtaining canonical variables, but for many general cases the canonical variables have already been found. In [1.7], for example, well-known canonical Clebsch variables describing eddy flows and potential barotropic flows of the ideal liquid are given. *Zakharov* and *Filonenko* [1.8] have found canonical variables for waves on the liquid surface, and *Pokrovskii* and *Khalatnikov* introduced four pairs of canonical variables describing the flow of normal and superfluid components of liquid helium with the framework of two-liquid hydrodynamics. In his survey [1.3] *Zakharov* describes canonical variables for relativistic hydrodynamics, for the hydrodynamics of charged liquid (plasma) interacting with the electromagnetic field, and for magnetic hydrodynamics. This is by no means a complete list of canonical variables for different nonlinear media.

One more remark: Equations (1.3.3, 5) are approximate. In some specific cases they may be invalid even if the conditions which we discussed above are satisfied. This is true, for instance, for the problem of the parametric excitation of waves which are scattered by statistic defects. In such cases fine nonlinear effects must be investigated more consistently. All the interactions, including interactions resulting in the damping of the nonlinear waves, must be treated in the greatest detail, which enables one to abandon the somewhat unjustified subdivision of the system into nonlinear waves and thermal bath. This microscopic approach based on the Wyld diagrammatic technique [1.9] is described in my book [1.1]. At present, at this first stage of investigating nonlinear phenomena we shall confine ourselves to the simple phenomenological Eqs. (1.3.3, 5).

1.4 Three-Wave Processes

Substitution of the Hamiltonian $\mathcal{H}$ given by (1.1.25) into (1.3.3) yields

$$\begin{aligned}&\partial b(\boldsymbol{k},t)/\partial t + [\omega(\boldsymbol{k}) + i\gamma(\boldsymbol{k})]b(\boldsymbol{k},t)\\ &\quad = -\frac{i}{2}\sum_{\mathbf{1}+\mathbf{2}=\boldsymbol{k}} V^*_{\boldsymbol{k},\mathbf{1},\mathbf{2}} b_{\mathbf{1}} b_{\mathbf{2}} - i \sum_{\boldsymbol{k}+\mathbf{2}=\mathbf{1}} V_{\mathbf{1},\boldsymbol{k},\mathbf{2}} b_{\mathbf{1}} b^*_{\mathbf{2}} - i \sum_{\mathbf{1}+\mathbf{2}+\boldsymbol{k}=\mathbf{0}} U_{\boldsymbol{k},\mathbf{1},\mathbf{2}} b^*_{\mathbf{1}} b^*_{\mathbf{2}} .\end{aligned} \tag{1.4.1}$$

This is the basic equation for describing different three-wave processes. Some of these will be discussed below.

1.4.1 Confluence of Two Waves and Other Induced Processes

Let two monochromatic waves propagate in some medium:

$$b(\boldsymbol{k},t) = b_1\delta(\boldsymbol{k}-\boldsymbol{k}_1)\exp[-i\omega(\boldsymbol{k}_1)t] + b_2\delta(\boldsymbol{k}-\boldsymbol{k}_2)\exp[-i\omega(\boldsymbol{k}_2)t] , \tag{1.4.2}$$

By virtue of (1.4.1) this will lead to the emergence of three additional waves

$$\begin{aligned} b(\boldsymbol{k},t) =& b_3\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\exp[-i\omega(\boldsymbol{k}_1)t - i\omega(\boldsymbol{k}_2)t]\\ &+ b_4\delta(\boldsymbol{k}+\boldsymbol{k}_2-\boldsymbol{k}_1)\exp[-i\omega(\boldsymbol{k}_1)t + i\omega(\boldsymbol{k}_2)t]\\ &+ b_5\delta(\boldsymbol{k}+\boldsymbol{k}_2+\boldsymbol{k}_1)\exp[i\omega(\boldsymbol{k}_1)t + i\omega(\boldsymbol{k}_2)t] .\end{aligned} \tag{1.4.3}$$

with the amplitudes

$$\begin{aligned} b_3 =& \frac{V^*(\mathbf{1}+\mathbf{2},\mathbf{1},\mathbf{2})b(\mathbf{1})b(\mathbf{2})}{2[\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)-\omega(\boldsymbol{k}_1+\boldsymbol{k}_2)-i\gamma(\boldsymbol{k}_1+\boldsymbol{k}_2)]} ,\\ b_4 =& -\frac{V^*(\mathbf{1},(\mathbf{1}-\mathbf{2}),\mathbf{2})b(\mathbf{1})b(\mathbf{2})}{[\omega(\boldsymbol{k}_1)-\omega(\boldsymbol{k}_2)-\omega(\boldsymbol{k}_1-\boldsymbol{k}_2)-i\gamma(\boldsymbol{k}_1-\boldsymbol{k}_2)]} ,\\ b_5 =& -\frac{U(-\mathbf{1}-\mathbf{2},\mathbf{1},\mathbf{2})b^*(\mathbf{1})b^*(\mathbf{2})}{[\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)-\omega(\boldsymbol{k}_1+\boldsymbol{k}_2)+i\gamma(\boldsymbol{k}_1+\boldsymbol{k}_2)]} ,\end{aligned} \tag{1.4.4}$$

If the frequencies $\omega(\boldsymbol{k}_1)$ and $\omega(\boldsymbol{k}_2)$ satisfy the condition

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_1+\boldsymbol{k}_2) , \tag{1.4.5}$$

the amplitude b_3 is large in comparison with b_4 and b_5. In this case we deal with the resonance process of confluence of two waves. If instead of (1.4.5) the condition

$$\omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_1 - \boldsymbol{k}_2) \tag{1.4.6}$$

is satisfied then, on the contrary, $|b_4| \gg |b_{3,5}|$. From the viewpoint of quantum mechanics and quasi-particles, this fact cannot be easily accounted for, but classically, in terms of amplitudes, it can be accounted for by resonance excitation by a wave with the wave vector $\boldsymbol{k}_4 = \boldsymbol{k}_1 - \boldsymbol{k}_2$ at one of the combination frequencies $\omega_4 = \omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2)$. If the medium in which the waves are propagating is in a state of thermodynamic equilibrium these waves have positive frequencies. Consequently, the condition

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) + \omega(-\boldsymbol{k}_1 - \boldsymbol{k}_2) = 0 \ . \tag{1.4.7}$$

required for the amplitude b_5 in (1.4.4) to be high is not satisfied under equilibrium. These processes can be neglected if the parameter equal to the Q-factor of the waves is large

$$Q(\boldsymbol{k}) = \omega(\boldsymbol{k})/\gamma(\boldsymbol{k}) \gg 1 \ . \tag{1.4.8}$$

From the above examples we can draw the very important conclusion that the parameter $1/Q(\boldsymbol{k})$ is small and nonresonant processes can be neglected in most cases, if we truncate the corresponding terms at the very first stage of the problem solution when formulating the interaction Hamiltonian. Only in studying relatively weak four-wave processes (see Sect. 1.5) must the contribution of non-resonance three-wave processes be allowed for in the second order perturbation theory.

