Victor S. L'vov

Wave Turbulence Under Parametric Excitation

Applications to Magnets

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5 Stationary Nonlinear Behavior of Parametrically Excited Waves. Basic *S*-Theory

5.1 History of the Problem

The development of the parametric instability in a continuous medium (if the system's size is big enough in comparison with the wavelength of the excited waves) results in a great number of waves simultaneously excited and intensively interacting. The state of the waves is determined mostly by some particular factors: the dispersion law of the waves and the nonlinear and dissipative properties of the medium. These factors can vary considerably in different cases.

It became clear in the early seventies that there is one simple and at the same time very important specific case when one can formulate a general nonlinear theory of the parametric excitation of waves. This is the case of pumping by a spatially homogeneous field with $\boldsymbol{k}_{\mathrm{p}} = 0$ or by a very long wave $k_{\mathrm{p}} \ll k', k''$. Then the characteristics of the state above threshold may be assumed to be statistically homogeneous. This is the case in most experiments on the parametric excitation of spin waves in magnetically ordered dielectrics. Note that these are the most "pure" experiments in the physics of nonlinear waves because it is relatively simple to carry them out (in comparison, say, with experiments on plasma or the experimental study of nonlinear optics) and the high quality of the dielectrics. Experiments on the ferrimagnetic YIG - Yttrium Iron Garnet - are the most successful because of its many unique properties. Most of the experiments that are described in this book have been performed on YIG. Experiments on antiferromagnets $MnCO_3$, $CsMnF_3$ and $FeBO_3$ have also produced valuable results and are described here.

The first experimental data on the behavior of the spin waves above the threshold as well as the first models of their behavior above the threshold were obtained in the early 60s. The researchers who tried to suggest such models primarily aimed at obtaining the mechanism limiting the increasing amplitude of unstable spin waves. The first step in this direction was made by *Suhl* in [5.1]. He showed that under the pumping of spin waves by the homogeneous precession of the magnetization the principal limiting mechanism is the feedback effect on the pumping. This leads to "freezing" of the spin wave amplitude at the threshold level. However to consistently explain the nonlinear behavior of spin waves under parallel pumping was not so easy. The "traditional" mechanism limiting the parametric instability under parametric resonance with a small number of degrees of freedom, i.e. the nonlinear damping and nonlinear frequency shift, proven to be inadequate. The nonlinear damping in most cases is too weak and sensitive to the magnitude of the constant magnetic field to account for the observed level of the spin waves. The nonlinear frequency shift does not limit the parametric resonance in the continuous medium at all since at any amplitude the renormalized frequencies of some waves completely satisfy the resonance conditions. *Schlömann* in 1962 [5.2] made an important contribution to understanding the behavior of the spin waves above the threshold. He showed that the nonlinear interaction of the spin waves must be allowed for and presumed that the most important role in this interaction is played by the nonlinear processes satisfying the conditions

$$\omega(\boldsymbol{k}) + \omega(-\boldsymbol{k}) = \omega(\boldsymbol{k}_1) + \omega(-\boldsymbol{k}_1) \qquad (5.1.1)$$

without taking the waves out of parametric resonance. Pioneering works by *Zakharov*, *L'vov* and *Starobinets* published in 1969–70 [5.3 – 6], opened up a new stage in the study of nonlinear processes under parametric excitation. The processes (5.1.1) were shown to retain phase correlation within each parametrically excited pair of waves with opposite wave vectors and to result in the self-consistent change of the total phase of waves in each pair. This decreases the energy flux from the pumping to the system of waves and leads to the limitation on their amplitudes. The waves whose renormalized frequencies fully satisfy the parametric resonance conditions in this case prove to be excited. This *phase mechanism* for limitation on the wave amplitude is typical in a continuous medium. It is essential in systems with the large sizes in comparison with the wavelength. Phase mechanism is of principal importance under the parallel pumping of spin waves in the non-decay part of spectrum. It is convenient to study the processes (5.1.1) (allowing for the necessary phase correlations) in the mean–field approximation where the role of the nonlinear wave interaction is reduced to the renormalizing coefficients of the linear equations (the wave frequency $\omega(\boldsymbol{k})$ and the pumping amplitude $hV(\boldsymbol{k})$), describing the parametric instability. The theory based on this approximation (*Zakharov*, *L'vov*, *Starobinets* [5.6]) was later called the *S-theory*. In their works published in 1970 – 74 these authors and their colleagues *Cherepanov*, *Musher*, *Zautkin*, *Rubentchik* and some others managed to develop considerably the study of spin wave behavior above the threshold within the S-theory, and its generalizations. A qualitative explanation has been given to numerous experimentally observed effects and good agreement between theory and experiment has been achieved [5.7–21]. For example, within the frame of the S-theory the giant auto-oscillations

of the magnetization under parametric excitation of spin waves discovered by *Hartwick*, *Peressini* and *Weiss* [5.22] have been explained. A first presentation of the S-theory has been made in the review by *Zakharov*, *L'vov* and *Starobinets* published in [5.23] (based primary on their theoretical and experimental results).