Let us consider the process (1.4.5) of two-wave confluence in more detail. It is clear from the first expression of (1.4.4) that the resonance curve is Lorentzian. If the function $V(\mathbf{3},\mathbf{1},\mathbf{2})$ is real, the phase shift in the resonance is $\Delta\varphi = \varphi_1 + \varphi_2 - \varphi_3 = \pi/2$. Far from the resonance $\Delta\varphi = 0$. In order to evaluate the efficiency of the two-wave confluence process, we must know the value of the interaction amplitude $V(\mathbf{3},\mathbf{2},\mathbf{1})$, the law of wave dispersion $\omega(\boldsymbol{k})$ and the decrement $\gamma(\boldsymbol{k})$. By way of example consider first the confluence of two sound waves for the case when the condition of 3-wave resonance (1.4.5) is satisfied. Let two sound waves $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ be excited in the continuum. They cause density oscillations of the continuum

$$\rho(\boldsymbol{r},t) = \rho_1 \cos(\omega_1 t - \boldsymbol{k}_1\boldsymbol{r}) + \rho_2 \cos(\omega_2 t - \boldsymbol{k}_2\boldsymbol{r}) \ .$$

Employing first the connection (1.2.10) between natural $\rho_{1,2}$ and canonical $b_{1,2}$ variables and also the estimate (1.2.8) for the 3-phonon interaction amplitude $V(\mathbf{1};\mathbf{2},\mathbf{3})$ we obtain from (1.4.4) at resonance:

$$x_3 \simeq x_1 x_2(\omega/\gamma_3) \,, \qquad x_j = \rho_j/\rho_0 \,. \tag{1.4.9a, b}$$

The ratio $Q = \omega/\gamma$ (for sound waves in acoustic media the Q-factor may reach $10^3 \div 10^4$ and more) describes resonance amplification of processes of wave conversion. Making use of (1.2.15, 16) and (1.4.4) similar estimations may be carried out for resonance three-wave conversion of the capillary waves on the liquid surface. Two capillary waves with the amplitudes μ_1, μ_2 (μ denotes the deviation of the liquid surface from the unperturbed state) in resonance produce a third wave with the amplitude μ_3:

$$\mu_3 \simeq \mu_1 \mu_2(\omega/\gamma_3) \,, \qquad \omega = \sqrt{ck^3/\rho} \,. \tag{1.4.10a, b}$$

If the wave amplitude is specified by a dimensionless parameter $x = k\mu$ (x is the angle of the deflection of the normal to the liquid surface from the vertical line), then the formula (1.4.10a) for capillary waves is rewritten so that its form coincides with (1.4.9a) for sound:

$$x \simeq x_1 x_2(\omega/\gamma_3) \,, \qquad x_j = k_j \mu_j \,. \tag{1.4.11a, b}$$

It must be emphasized that expressions similar to (1.4.9a, 11a) are, generally speaking, common to any problem of resonance conversion of waves, provided it contains no dimensionless parameters characterizing the relations of different interactions. In particular, the exact formula (4.1.4) for the resonance conversion of magnons which will be obtained in Chap. 4 coincides with (1.4.9a, 11a) to an accuracy of numeric coefficients and angular dependence, if x denotes the precession angle of the magnetic moment.

1.4.2 Decay Instability

Here we consider the instability of a plane monochromatic wave $b(\boldsymbol{k}, t)$ with respect to decay into two other waves. Examples of such a process are the decay of homogeneous precession of magnetization (or electromagnetic radiation of microwave frequencies) into two magnons, decay of a phonon into two magnons, etc. Among these processes are also included induced light scattering: photon decaying into a photon and phonon, into a photon and magnon, and a number of other processes. Thus, let a monochromatic wave with the frequency $\omega_0 = \omega(\boldsymbol{k}_0)$ and with the amplitude $b(\boldsymbol{k}, t)$ propagate in the medium:

$$b(\boldsymbol{k}, t) = b\,\delta(\boldsymbol{k} - \boldsymbol{k}_0) \exp(-i\omega_0 t) \,. \tag{1.4.12}$$

Then (1.4.1) for small amplitude waves has the following form:

$$\partial b_1/\partial t + [\gamma_1 + i\omega(\boldsymbol{k}_1)] b_1 + iVbb_2^* \exp(-i\omega_0 t) = 0 \,,$$

$$\partial b_2^*/\partial t + [\gamma_2 + i\omega(\boldsymbol{k}_2)] b_2^* - iV^* b^* b_1 \exp(-i\omega_0 t) = 0 \,. \tag{1.4.13}$$

The second nonlinear term in (1.4.1) is retained here. The first and third terms for the decay process of interest for us are not resonant, and with small 1/Q-factor (1.4.8) they can be neglected. A solution of these equations is sought in the following form:

$$b_1(t) = b_1 \exp[(\nu - i\omega_1)t] \ , \qquad b_2^*(t) = b_2^* \exp[(\nu - i\omega_2)t] \ , \tag{1.4.14}$$

$$\omega_1 + \omega_2 = \omega_0 \ . \tag{1.4.15}$$

The instability exponent in (1.4.14b) is not complex conjugation, even if it is a complex value. Only in this case and if the condition (1.4.15) is satisfied the substitution (1.4.14) converts the differential equation (1.4.11) to an algebraic one with constant coefficients. It has non-zero solutions if

$$\begin{bmatrix} \gamma_1 + \nu + i[\omega(\boldsymbol{k}_1) - \omega_1] & iVb \\ -iVb^* & \gamma_2 + \nu - i[\omega(\boldsymbol{k}_2) - \omega_2] \end{bmatrix} = 0 \ . \tag{1.4.16}$$

It must be noted that the condition (1.4.15) determines only the sum of $\omega_1 + \omega_2$, whereas their difference remains arbitrary. This arbitrariness implies that if a certain frequency δ is added to ω_1 and subtracted from ω_2, then in accordance with (1.4.14) this will lead to the substitution $\nu \to \nu - i\delta$. We may use this arbitrariness in order to simplify the analysis of the (1.4.16). Thus, ω_1 and ω_2 will be selected so that

$$(\gamma_1 + \nu)[\omega(\boldsymbol{k}_2) - \omega_2] = (\gamma_2 + \nu)[\omega(\boldsymbol{k}_1) - \omega_1] \ . \tag{1.4.17}$$

The imaginary part in (1.4.16) then becomes zero and for ν from (1.4.16, 17) we obtain:

$$\begin{aligned} 2\nu &= -\gamma_1 - \gamma_2 + \sqrt{B + \sqrt{B^2 + 2(\Delta\gamma\Delta\omega)^2}} \ , \\ 2B &= 4|Vb|^2 + (\Delta\gamma)^2 - 4(\Delta\omega)^2 \ , \\ \Delta\gamma &= \gamma_1 + \gamma_2 \ , \qquad 2\Delta\omega = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) - \omega_0 \ . \end{aligned} \tag{1.4.18}$$

Then, substituting μ from (1.4.18) into (1.4.17) ω_1 and ω_2 can be found from the solution of the system (1.4.15, 17). It is clear that ω_1 will approximate $\omega(\boldsymbol{k}_1)$, and ω_2 will be close to $\omega(\boldsymbol{k}_2)$. For the simple case when $\gamma_1 = \gamma_2 = \gamma$,

$$\begin{aligned} \nu &= -\gamma + \sqrt{|Vb|^2 - (\Delta\omega)^2} \, , \\ \omega_1 &= \omega(\boldsymbol{k}_1) - \Delta\omega/2 \ , \quad \omega_2 = \omega(\boldsymbol{k}_2) + \Delta\omega/2 \ , \end{aligned} \tag{1.4.19}$$

At Re $\nu > 0$ the wave amplitudes exponentially increase in time (according to (1.4.16)). The source wave in this case is said to be unstable with respect to decay into two waves b_1 and b_2. ν is called an increment of decay instability. This instability is often (especially if $\boldsymbol{k}_0 = 0$) also called parametric. The instability increment is maximum at resonance $\Delta\omega = 0$:

$$2\nu_{max} = -(\gamma_1 + \gamma_2) + \sqrt{4|Vb|^2 + (\Delta\gamma)^2} \ . \tag{1.4.20}$$

There is a threshold of the decay instability due to the damping of the waves

$$|Vb_{\mathrm{cr}}|^2 = \gamma_1\gamma_2 \ . \tag{1.4.21}$$

If the amplitudes V are known, by measuring b we may find the wave damping. In this way spin wave damping in ferromagnets is usually obtained. The interface of the instability region can be found from (1.4.18) at $\nu = 0$. If $\gamma_1 = \gamma_2$ the instability region is

$$(\varDelta\omega)^2 < 4|Vb|^2 - \gamma^2 \ . \tag{1.4.22}$$

It will be recalled that the resonance conditions for the decay

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_1 - \boldsymbol{k}_2) \ . \tag{1.4.23}$$

are not always satisfied and accordingly the laws of dispersion $\omega(\boldsymbol{k})$ are subdivided into decay and nondecay ones.

As an example, consider the decay instability in two systems: (i) sound in the continuum; (ii) capillary waves on the liquid surface. Using (1.2.6, 8, 9) for the sound and (1.2.15, 16) for the capillary waves we estimate the increment of the decay instability of waves:

$$\begin{aligned} &\nu(\boldsymbol{k},\boldsymbol{\kappa}) \simeq |V(\boldsymbol{k},\boldsymbol{k}_+,\boldsymbol{k}_-)b(\boldsymbol{k})| \simeq \gamma(\boldsymbol{k})\omega(\boldsymbol{k}) \ , \\ &\quad \boldsymbol{k}_\pm = \boldsymbol{k}/2 \pm \boldsymbol{\kappa} \ , \quad \kappa \le k \ . \end{aligned} \tag{1.4.24}$$

For sound the wave frequency $\omega(\boldsymbol{k})$ and the dimensionless amplitude of the wave $x(\boldsymbol{k})$ are given by (1.2.6) and (1.4.9b), for the capillary waves they are specified by (1.4.10a, 11b).

In Chap. 3 we shall deal with the decay instability of magnons in magnetodielectrics in detail. The exact formulae obtained there agree with the results of (1.4.24), if $\omega(\boldsymbol{k})$ denotes the frequency of magnons and $x(\boldsymbol{k})$ designates the angle of magnetic moment precession.

1.4.3 Interaction of Three Waves with Finite Amplitude

Consider the interaction of three wave packages with narrow spectrum whose characteristic vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and frequencies $\omega(\boldsymbol{k}_1)$, $\omega(\boldsymbol{k}_2)$, $\omega(\boldsymbol{k}_3)$ satisfy the resonance condition $\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$ (*Bloembergen* problem [1.10]),

$$\omega(\boldsymbol{k}_1) = \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) \ . \tag{1.4.25a}$$

Assume

$$b(\boldsymbol{k}) = \sum_{j=1}^{3} b_j(\boldsymbol{k}) \tag{1.4.25b}$$

and take b_j to be non-zero for small $\boldsymbol{\kappa} = \boldsymbol{k} - \boldsymbol{k}_j$. Within each packet the strong $\boldsymbol{r}, t$-dependence is excluded by the substitution

$$a_j(\boldsymbol{\kappa}) = b_j(\boldsymbol{k}_j + \boldsymbol{\kappa}) \exp i[\omega(\boldsymbol{k}_j)t - \boldsymbol{k}_j\boldsymbol{r}] \; . \tag{1.4.26}$$

Equations (1.4.1) are reduced by taking

$$\omega(\boldsymbol{k}_j + \boldsymbol{\kappa}) = \omega(\boldsymbol{k}_j) + (\boldsymbol{\kappa}\boldsymbol{v}_j) \; . \tag{1.4.27}$$

Further change over to $\boldsymbol{r}$-representation according to the formula

$$a_j(\boldsymbol{r}) = (2\pi)^{-3/2} \int a_j(\boldsymbol{k}) \exp(i\boldsymbol{k}\boldsymbol{r})\, d\boldsymbol{k} \; . \tag{1.4.28}$$