In 1975–80 some interesting works developing the S-theory were published (see, for example, [5.24–30]). But the process of theoretical investigation of nonlinear phenomena under spin-wave parametric excitation slowed down. This was mainly due first to the fact that in 1974 the theory got ahead of experimental studies, and second, that it was not yet commonly accepted by physicists. At that time several publications appeared which either disproved the S-theory (see, e.g., [5.31]) or, on the contrary, obtained again its results using different methods (e.g. [5.30–32]). Incorrect results obtained in some theoretical publications of a transient character making use of the S-theory may be due to the insufficient understanding of the limits of its applicability (e.g., [5.33, 34]). At the same time *Melkov*, *Prozorova*, *Smirnov*, *Ozhogin*, *Zautkin* and their colleagues [5.35–68] obtained interesting and important experimental results on the nonlinear behavior of parametric spin waves. These works were based on the new understanding of the physical processes and phenomena beyond the threshold of the parametric excitation of waves. The basic deductions of the S-theory were confirmed, the details of the nonlinear behavior of the spin waves were clarified and qualitatively new nonlinear effects were discovered that would have been difficult to predict. Two comparatively short reviews devoted to the S-theory were published recently. The first one (*L'vov* [5.69]) deals mostly with theory, the second one (*L'vov* and *Prozorova* [5.70] is devoted primarily to experimental studies of the parametric excitation of magnons in ferro- and antiferromagnets. Chapters 5-7 of the present book give a detailed review of the current state of the S-theory describing the nonlinear behavior of parametrically excited waves in the mean–field approximation. A systematic presentation of some generalizations and modifications of the S-theory for more complicated conditions of parametric excitation (in the presence of the nonlinear wave damping, under violated space homogeneity, for incoherent pumping, under a swept pumping frequency, under simultaneously excited waves of two types etc.) is given for the first time. Much attention will also be given to the description and discussion of various experimental studies of the nonlinear behavior of parametrically excited magnons from the viewpoint of the S-theory. These results (theoretical as well as experimental) may be significant not only for a certain branch of the physics of magnetodielectrics. They are and most certainly will be of great importance for the development of the physics of nonlinear waves in other media. That is why we did our best to separate the results of the general nonlinear theory of the parametric wave excitation in our presentation from the results characteristic of the magnetics. Similarly, discussing the experimental studies of the nonlinear magnons I have tried to emphasize results generally significant for the physics of nonlinear waves. The introduction to the nonlinear theory of the parametric excitation of waves (Chaps. 5 and 7) proceeding from the Hamiltonian equations of motion is based on the same approach and gives no detailed treatment of the nature of specific waves or the medium in which they are excited. This chapter studies the stationary state of the parametrically excited waves.

5.2 Statement of a Problem of Nonlinear Wave Behavior

The following assumptions, simplifications and approximations were made in formulating the nonlinear theory of the parametric excitation of waves:

1. The medium in which waves are propagating (spin waves in magnetodielectrics, various waves in plasma, waves on a liquid surface etc.) is unbounded and spatially homogeneous. To this end the samples of magnetodielectrics, plasma etc. must be of sufficiently high quality, and their linear size L must significantly exceed the mean free path of the waves: $l \simeq \partial\omega/\gamma\partial k$. The effect of statistical random inhomogeneities on the propagation of the waves and their nonlinear behavior will be considered in Chap. 10.

2. Waves in the medium will be described within the frame of the classical Hamiltonian formalism described in Chap. 1, i.e. using the equations of motion (1.3.1) for the complex amplitudes of the travelling waves $b(\boldsymbol{k}), b^*(\boldsymbol{k})$, for which the quadratic Hamiltonian $\mathcal{H}_2$ is diagonal. This approach is in most cases indisputable. However, as mentioned in Chap. 3, it is not evident when the spin waves (magnons) are described at temperatures not small in comparison with the Curie temperature. The exact method for the description of magnons interacting under finite temperatures must be based on the spin diagram techniques for the non-equilibrium processes (*Belinicher, L'vov* [5.71]). However for simplicity we shall proceed here from the Bloch equations for the canonical variables assuming them to give considerable evidence for spin waves.

3. As in Chap. 1 the wave damping at this stage of investigation will be taken into account phenomenologically via the dissipative term $\gamma(\boldsymbol{k})b(\boldsymbol{k},t)$ in (1.3.3). A doubt may arise as to whether this procedure is justified in describing coherent wave systems where the phase relations are significant. Later in Chap. 10 the justification of this procedure will be presented and it will be shown that the decrement of the wave damping in (1.3.3) can be obtained by means of the ordinary kinetic equation. This, however, does not refer to the special case (considered in Chap. 10) when the damping of waves is due to scattering by random heterogeneities.