Then

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + \gamma_1 + \boldsymbol{v}_1 \cdot \nabla\Big) a_1(\boldsymbol{r}, t) &= -iVa_2(\boldsymbol{r}, t) a_3(\boldsymbol{r}, t) \; , \\ \Big(\frac{\partial}{\partial t} + \gamma_2 + \boldsymbol{v}_2 \cdot \nabla\Big) a_2(\boldsymbol{r}, t) &= -iVa_1(\boldsymbol{r}, t) a_3^*(\boldsymbol{r}, t) \; , \\ \Big(\frac{\partial}{\partial t} + \gamma_3 + \boldsymbol{v}_3 \cdot \nabla\Big) a_3(\boldsymbol{r}, t) &= -iVa_2^*(\boldsymbol{r}, t) a_1(\boldsymbol{r}, t) \, , \end{aligned} \tag{1.4.29}$$

where $V = (2\pi)^{3/2} V(\mathbf{1}\,; \mathbf{2}, \mathbf{3})$, variables $a(\boldsymbol{r}, t)$ denote complex envelopes of the waves $\exp[i(\boldsymbol{k}_i\boldsymbol{r} - \omega_i t)]$. Equation (1.4.29) under $\gamma_i = 0$ is a Hamiltonian equation with the Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{3} \frac{\boldsymbol{v}_j}{2i} \int [a_j^*(\boldsymbol{r}) \nabla a_j(\boldsymbol{r})]\, d\boldsymbol{r} + V \int [a_1^* a_2 a_3 + \text{c.c.}]\, d\boldsymbol{r} \; . \tag{1.4.30}$$

In addition to $\mathcal{H}$, Eq. (1.4.29) under $\gamma_i = 0$ has two other independent integrals ($Manly - Row$ integrals)

$$N_1 + N_2 = \text{const.}\, , \; N_2 + N_3 = \text{const.}\; , \quad N_j = \int a_j^*(\boldsymbol{r}) a_j(\boldsymbol{r})\, d\boldsymbol{r} \; . \tag{1.4.31}$$

The values N_i designate the total number of particles of j type. Employing three motion integrals enables one to study these equations effectively (see, e.g., [1.10]). Furthermore, the method of the *inverse scattering problem* (see, for instance, the survey by *Zakharov* [1.11], *Novikov* et al. [1.12]) makes it possible to study the evolution of three waves within (1.4.29) for arbitrary, localized initial conditions. Here again we consider a simple but interesting example, the generation of the second harmonic ($a_2 = a_3$) in the stationary case. Let us assume for the sake of simplicity $\gamma_j = 0\, , \boldsymbol{v}_j = \boldsymbol{v}$ and at $z = 0$, $a_1 = 0$, $a_2 = a_3 = A_2$ and is real. Substituting into (1.4.29) $a_1 = iA_1$, $a_2 = a_3 = A_2$ yields

$$v\partial A_1/\partial z = VA_2^2 \; , \qquad v\partial A_2/\partial z = -VA_1A_2 \; . \tag{1.4.32}$$

Solution of these equations with conditions $A_1(0) = 0$, $A_2(0) = B$ on the interface ($z = 0$) has the following form:

$$A_1(z) = B \tanh(VBz/v) \ , \qquad A_2(z) = B/\cosh(VBz/v) \ . \tag{1.4.33}$$

The characteristic length $L = v/(VB)$ is called the length of interaction. At $z \ll L$ all the energy of the initial wave is transferred into the second harmonic A_1. Experimental generation of the second harmonic is the most impressive experiment of nonlinear optics: a red beam enters a crystal and leaves it as a green beam. Complete conversion is impossible because the conditions of resonance are not completely satisfied , etc. In the most successful experiment $K_{\mathrm{tz}} \simeq 90\%$.

1.4.4 Explosive Three–Wave Instability

If waves with negative energy can exist in the medium (active medium), the resonance conditions may be satisfied for "three waves generation from the vacuum"

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) = 0 \ , \qquad \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = 0 \ . \tag{1.4.34}$$

This process is described by the last term in (1.4.1). Retaining only this term in the vicinity of the resonance (1.4.34) we seek the solution in the form of

$$b(\boldsymbol{k}, t) = \sum_{j=1}^{3} a_j(t)\delta(\boldsymbol{k} - \boldsymbol{k}_j) \exp[-i\omega(k_j)t] \ .$$

The fastest increase of the wave amplitudes will take place if $\omega(\boldsymbol{k}_j)$ and $\boldsymbol{k}_j$ exactly satisfy the condition (1.4.34). In this case (1.4.1) yields

$$\begin{aligned} \partial a_1/\partial t + \gamma_1 a_1 &= -2iU a_2^* a_3^* \ , \\ \partial a_2/\partial t + \gamma_2 a_2 &= -2iU a_1^* a_3^* \ , \\ \partial a_3/\partial t + \gamma_3 a_3 &= -2iU a_1^* a_2^* \ . \end{aligned} \tag{1.4.35}$$

At $\gamma_j = \gamma$ these equations have a solution $|A_j| = A$, where

$$A(t) = A_0 \gamma \exp(-\gamma t)\{\gamma + 2UA[\exp(-\gamma t) - 1]\}^{-1} \ . \tag{1.4.36}$$

If the initial amplitude A_0 is sufficiently large $2UA_0 > \gamma$, the amplitude $A(t)$ becomes infinite over a finite time t, under small $t_0 = 1/(2UA_0)$. Such an instability is called *explosive*. In this simplest solution "explosion" occurs simultaneously over the whole space. Actually, any spatial inhomogeneity of initial conditions will lead to "explosions" at isolated points. This phenomenon has been consistently explained theoretically by *Zakharov* and *Manakov* [1.13] within the frame of equations in partial derivatives similar to the (1.4.9) and also integrated by the inverse scattering problem.

Note that the complete realization of conditions (1.3.4) for explosive instability is impossible in magnetodielectrics. However, for small magnetic fields in ferromagnets the conditions under which homogeneous precession of magnetization decays into 3 magnons

$$\omega_0 = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) \ , \qquad \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = 0 \tag{1.4.37}$$

can be fulfilled. This process is described by a four-magnon Hamiltonian

$$\mathcal{H}_4 = \frac{1}{3} \sum_{\mathbf{1+2+3=0}} (T_{\mathbf{0};\mathbf{1},\mathbf{2},\mathbf{3}} b_{\mathbf{0}} b_{\mathbf{1}}^* b_{\mathbf{2}}^* b_{\mathbf{3}}^* + \text{c.c.}) \ . \tag{1.4.38}$$

Evidently the amplitudes of the magnons b_1,b_2,b_3 when b_0 is large will evolve according to (1.4.36) (with the substitution $U \Rightarrow Tb_0$) until it is possible to neglect the inverse influence of magnons on the homogeneous precession. The curious reader could obtain a fourth equation to (1.4.35) for the amplitude of the homogeneous precession in the external field h and take this inverse influence into account.