4. Simplifications of items 1–3 are not specific to the problem of the nonlinear behavior of parametrically excited waves; they are employed in the S-theory as in many other problems of the physics of nonlinear waves. Here we shall discuss a more specific problem: how to select the Hamiltonian of the wave interaction in the S-theory. The frequencies of all parametric waves are almost the same (in first-order processes of parametric excitation they equal half the pumping frequency) so only the four-wave scattering processes of the $2 \to 2$ type are in resonance with parametric waves. These processes were described by the Hamiltonian (1.1.32). The remaining part of $\mathcal{H}_{\text{int}}$ specifies the interaction of parametric waves with the thermal bath of the thermally excited waves and results in the damping of the parametric waves phenomenologically allowed for in (1.3.3). Running a little ahead it can be noted that this procedure has a large region of applicability with respect to the pumping amplitude if three-wave decay processes of one parametric wave into two thermal waves are forbidden. If these processes are allowed the region of applicability is narrower. Sections 5.6.2, 11.2.1 and 11.4.3 consider in detail the influence of the three-wave processes of the interaction of parametric waves with thermal waves on the nonlinear behavior of the parametric waves.

5. The external action on the medium – the parametric pumping – will be considered as monochromatic and spatially homogeneous. These assumptions prove to be correct for all methods of parametric excitation of spin waves considered in 5.4.3. Indeed, the characteristic values of the pumping frequencies ω_{p} of the spin waves are within the range from 10 to 36 gHz, which corresponds to the wavelengths of electromagnetic radiation from 3.0 to 0.8 cm. They considerably exceed the usual experimental values of the parametric spin wave lengths – $(10^{-1}$–$10^{-4})$ cm. As to the degree of pumping incoherence, the frequency linewidth of the experimentally used SHF radiation sources – magnetrons and klystrons – is at least by a factor of 10^2 less than the smallest attained linewidth of the spin waves $\gamma(\boldsymbol{k}) = 10^5$ Hz.

For first-order parametric processes (the decay of one "quantum" of the pumping field into two waves) the Hamiltonian of the interaction with the pumping $\mathcal{H}_{\text{p}}$ has the form (1.4.18):

$$\mathcal{H}_{\text{p}} = \frac{1}{2}\sum_{\boldsymbol{k}} \left[hV(\boldsymbol{k})b^*(\boldsymbol{k})b^*(-\boldsymbol{k})\exp(i\omega t) + \text{c.c.}\right] . \tag{5.2.1}$$

For the Suhl second-order processes (the decay of two magnons with $\boldsymbol{k} = 0$ into two magnons with $\boldsymbol{k} \neq 0$) the Hamiltonian $\mathcal{H}_{\text{p}}$ can be derived from (5.2.1) by substituting $hV(\boldsymbol{k})\exp(i\omega t) \to S(0,\boldsymbol{k})b_0^2$. We shall therefore take $\mathcal{H}_{\text{p}}$ to have the form (5.2.1) and if necessary this substitution will only in the final formula actually be carried out.

6. Here we shall write the equations of the wave motion that will be subsequently used to analyze the nonlinear behavior of the parametrically excited waves. Substituting into dynamic equations of motion (1.1.3) the

Hamiltonian of the waves $\mathcal{H}_{\rm int} = \mathcal{H}_{\rm p} + \mathcal{H}_4$ where $\mathcal{H}_{\rm p}$ and $\mathcal{H}_4$ are given by (5.2.1) and (1.1.32) respectively, we finally get:

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\gamma(\boldsymbol{k})+i\omega(\boldsymbol{k})\right]b(\boldsymbol{k},t)+ihV(\boldsymbol{k})\exp(i\omega t)b^*(-\boldsymbol{k},t)\\ &\quad=-i\frac{\partial\mathcal{H}_4}{\partial b^*(\boldsymbol{k},t)}=-i\sum_{k+1=2+3}T(\boldsymbol{k},1;2,3)b_1^*b_2b_3\,.\end{aligned} \tag{5.2.2}$$

5.3 Phase Relations and Mechanisms for Amplitude Limitation

5.3.1 Analysis of Phase Relations

The initial equations of motion (5.2.2) in the linear approximation (i.e. at $\mathcal{H}_4 = 0$) split into independent pairs of equations for waves with equal and oppositely directed wave vectors $\pm\boldsymbol{k}$. On eliminating the fast time-dependence, i.e. in the "slow" variables

$$a(\boldsymbol{k},t)=b(\boldsymbol{k},t)\exp(i\omega t/2)\,, \tag{5.3.1}$$

they assume the form

$$\begin{aligned}&\{\partial/\partial t+\gamma(\boldsymbol{k})+i[\omega(\boldsymbol{k})-\omega_{\rm p}/2]\}a(\boldsymbol{k},t)+ihV(\boldsymbol{k})a^*(-\boldsymbol{k},t)=0\,,\\ &\{\partial/\partial t+\gamma(\boldsymbol{k})-i[\omega(\boldsymbol{k})-\omega_{\rm p}/2]\}a^*(-\boldsymbol{k},t)+ihV(\boldsymbol{k})a(\boldsymbol{k},t)=0\,.\end{aligned} \tag{5.3.2}$$

Taking $a(\boldsymbol{k},t)$, $a(-\boldsymbol{k},t)\propto \exp[\nu(\boldsymbol{k})t]$, we have

$$\nu(\boldsymbol{k})=-\gamma(\boldsymbol{k})\pm\sqrt{|hV(\boldsymbol{k})|^2-[\omega(\boldsymbol{k})-\omega_{\rm p}/2]^2}\,. \tag{5.3.3}$$

Obviously, this expression coincides with (1.4.19) which was obtained in Chap. 1, where the parametric instability had been considered as a special case of three-wave decay instability of the wave with $\boldsymbol{k}_{\rm p}=0$. The minimum threshold corresponding to the parametric resonance $2\omega(\boldsymbol{k})=\omega_{\rm p}$ is obtained from the condition $|hV(\boldsymbol{k})|=\gamma(\boldsymbol{k})$, which has a simple meaning of energy balance. Indeed, the energy flux W_+ from the pumping to the wave pair $\pm\boldsymbol{k}$ is

$$\begin{aligned}W_+&=-\partial\mathcal{H}_{\rm p}/\partial t=i\omega_{\rm p}[hV(\boldsymbol{k})a^*(\boldsymbol{k})a^*(-\boldsymbol{k})+\text{c.c.}]\\ &=2|hV(\boldsymbol{k})|\omega_{\rm p}|a(\boldsymbol{k})|^2\sin[\Psi_{\rm p}(\boldsymbol{k})-\Psi(\boldsymbol{k})]\,.\\ a(\boldsymbol{k})&=|a(\boldsymbol{k})|\exp[-i\varphi(\boldsymbol{k})],\end{aligned} \tag{5.3.4}$$

where $\Psi(\boldsymbol{k})=\varphi(\boldsymbol{k})+\varphi(-\boldsymbol{k})$ is the common phase of the pair and $\Psi_{\rm p}(\boldsymbol{k}) = \arg\{hV(\boldsymbol{k})\}$ is the pumping phase. On the other hand, the energy W_- dissipated by the pair per unit time is

$$W_-=2\gamma(\boldsymbol{k})[\omega(\boldsymbol{k})|a(\boldsymbol{k})|^2+\omega(-\boldsymbol{k})|a(-\boldsymbol{k})|^2]=2\omega_{\rm p}\gamma(\boldsymbol{k})|a(\boldsymbol{k})|^2\,. \tag{5.3.5}$$

At the threshold point $W_+=W_-$. The greatest energy flux and the lowest instability threshold are characteristic of the pair with the most advantageous phase relation: $\Psi(\boldsymbol{k})=\Psi_{\rm p}(\boldsymbol{k})+\pi/2$. For the threshold in this case the relation $|hV(\boldsymbol{k})|=\gamma(\boldsymbol{k})$ holds true again. The condition for parametric resonance can evidently be simultaneously fulfilled for a great number of pairs whose wave vectors are on the resonance surface. The pairs with the minimum ratio $\gamma(\boldsymbol{k})/|hV(\boldsymbol{k})|$ have the minimum threshold of excitation: $h_{\rm th}=\min[\gamma(\boldsymbol{k})/|hV(\boldsymbol{k})|]$. At $h>h_{\rm th}$ the amplitudes of the pairs begin to increase exponentially:

$$\begin{aligned}a(\boldsymbol{k},t)&=a(\boldsymbol{k})\exp[\nu(\boldsymbol{k})t-i\Psi(\boldsymbol{k})/2]\,,\\ a^*(-\boldsymbol{k},t)&=a^*(-\boldsymbol{k})\exp[\nu(\boldsymbol{k})t+i\Psi(\boldsymbol{k})/2]\end{aligned} \tag{5.3.6}$$

with the increment (5.3.3). It follows from (5.3.2) that

$$\cos[\Psi(\boldsymbol{k})-\Psi_{\rm p}(\boldsymbol{k})]=[\omega(\boldsymbol{k})-\omega_{\rm p}/2]/|hV(\boldsymbol{k})|\,. \tag{5.3.7}$$

This means that in the linear stage of the parametric instability a certain relation between the phases of the waves in pairs is established. The phase correlation of waves with equal and oppositely directed wave vectors may, by analogy with superconductivity, be called *coupling*. Unlike superconductivity, the physical cause for the wave coupling is that the pumping separates pairs of waves with the maximum instability increment from the thermal bath of waves with chaotic phases. Phase correlation will later be shown to be complete at the nonlinear stage of the development of the instability. This means that although the value $a(\boldsymbol{k},t)$ is random, the value $a(\boldsymbol{k},t)a(-\boldsymbol{k},t)$ will be dynamic. In this case

$$\langle a(\boldsymbol{k},t)a(-\boldsymbol{k},t)\rangle=a(\boldsymbol{k},t)a(-\boldsymbol{k},t)\,,\quad \langle\Psi(\boldsymbol{k},t)\rangle=\Psi(\boldsymbol{k},t).$$

5.3.2 Nonlinear Mechanisms for Limiting Parametric Instability

Nonlinear damping, i.e. a dependence of $\gamma(\boldsymbol{k})$ on the squared amplitudes of parametric waves $|a(\boldsymbol{k})|^2$ (*Schlömann* [5.72], *Gottlib* and *Suhl* [5.73]) can serve as the simplest mechanism of this kind. The stationary amplitudes of waves are determined by the well-known condition of energy balance: $|hV(\boldsymbol{k})|=\gamma(\boldsymbol{k})$. The simplest dependence is chosen for the qualitative analysis: $\gamma=\gamma_0+\eta\sum|a(\boldsymbol{k})|^2$. Then

$$\sum_{\boldsymbol{k}}|a(\boldsymbol{k})|^2=(|hV|-\gamma_0)/\eta=V(h-h_{\rm th})/\gamma_0\,. \tag{5.3.8}$$

The phases $\Psi(\boldsymbol{k})$ in this case are obtained from the conditions of the parametric resonances and are shifted by $\pi/2$ from the pumping phase.