1.5 Four–Wave Processes

When three–wave processes are forbidden by conservation laws the equation of motion (1.3.1) must allow for additional terms due to four-wave $\mathcal{H}_{\text{int}}$ (1.1.26). One cannot avoid completely taking into account the three-wave terms, since according to (1.1.32) in the second order of the perturbation theory they result in allowed four-wave processes. It would be interesting to find out which contribution into the four-wave interaction amplitudes is larger: the original W or that arising because of the interaction of the three-wave processes $\tilde{T} \simeq V^2/\Delta\omega$. If the initial Hamiltonian is expanded in terms of a single small parameter, then $\Delta\omega W \simeq V^2$. Then the contributions under consideration are of the same order of magnitude. This is the case (for instance) for the dipole-dipole contribution to $\mathcal{H}_{\text{int}}$ for magnons in ferromagnets. When the situation is more complicated, any of these contributions may prevail for special reasons.

Recall one of the results of Sect. 1.1.4: via an appropriate nonlinear canonical transformation (similar to (1.1.31)) all the terms of $\mathcal{H}_{\text{int}}$ which are responsible for the processes forbidden by conservation laws can be eliminated. Therefore the four-wave scattering of the $2 \Rightarrow 2$ type waves is of particular interest

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_4) \ , \qquad \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 \ , \tag{1.5.1}$$

since it is allowed by all dispersion laws if all $\boldsymbol{k}_j$ are close. Henceforth, only such processes will be considered. Equations (1.3.3) in this case are:

$$\frac{\partial b(\boldsymbol{k},t)}{\partial t} + [i\omega(\boldsymbol{k}) + \gamma(\boldsymbol{k})]b(\boldsymbol{k},t) = -\frac{1}{2}\sum_{\mathbf{123}} T_{\boldsymbol{k},\mathbf{1};\mathbf{2},\mathbf{3}} b_{\mathbf{1}}^{*} b_{\mathbf{2}} b_{\mathbf{3}} \delta(\boldsymbol{k}+\mathbf{1}-\mathbf{2}-\mathbf{3}) \ , \tag{1.5.2}$$

where $T = W + \tilde{T}$ is the total amplitude of the processes (1.1.32b). Let us begin the consideration of these equations with the simplest problem.

1.5.1 Modulation Instability of the Plane Wave

Let $b(\boldsymbol{k}) = b_0\,\delta(\boldsymbol{k}-\boldsymbol{k}_0)$ be a wave of finite amplitude propagating in a medium. Equations (1.5.2) have the form (at $\gamma(\boldsymbol{k}) = 0$):

$$\partial b_0/\partial t + i\omega_{\mathrm{NL}}(\boldsymbol{k}_0) b_0 = 0 \ , \tag{1.5.3}$$

$$\omega_{\mathrm{NL}}(\boldsymbol{k}_0) = \omega(\boldsymbol{k}_0) + T_{\mathbf{0}\,\mathbf{0}}\,|b_0|^2 \ , \quad T_{\mathbf{0}\,\mathbf{0}} = \frac{1}{2}T(\boldsymbol{k}_0,\boldsymbol{k}_0;\boldsymbol{k}_0,\boldsymbol{k}_0), \tag{1.5.4}$$

where $\omega_{\mathrm{NL}}(\boldsymbol{k}_0)$ is the frequency of the wave b_0 dependent on its amplitude. In the presence of the wave b_0 write (1.5.2) for the waves with small amplitudes whose wave vectors and frequencies satisfy the resonance conditions:

$$2\omega(\boldsymbol{k}_0) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) \ , \qquad 2\boldsymbol{k}_0 = \boldsymbol{k}_1 + \boldsymbol{k}_2 \ , \tag{1.5.5}$$

$$\begin{aligned} &\partial b_{\mathbf{1}}/\partial t + i\omega_{\mathrm{NL}}(\boldsymbol{k}_{\mathbf{1}}) b_{\mathbf{1}} + iT_{1,\mathbf{2}}\, b_0^2 b_{\mathbf{2}}^{*} = 0 \ , \\ &\partial b_{\mathbf{2}}/\partial t + i\omega_{\mathrm{NL}}(\boldsymbol{k}_{\mathbf{2}}) b_{\mathbf{2}} + iT_{1,\mathbf{2}}\, b_0^{*} b_{\mathbf{1}}^2 = 0 \ , \end{aligned} \tag{1.5.6}$$

Here $b_{\boldsymbol{j}} = b(\boldsymbol{k}_j)$, $T_{\mathbf{10}} = T_{\mathbf{10},\mathbf{10}}/2$, $T_{\mathbf{12}} = T_{\mathbf{00},\mathbf{12}}/2$, $T_{\mathbf{20}} = T_{\mathbf{20},\mathbf{20}}/2$ and

$$\omega_{\mathrm{NL}}(\boldsymbol{k}_1) = \omega(\boldsymbol{k}_1) + 2T_{\mathbf{10}}|b_{\mathbf{0}}|^2 \ , \quad \omega_{\mathrm{NL}}(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_2) + 2T_{\mathbf{20}}|b_{\mathbf{0}}|^2 \ , \tag{1.5.7}$$

are dependencies of the frequencies of $b_{\mathbf{1}}$,$b_{\mathbf{2}}$ -waves (i.e. waves with amplitudes $b_{\mathbf{1}}$ and $b_{\mathbf{2}}$) on the amplitude of another wave b_0. Note that (1.5.7) differs from (1.5.4) in factor 2. The instability increment is calculated similar to the three-wave case. Comparing (1.5.6) and (1.4.11) we may immediately write (by analogy with (1.4.17)):

$$\nu^2 = |T_{\mathbf{12}}\, b^2|^2 - [\omega_{\mathrm{NL}}(\boldsymbol{k}_2) - 2\omega_{\mathrm{NL}}(\boldsymbol{k}_0)]^2/4 \ . \tag{1.5.8}$$