One more evident limiting mechanism for parametric instability is connected with the feedback of parametrically excited waves on the pumping. The energy flux into the system of parametric waves required for maintaining their number at a finite level is taken off from the pumping and decreases its amplitude. In the simplest case when there are no other limiting mechanisms the pumping amplitude is "frozen" at the threshold level, and the number of excited parametric waves is obtained from the energy balance condition in the system of the pumping. In some cases when, say, the homogeneous precession of magnetization excited in resonance by the external SHF field serves as the pumping for spin waves in the ferromagnets, the *feedback mechanism* is really important and will be treated in detail in Sect. 5.6.3. In other cases, for example, under the parallel pumping when the spin waves in a small sample of a magnetodielectric are excited by the SHF field of a big cavity-resonator the feedback of the spin waves on the field in the resonant cavity, as a rule, can be neglected.

The third *phase limiting mechanism* suggested by *Zakharov*, *L'vov* and *Starobinets* [5.4] plays the key role in the parametric excitation of waves. The present chapter will deal with its detailed treatment. Now it will only be noted that the phase limiting mechanism is connected with the coupling which leads to the misphasing ($\sin\Psi(\boldsymbol{k}) < 1$) between the pairs and the external pumping, i.e. to the decreased energy flux into the system.

The fourth limiting mechanism due to the generation and the collapse of solitons is active when the amplitude of waves interacting with the pumping $V(\boldsymbol{k})$ in the Hamiltonian (5.2.1) is maximum at a single pair of points $\pm\boldsymbol{k}$ so that above the threshold a narrow wave packet is excited. This limiting mechanism will be described in Sect. 8.3.

5.4 Basic Equations of Motion in the S-Theory

5.4.1 Statistical Properties of a Non-Interacting Field

As known from statistical physics, in the absence of interaction the statistical properties of the non-interacting wave field are Gaussian. This means that all the odd-order correlators are zero:

$$\langle b(\boldsymbol{k},t)\rangle = 0\,, \quad \langle b^*(\boldsymbol{k},t)b(\boldsymbol{k}_1,t)b(\boldsymbol{k}_2,t)\rangle = 0, \tag{5.4.1}$$

and the even-order correlators are expressed in terms of various products of the double correlators (correlation functions)

$$\langle b(\boldsymbol{k},t)b^*(\boldsymbol{k}',t)\rangle = n(\boldsymbol{k},t)\delta(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{5.4.2}$$

In particular, the splitting rule of the fourth-order correlators has the form

$$\langle b_1^*b_2^*b_3b_4\rangle = n_1n_2[\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)+\delta(\boldsymbol{k}_1-k_4)\delta(\boldsymbol{k}_2-\boldsymbol{k}_3)]\,. \tag{5.4.3}$$

Now and further on we shall use the short notation $b_j = b(\boldsymbol{k}_j,t)$, $n_j = n(\boldsymbol{k}_j,t)$. We shall also use the splitting rule of the sixth-order correlators

$$\begin{aligned}\langle b_1^*b_2^*b_3^*b_4b_5b_6\rangle &= n_1n_2n_3\\ &\times\{\delta(\boldsymbol{k}_1-\boldsymbol{k}_4)[\delta(\boldsymbol{k}_2-\boldsymbol{k}_5)\delta(\boldsymbol{k}_3-\boldsymbol{k}_6)+\delta(\boldsymbol{k}_2-\boldsymbol{k}_6)\delta(\boldsymbol{k}_3-\boldsymbol{k}_5)]\\ &+\delta(\boldsymbol{k}_1-\boldsymbol{k}_5)[\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)\delta(\boldsymbol{k}_3-\boldsymbol{k}_6)+\delta(\boldsymbol{k}_2-\boldsymbol{k}_6)\delta(\boldsymbol{k}_3-\boldsymbol{k}_4)]\\ &+\delta(\boldsymbol{k}_1-\boldsymbol{k}_6)[\delta(\boldsymbol{k}_2-\boldsymbol{k}_4)\delta(\boldsymbol{k}_3-\boldsymbol{k}_5)+\delta(\boldsymbol{k}_2-\boldsymbol{k}_5)\delta(\boldsymbol{k}_3-\boldsymbol{k}_4)]\}.\end{aligned} \tag{5.4.4}$$

Equations (5.3.4, 5) are the classical limit of the well-known Week theorem for the Bose-operators. It must also be noted that the δ-functions of the momenta differences in the right-hand sides of (5.4.2–4) are the consequence of the spatial homogeneity of the problem. And, finally, the correlators $n(\boldsymbol{k},t)$ have the dimensionality of the action (erg·s). They are the classical analogues of the dimensionless quantum-mechanical occupation numbers $n_{\rm qm}(\boldsymbol{k},t)$. Within the limit $n_{\rm qm} \gg 1$, $\hbar n_{\rm qm}(\boldsymbol{k},t) = n(\boldsymbol{k},t)$.

5.4.2 Mean–Field Approximation

In this section the mean–field approximation will be formulated and statistical equations of motion for conjugate correlators describing the system of parametrically excited waves will be obtained in first-order perturbation theory with respect to the interaction Hamiltonian $\mathcal{H}_{\rm int}$. Differentiating (5.4.2) for $n(\boldsymbol{k},t)$ with respect to time and using the equations of motion (5.2.2, 3), we obtain:

$$\begin{aligned}&\left\{\left[\frac{\partial}{\partial t}+2\gamma(\boldsymbol{k})\right]n(\boldsymbol{k},t)+2\,\mathrm{Im}\left[hV^*(\boldsymbol{k})\sigma(\boldsymbol{k},t)\right]\right\}\delta(\boldsymbol{k}-\boldsymbol{k}_1)\\ &\qquad = \mathrm{Im}\left\{\sum_{k+2=3+4} T(\boldsymbol{k},\boldsymbol{2};\boldsymbol{3},\boldsymbol{4})\langle a_1^*a_2^*a_3a_4\rangle\right\}.\end{aligned} \tag{5.4.5}$$

where $a_j = a(\boldsymbol{k}_j,t)$ are slow variables (5.3.1). In (5.4.5) for the first time in our theory appeared a new object - the *anomalous double correlator* $\sigma(\boldsymbol{k},t)$, which will be given by the following formula

$$\langle a(\boldsymbol{k},t)a(\boldsymbol{k}_1,t)\rangle = \sigma(\boldsymbol{k},t)\delta(\boldsymbol{k}+\boldsymbol{k}_1)\,. \tag{5.4.6}$$

With such a definition of $\sigma(\boldsymbol{k},t)$ via slow variables, the explicit time dependence in (5.4.5) is absent. Recall that in the free wave field the anomalous correlator σ is zero because it vanishes under averaging over the wave phases. Indeed,

$$\sigma(\boldsymbol{k}) \propto \langle|a(\boldsymbol{k})||a(-\boldsymbol{k})|\rangle\exp\{i[\varphi(\boldsymbol{k})+\varphi(-\boldsymbol{k})]\}\,. \tag{5.4.7}$$

In the presence of pumping, however, at the linear stage of the parametric instability the sum of phases $[\varphi(\boldsymbol{k})+\varphi(-\boldsymbol{k})]$ according to (5.3.7) is rigidly fixed. Therefore at the linear stage $\sigma(\boldsymbol{k},t) \neq 0$ and just in case must be allowed for in the nonlinear equations, too.

Differentiating $\sigma(\boldsymbol{k},t)$ (5.4.6) with respect to time and employing the equations of motion (5.2.2) and (5.2.3), we obtain the equation of motion for $\sigma(\boldsymbol{k},t)$:

$$\begin{aligned}\Big\{\Big[\frac{\partial}{\partial t} + \big[\gamma(\boldsymbol{k}) + \gamma(-\boldsymbol{k})\big] + i\big[\omega(\boldsymbol{k}) + \omega(-\boldsymbol{k}) - \omega_{\mathrm{p}}\big]\Big]\sigma(\boldsymbol{k},t) \\ + ihV(\boldsymbol{k})\big[n(\boldsymbol{k},t) + n(-\boldsymbol{k},t)\big]\Big\}\delta(\boldsymbol{k}+\boldsymbol{k}_1) \\ = \frac{1}{2}\sum_{\mathbf{2},\mathbf{3},\mathbf{4}}\Big[T(\boldsymbol{k},\mathbf{2};\mathbf{3},\mathbf{4})\langle a_1 a_2^* a_3 a_4\rangle + T(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4})\langle a_k a_2^* a_3 a_4\rangle\Big]\,.\end{aligned} \tag{5.4.8}$$

Equations (5.4.5, 8) are accurate but not constructive since they express double correlators $n(\boldsymbol{k})$ and $\sigma(\boldsymbol{k})$ via the fourth-order correlators. The equations for the fourth-order correlators, in turn, will contain the sixth-order correlators, etc.