If $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ differ significantly from $\boldsymbol{k}_0$ and at the same time $\Delta\omega = 0$, then the order of magnitude of the instability increment $\nu = |T_{12}||b_0|^2$ also coincides with the nonlinear addition to the wave frequency. In a dissipative medium when the damping decrement of waves $\boldsymbol{k}_{1,2}$ is $\gamma_{1,2}$ an instability threshold arises. The threshold wave amplitude (similar to (1.4.19)) is

$$|T_{\mathbf{12}}\, b^2| = \sqrt{\gamma_1\gamma_2} \ . \tag{1.5.9}$$

When $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ approximate $\boldsymbol{k}_0$ the conservation law (1.5.5) must allow for nonlinear additions to the frequency

$$\omega_{\rm NL}(\boldsymbol{k}_{1,2}) = \omega(\boldsymbol{k}_0) \mp \boldsymbol{v}\cdot\boldsymbol{\kappa} + \frac{\partial^2\omega}{2\partial k_i \partial k_j}\kappa_i\kappa_j + 2T_{\mathbf{0}\mathbf{0}}\,|b_0|^2 \ , \tag{1.5.10}$$

where $\boldsymbol{k}_{1,2} = \boldsymbol{k}_0 \mp \boldsymbol{\kappa}$, $\boldsymbol{v} = (\partial\omega/\partial\boldsymbol{k})_{\boldsymbol{k}=\boldsymbol{k}_0}$. The function $\omega_{\rm NL}(\boldsymbol{k}_{1,2})$ was expanded in terms of $\boldsymbol{\kappa}$; and $\boldsymbol{k}$-dependence in the matrix elements of T has been neglected. Now

$$\nu^2(\boldsymbol{\kappa}) = -\hat{L}\kappa^2(\hat{L}\kappa^2 + 2T_{\mathbf{0}\mathbf{0}}|b_{\mathbf{0}}|^2) \ , \quad \hat{L}\kappa^2 = \frac{1}{2}\sum_{i,j}\frac{\partial^2\omega(\boldsymbol{k})}{\partial k_i\partial k_j}\kappa_i\kappa_j \ . \tag{1.5.11}$$

If $T_{\mathbf{0}\mathbf{0}}\hat{L}\kappa^2 > 0$, then $\nu < 0$, and there is no instability. This is the difference between the considered situation and the case of the three waves when instability emerged already when the conservation laws were satisfied and the sign of the matrix element was of no significance. When the quadratic form of (1.5.11) is not of fixed sign, the instability criterion

$$T_{\mathbf{0}\mathbf{0}}\hat{L}\kappa^2 < 0 \tag{1.5.12}$$

can be obtained for any sign of $T_{\mathbf{0}\mathbf{0}}$ by changing the direction of $\boldsymbol{\kappa}$. If the form $\hat{L}\kappa^2$ is of fixed sign, the instability region is restricted with respect to κ from 0 to $\hat{L}\kappa^2 = -4T_{\mathbf{0}\mathbf{0}}|b_0|^2$. The instability is maximum for $\hat{L}\kappa^2 = -2T_{\mathbf{0}\mathbf{0}}|b_0|^2$. As the instability (1.5.11) of the $\boldsymbol{k}$-wave increases, new waves arise with $\boldsymbol{k}_{1,2}$ approximating $\boldsymbol{k}_0$. This may be interpreted as the appearance of modulation of the amplitude and the phase of the initial $\boldsymbol{k}_0$-wave. Therefore, such an instability is called by *Zakharov* in [1.14] a *modulation instability.*

Such an instability is a common phenomenon. For instance, for Langmuir waves in plasma $\hat{L}\kappa^2 > 0$, $T_{\mathbf{0}\mathbf{0}} < 0$, which brings about modulation instability leading to Langmuir collapse. In nonlinear dielectrics where $\partial n/\partial|E|^2$ (refractive index n increases with the increase of the electric field strength E), $T_{\mathbf{0}\mathbf{0}} < 0$, since $\omega = kc/\sqrt{n}$ and $\partial\omega/\partial|E|^2 \simeq -\partial n/\partial|E|^2$. Considering that for the linear law of dispersion $\omega'' > 0$ and $T_{\mathbf{0}\mathbf{0}}\omega'' < 0$, this leads to modulation instability whose increase results in self-focusing of light.

By way of example consider the modulation instability of gravitational waves on the liquid surface. The law of dispersion of these waves (see (1.4.12)) has the form $\omega(\boldsymbol{k}) = \sqrt{gk}$. Find the square form of (1.5.12):

$$\hat{L}\kappa^2 = \Big(\kappa_\parallel^2 - \frac{\kappa_\perp^2}{2}\Big)\frac{\omega(k)}{2k^2} \ , \quad \boldsymbol{\kappa}_\parallel = \frac{\boldsymbol{k}(\boldsymbol{\kappa}\cdot\boldsymbol{k})}{k^2} \ , \qquad \boldsymbol{\kappa}_\perp = \boldsymbol{\kappa} - \boldsymbol{\kappa}_\parallel \ . \quad 1.5.13)$$

It can be seen that for $\boldsymbol{\kappa} = \boldsymbol{\kappa}_\perp$, $\hat{L}\kappa^2 < 0$ and at $\boldsymbol{\kappa} = \boldsymbol{\kappa}''$, $\hat{L}\kappa^2 > 0$. Thus, the square form is not of fixed sign and modulation instability will increase whatever the sign of $T_{\mathbf{0}\mathbf{0}}$.

We now estimate the threshold amplitude of the modulation instability $\mu_{\rm cr}$ and its increment ν. It follows from (1.2.13, 14) and (1.5.9) that

$$\omega(k)(k\mu_{\rm cr})^2 \simeq \gamma(k) \ . \tag{1.5.14}$$

Here $\gamma(k)$ is the damping decrement of the gravitational waves. For estimation take $\gamma(k)$ as the damping decrement $\gamma = 2\nu_c k^2$ due to viscosity (see 25 in [1.15]). The kinematic viscosity ν_c is equal to $10^{-2}\,\mathrm{cm}^2\,\mathrm{s}^{-1}$ for water. Hence and from (1.5.14) for waves of the wavelength 100 cm we obtain $\mu_{\mathrm{cr}} \simeq (10^{-2} \div 10^{-3})$ cm. Thus, the threshold amplitude of the wave is vanishingly small and modulation instability of long sea waves must always be in progress. From (1.2.13, 14) and (1.5.8) the increment is easily estimated

$$\nu \simeq \omega(k)(k\mu)^2 \ . \tag{1.5.15}$$

For $\lambda = 100$ cm and $\mu \simeq (10 \div 20)$ cm the increment of the modulation instability (1.5.15) approximates the wave frequency, i.e. the instability is fast increasing. What is the characteristic modulation period L? It may be estimated as $2\pi/\kappa_m$, where κ_m corresponds to the maximum $\nu(\kappa)$. It follows from (1.5.11) and (1.5.13) that in the plane $\kappa_\parallel$, $\kappa_\perp$ the instability region is in a narrow band in the vicinity of the lines $\sqrt{2}\kappa_\perp = \pm\kappa_\parallel$ which bend at κ comparable to the value of k. To find the region of maximum increment on this band (see (1.5.7, 8)) the explicit $\boldsymbol{k}$-dependence of T_{12} must be known. It can be shown that $\kappa_m \simeq 0.1k$. This implies that modulation instability results in long-period longitudinal and lateral modulation of the sea wave amplitude which is known as the "tenth wave" phenomenon.