At the first stage of the theory development this chain of equations should be closed in the simplest way taking the interaction Hamiltonian to be small in any required sense. This enables us to confine ourselves to first-order perturbation theory with respect to H_{int}. To formulate this approximation, note that the first parts of (5.4.5, 8) explicitly contain amplitudes of the interaction Hamiltonian $T(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4})$. Therefore in the linear approximation with respect to $\mathcal{H}_4$ the fourth-order correlators must be calculated in the zeroth approximation with respect to $\mathcal{H}_4$, as for the free wave field in the presence of pumping. This implies that the fourth-order correlators can be expressed in terms of $n(\boldsymbol{k},t)$ and $\sigma(\boldsymbol{k},t)$ using the following splitting procedure generalizing the standard procedure (5.4.3):

$$\begin{aligned}\langle a_1^* a_2^* a_3 a_4\rangle =& n_1 n_2[\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\delta(\boldsymbol{k}_2-\boldsymbol{k}_4) \\ &+ \delta(\boldsymbol{k}_1-\boldsymbol{k}_4)\delta(\boldsymbol{k}_2-\boldsymbol{k}_3)] \\ &+ \sigma_1^*\sigma_3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2)\delta(\boldsymbol{k}_3+\boldsymbol{k}_4),\end{aligned} \tag{5.4.9}$$

$$\begin{aligned}\langle a_1^* a_2 a_3 a_4\rangle =& n_1[\sigma_3\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)\delta(\boldsymbol{k}_3+\boldsymbol{k}_4) \\ &+ \sigma_2\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\delta(\boldsymbol{k}_2+\boldsymbol{k}_4)] \\ &+ \sigma_2\delta(\boldsymbol{k}_1-\boldsymbol{k}_4)\delta(\boldsymbol{k}_2+\boldsymbol{k}_3)]].\end{aligned} \tag{5.4.10}$$

Similar relations are well known in theoretical physics, i.e. in the theory of superfluidity of a weakly non-ideal Bose gas [5.74]. Relations (5.4.9, 10) enable us to close the system of equations (5.4.5, 8):

$$\begin{aligned}\frac{\partial\sigma(\boldsymbol{k},t)}{\partial t} =& \{-2\gamma(\boldsymbol{k}) + i[2\omega_{\mathrm{NL}}(\boldsymbol{k},t) - \omega_{\mathrm{p}}]\}\sigma(\boldsymbol{k},t) \\ &- i[n(\boldsymbol{k},t) + n(-\boldsymbol{k},t)]P(\boldsymbol{k},t)\,, \\ \frac{1}{2}\cdot\frac{\partial n(\boldsymbol{k},t)}{\partial t} =& -\gamma(\boldsymbol{k})n(\boldsymbol{k},t) - \mathrm{Im}\{P^*(\boldsymbol{k},t)\sigma(\boldsymbol{k},t)\}\,, \\ \frac{1}{2}\cdot\frac{\partial n(-\boldsymbol{k},t)}{\partial t} =& -\gamma(\boldsymbol{k})n(-\boldsymbol{k},t) - \mathrm{Im}\{P^*(\boldsymbol{k},t)\sigma(\boldsymbol{k},t)\}\,, \\ P(\boldsymbol{k},t) =& hV(\boldsymbol{k}) + \sum_{\mathbf{1}} S(\boldsymbol{k},\boldsymbol{k}_1)\sigma(\boldsymbol{k}_1,t)\,, \\ \omega_{\mathrm{NL}}(\boldsymbol{k},t) =& \omega(\boldsymbol{k}) + 2\sum_{\mathbf{1}} T(\boldsymbol{k},\boldsymbol{k}_1)n(\boldsymbol{k}_1,t)\,,\end{aligned} \tag{5.4.11}$$

where $S(\boldsymbol{k},\boldsymbol{k}_1) = S^*(\boldsymbol{k}_1,\boldsymbol{k}) = T(\boldsymbol{k},-\boldsymbol{k},\boldsymbol{k}_1,-\boldsymbol{k}_1)/2$, and $T(\boldsymbol{k},\boldsymbol{k}_1) = T^*(\boldsymbol{k},\boldsymbol{k}_1) = T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{k}_1)/2$. The value $P(\boldsymbol{k},t)$ will be called the *complete pumping.* It differs from the *external pumping* $hV(\boldsymbol{k})$, describing the energy flux from the external fluid of the pumping h into the pair of waves with the wave vectors $\pm\boldsymbol{k}$ in the *self-consistent pumping* $\sum S(\boldsymbol{k},\boldsymbol{k}_1)\sigma(\boldsymbol{k}_1,t)$, specifying the energy exchange between the pairs due to their interaction under pairing (i.e. when $\sigma \neq 0$). The function $\omega_{\mathrm{NL}}(\boldsymbol{k},t)$ is the wave frequency renormalized due to the interaction. It differs from the frequency of the non-interacting wave field $\omega(\boldsymbol{k})$ in the nonlinear term proportional to $T(\boldsymbol{k},\boldsymbol{k}_1)$. We have already discussed such a renormalization when dealing with the four-wave processes in Sect. 1.5.

Equations (5.4.11) obtained in the present section will be referred to as the *basic equations of the S-theory.* The content of this theory presented in Chaps. 5–8 essentially consists of the analysis of different solutions of the equations (and also modifications for more complicated conditions of parametric excitation). The very name of "S-theory" reflects the decisive influence of the $S(\boldsymbol{k},\boldsymbol{k}_1)$ function (specifying the rate of energy exchange between $\pm\boldsymbol{k}$ and $\pm\boldsymbol{k}_1$ pairs) on the nonlinear behavior of the system of interacting parametric waves.