Further in Chap. 3 we shall treat the modulation instability of spin waves in ferromagnets and antiferromagnets.

1.5.2 Equation for the Envelopes

In studying the nonlinear stage of modulation instability one must take advantage of the fact that all the secondary waves are in a narrow region near $\boldsymbol{k}$, i.e. the propagation of a narrow wave package must be studied or, equivalently, one must study the slow changes of the complex amplitude (*envelope*) of a monochromatic wave.

Here we shall obtain an equation for the envelope. In (1.5.6) $\omega(\boldsymbol{k})$ will be expanded in terms of κ up to the terms $\propto \kappa^2$ and we shall take

$$T(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4}) = T(\boldsymbol{k}_0,\boldsymbol{k}_0;\boldsymbol{k}_0,\boldsymbol{k}_0) = T/4\pi^3 \ . \tag{1.5.16}$$

This approach is correct if there are no long-range forcing in the medium and if $T(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4})$ is a continuous function of its arguments. Next, as in (1.4.3), we pass to the slow variables

$$a(\boldsymbol{k}) = b(\boldsymbol{k}+\boldsymbol{\kappa})\exp[i(\boldsymbol{k}\boldsymbol{r} - \omega_0 t)] \tag{1.5.17}$$

and then to $\boldsymbol{r}$-representation according to (1.4.28). We have as a result:

$$\left[i\frac{\partial}{\partial t} + \boldsymbol{v}\nabla + \hat{L} - T|a|^2\right] a(\boldsymbol{r},t) = 0 \ , \tag{1.5.18}$$

$$\hat{L} = \frac{1}{2}\frac{\partial^2 \omega(\boldsymbol{k})}{\partial k_i \partial k_j}\frac{\partial^2}{\partial x_i \partial x_j} \ . \tag{1.5.19}$$

In an isotropic medium $\omega(\boldsymbol{k}) = \omega(k)$ and

$$\hat{L} = \frac{\omega''}{2}\frac{\partial^2}{\partial z^2} + \frac{v}{2k}\Delta_\perp \ , \qquad \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ . \tag{1.5.20}$$

Here $v = \partial\omega/\partial k$, $\omega'' = \partial^2\omega/\partial k^2$. Equation (1.5.18) describes the evolution of a narrow wave package: the term $\boldsymbol{v}\nabla$ describes its motion as a whole with a group velocity, the terms $\partial^2/\partial z^2$ and $\Delta_\perp$ describe the dispersion and the diffraction, and the last term describes the nonlinear self-action of the waves in the package. Optically, this term describes the dependence of the refractive index of the medium on the square of amplitude of the electric field. When (1.5.19) is treated as a Schrödinger equation the nonlinear term may be interpreted as a potential of a self-consistent attraction (at $T < 0$) or repulsive potential (at $T > 0$), which is proportional to the density of particles. Equations (1.5.19, 20) are Hamiltonian equations with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int [i\boldsymbol{v}(a^*\nabla a - a\nabla a^*) + \omega''(\frac{\partial a}{\partial z})^2 + \frac{v}{k}|\Delta_\perp a|^2 + 2T|a|^4]\, d\boldsymbol{r} \ . \tag{1.5.21}$$

This equation has one more integral of motion $N = \int |a|^2 d\boldsymbol{r}$, meaning "the total number of particles".

There are two problem statements for (1.5.19). Firstly the wave amplitude may be given at a boundary of the medium, e.g. light is incident on a nonlinear dielectric. In this case the term $\omega''\partial^2 a/\partial z^2$ may be neglected compared to $v\partial a/\partial z$ (z is the $\boldsymbol{v}$-direction). Secondly, if the problem is stationary, then $\partial a/\partial t = 0$. In this case:

$$\left[iv\frac{\partial}{\partial z} + \frac{v}{2k}\Delta_\perp - T|a|^2\right] a(\boldsymbol{r}, t) = 0 \ . \tag{1.5.22}$$

Considering $\tau = z/v$ as a new time (1.5.22) can be treated as a two-dimensional Schrödinger equation ($\boldsymbol{r} = x, y$).

1.5.3 Package Evolution in Unbounded Media

Let us change over to a reference system moving with a group velocity. Then the term $\boldsymbol{v}\cdot\nabla a$ vanishes. The dispersion term ω'' must be retained. Thus (1.5.19) takes the form:

$$\left[i\frac{\partial}{\partial t} + \frac{\omega''}{2}\frac{\partial^2}{\partial z^2} + \frac{v}{2k_0}\Delta_\perp - T|a|^2\right] a(\boldsymbol{r}, t) = 0 \ . \tag{1.5.23}$$

Consider qualitatively the solution of this problem in the cases of three dimensions ($d = 3$), two dimensions ($d = 2$) and one dimension. The case $d = 3$ within (1.5.23) corresponds to the evolution of the 3-dimensional bunch moving with a group velocity. If $d = 2$ we deal with the problem of the plane package $a(x, y, t)$ moving with a group velocity or the problem of stationary self-focusing of the package $a(x, y, t)$ within the scope of (1.5.22) after the substitution $z/v = t$. The case $d = 1$ corresponds, for example, to the stationary plane self-focusing $a(x, z)$. Further on, for definiteness we shall always assume $\omega'' > 0$ and will consider the case of particles "attraction" $T < 0$. Let the package (at a certain time) have a characteristic size ℓ and amplitude in the center a. The number of particles in it is given by $N = \int |a|^2 d\boldsymbol{r} \simeq |a|^2 \ell^d$, where d is the dimensionality of the package. Since N is an integral of motion,

$$a(t) = \sqrt{N} \ell^{-d}(t) \; . \tag{1.5.24}$$

Estimate with the help of (1.5.21) and (1.5.24) the energy of the package:

$$\mathcal{H} \simeq \omega'' N \ell^{-2} - |T| N^2 \ell^{-d} \; . \tag{1.5.25}$$

It is clear that in $d = 1$ there exists a stationary solution with

$$\ell = \ell_0 \simeq \omega''/(|T|N) \simeq \omega''/(|Ta^2|\ell_0) \; , \tag{1.5.26}$$

minimizing the energy (1.5.25). In this case the pressure of the particles due to their motion in the potential well balances the attractive force. In the case of 3 dimensions as $\ell \to 0$ the pressure increases slower than the attractive force which leads to collapse, and the particles fall on the center over some finite time. In this case the amplitude $a(t)$ and the size $\ell(t)$ are connected by the relation (1.5.24) where N is the number of the particles involved in collapse. The energy (1.5.25) of the collapsing particle decreases. Evidently the total energy (including the kinetic energy of the particles moving away, not drawn into the collapse) is retained. In the case of two dimensions the fate of the package of particles is determined by the initial conditions: at $\omega'' > TN \simeq T|a|^2/\ell$ the minimum (1.5.25) is achieved as $\ell \to \infty$, i.e. the particles are moved away. Under $\omega'' < TN$ as in the case of three dimensions, part of the package is involved in the collapse process.

Let us treat the stationary solutions of (1.5.18) in the case of one dimension in more detail. Substituting for $a(z, t)$ as $a(z, t) = \varphi(z) \exp(i\lambda^2 t)$, we obtain

$$\begin{aligned} \omega'' \varphi_{zz} &= 2(\lambda^2 \varphi - |T|\varphi^3) = -\partial U/\partial\varphi \; , \\ U &= |T|\varphi^4/2 - \lambda^2\varphi^2 \; , \qquad \varphi_{zz} = \partial^2\varphi/\partial z^2 \end{aligned} \tag{1.5.27}$$

After transformation $z \to t$ this equation may be treated as a Newtonian equation for particles with mass ω'' and coordinate φ moving in the field $U(\varphi)$. It has an integral of motion corresponding to the "energy":

$$E = \frac{1}{2}\omega^{''}\varphi_{zz}^2 + \frac{1}{2}|T|\varphi^4 - \lambda^2\varphi^2 \ . \tag{1.5.28}$$

The simplest solution is a particle at rest at the bottom of the well $\varphi =$ constant, $\lambda = |T|\varphi^2$ corresponds in the initial equation (1.5.23) to a plane wave with a nonlinear frequency shift -$|T|\varphi^2$. When $E > E_{\min}$ it results in periodical oscillations of a particle in the well, i.e. to periodical modulation of the wave $a(z)$. The solution with $E = 0$ is of great interest. It can be obtained from (1.5.28) at $E = 0$

$$\varphi(z,t) = \sqrt{\frac{2}{|T|}}\frac{\lambda\exp(i\lambda^2 t)}{\cosh(\sqrt{2}\lambda z/\sqrt{\omega''})} \ . \tag{1.5.29}$$

The characteristic size ℓ_0 of this localized solution, i.e. *soliton*, is given, as can be readily seen, by the expression (1.5.25). In addition, while the soliton (1.5.29) is at rest (in the reference system moving with group velocity) there exist also moving solitons as well as solutions with two, three, etc. solitons. N-soliton solutions are of great importance. Employing the method of the *inverse scattering problem* for the nonlinear Schrödinger equation (1.5.22, 23) *Zakharov* and *Shabat* [1.16] showed that an arbitrary (fairly smooth) initial distribution of $a(z,0)$ as $t \to \infty$ generally breaks into N solitons whose number, amplitudes and velocities may be obtained from $a(z,0)$. Solitons are often stable formations: on collision, both solitons retain their velocities and amplitudes.

Finally, the qualitative conclusion that in two- and three-dimensional cases the initial package is destroyed over the finite time will be verified by a direct calculation. For simplicity (1.5.23) will be written in a dimensionless form. At $\omega^{''} = 0$, $T < 0$

$$\begin{aligned} &i\frac{\partial\Psi}{\partial t} = \frac{\delta\mathcal{H}}{\delta\Psi}, \qquad \mathcal{H} = \frac{1}{2}\int[|\nabla\Psi|^2 - \frac{1}{2}|\Psi|^4]\,d\boldsymbol{r}\,, \\ &i\frac{\partial\Psi}{\partial t} + \Delta\Psi + |\Psi|^2\Psi = 0\ . \end{aligned} \tag{1.5.31}$$

Following the example of *Vlasov* et al. [1.17] we calculate

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\langle R^2\rangle &= \frac{\partial^2}{\partial t^2}\int r^2|\varphi|^2 d\boldsymbol{r} = i\frac{\partial}{\partial t}\int\sum_j x_j^2\nabla(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)d\boldsymbol{r} \\ &= -\,2i\frac{\partial}{\partial t}\sum_j x_j\Big(\Psi\frac{\partial\Psi^*}{\partial x_j} - \Psi^*\frac{\partial\Psi}{\partial x_j}\Big)d\boldsymbol{r}\ . \end{aligned}$$

Making use once more of the equation of motion and integrating by parts so that no terms proportional to x should be retained we obtain

$$\frac{\partial^2}{\partial t^2}\langle R^2\rangle = 4\int|\nabla\Psi|^2 d\boldsymbol{r} = 8\mathcal{H} + 2(2-d)\int|\Psi|^4 d\boldsymbol{r}\ , \tag{1.5.32}$$

where d is the dimensionality of the space, $\mathcal{H}$ is the Hamiltonian (1.5.30). Thus, at $d > 2$ and $8\mathcal{H} > \partial^2 \langle R^2 \rangle / \partial t^2$, we have

$$\langle R^2 \rangle < 4\mathcal{H}t^2 + C_1 t + C_2 \ . \tag{1.5.33}$$

If at the initial time $\mathcal{H} < 0$, i.e. $2\int |\Delta\varphi|^2 d\boldsymbol{r} < \int |\varphi|^4 d\boldsymbol{r}$, then over a finite time the solution becomes singular and collapse takes place because $\langle R^2 \rangle$ goes to zero. The stationary two-dimensional solution of (1.5.23) (round wave guide) would be of great practical interest since it would enable one to transmit the energy of laser radiation over large distances without the losses caused by diffraction divergence. Such a solution, as it clear from (1.5.25), corresponds to $\mathcal{H} = 0$. Unfortunately, this solution is not stable. If at the initial time $\mathcal{H}$ proves positive because of the fluctuations, then by virtue of (1.5.33) (it should be recalled that at $d = 2$ there is an equality sign in (1.5.33)) $\langle R^2 \rangle$ will increase and the waveguide will "blur". In the opposite case $\mathcal{H} < 0$, field intensity in the center will be very high. As a rule, this leads to irreversible destruction of the nonlinear dielectric accompanied by losses of radiation energy. For details see the review by *Akhmatov* et al. [1.18], *Zakharov* et al. [1.19].