Recall that the basic equations of the S-theory (5.4.11) have been obtained in first-order perturbation theory with respect to $\mathcal{H}_4$ in the approximation not allowing for the correlation of the wave field fluctuations due to the interaction. In theoretical physics such an approximation is often called the *mean–field approximation.* Classical examples of such an approximation are the Curie-Weiss theory of the molecular field, the Landau theory of second-order phase transitions as well as the BCS-theory of superconductivity. Those highly meaningful examples refer to the physics of the second-order phase transitions. In the physics of nonlinear waves, however, the mean–field approximation is usually trivial since it does not account for non-trivial wave dynamics. It can be seen from (5.4.8) that in the absence of pumping and wave coupling (i.e. at $hV(\boldsymbol{k}) = 0$, $\sigma(\boldsymbol{k},t) = 0$) the kinetics of the number of waves is trivial: $\partial n(\boldsymbol{k},t)/\partial t = -2\gamma(\boldsymbol{k})n(\boldsymbol{k},t)$.

Non-trivial wave kinetics in the mean–field approximation can be a result not only of the coupling, but also of the spatial inhomogeneity or nonhermitian amplitude of the wave interaction $T(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4})$. Theoretical treatment of the phenomena arising in this situation was presented in short e.g. in my "Lectures on the Physics of Nonlinear Phenomena" [5.75]. It is essential for explaining some facts of plasma physics, nonlinear optics and even environmental science. It has much in common with the S-theory.

5.4.3 General Analysis of Basic Equations of the S-Theory

Proceeding to the analysis of the basic equations of the S-theory (5.4.11), let us obtain from them the following relations

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+4\gamma(\boldsymbol{k})\right]\left[n(\boldsymbol{k},t)n(-\boldsymbol{k},t)-|\sigma(\boldsymbol{k},t)|^2\right]=0,\\&\left[\frac{\partial}{\partial t}+2\gamma(\boldsymbol{k})\right]\left[n(\boldsymbol{k},t)-n(-\boldsymbol{k},t)\right]=0\end{aligned} \tag{5.4.12}$$

showing that arbitrary initial distributions, over time of the order $1/\gamma(\boldsymbol{k})$, relax to the state (not necessarily stationary) in which $n(\boldsymbol{k},t)=n(-\boldsymbol{k},t)$ and $|\sigma(\boldsymbol{k},t)|=n(\boldsymbol{k},t)$. The conditions implying that the phases of wave pairs are fully correlated also at the nonlinear stage of the parametric instability make it possible to introduce instead of the variables $\sigma(\boldsymbol{k},t)$, $\sigma^*(\boldsymbol{k},t)$, $n(\boldsymbol{k},t)$ and $n(-\boldsymbol{k},t)$ only the two real variables $n(\boldsymbol{k},t)$ and $\Psi(\boldsymbol{k},t)$:

$$\begin{aligned}\frac{\partial n(\boldsymbol{k},t)}{2\partial t}&=n(\boldsymbol{k},t)\Big\{-\gamma(\boldsymbol{k})+\mathrm{Im}\{P^*(\boldsymbol{k},t)\exp[-i\Psi(\boldsymbol{k},t)]\}\Big\}\\\frac{\partial\Psi(\boldsymbol{k},t)}{\partial t}&=\omega_{\mathrm{NL}}(\boldsymbol{k},t)-\frac{\omega_{\mathrm{p}}}{2}+\mathrm{Re}\{P^*(\boldsymbol{k},t)\exp[-i\Psi(\boldsymbol{k},t)]\}\,,\\P(\boldsymbol{k},t)&=hV(\boldsymbol{k})+\sum_1 S(\boldsymbol{k},\boldsymbol{k}_1)n(\boldsymbol{k}_1,t)\exp[-i\Psi(\boldsymbol{k},t)]\,,\\\omega_{\mathrm{NL}}(\boldsymbol{k},t)&=\omega(\boldsymbol{k})+2\sum_1 T(\boldsymbol{k},\boldsymbol{k}_1)n(\boldsymbol{k}_1,t)\,.\end{aligned} \tag{5.4.13}$$

This may be written in a more compact form if instead of the two real variables, the wave number $n(\boldsymbol{k},t)$ and the phase of the pair $\Psi(\boldsymbol{k},t)$, we use one complex variable $c(\boldsymbol{k},t)$:

$$\begin{aligned}&n(\boldsymbol{k},t)=n(-\boldsymbol{k},t)=|c(\boldsymbol{k},t)|^2\,,\qquad\sigma(\boldsymbol{k},t)=c^2(\boldsymbol{k},t)\,,\\&c(\boldsymbol{k},t)=\sqrt{n(\boldsymbol{k},t)}\exp[-i\Psi(\boldsymbol{k},t)/2]\,.\end{aligned} \tag{5.4.14}$$

In those variables we have one complex equation instead of (5.4.11) for $n(\boldsymbol{k},t)$, $n(-\boldsymbol{k},t)$, $\sigma(\boldsymbol{k},t)$ and $\sigma^*(\boldsymbol{k},t)$ or (5.4.14) for $n(\boldsymbol{k},t)$ and $\Psi(\boldsymbol{k},t)$:

$$\frac{\partial c(\boldsymbol{k},t)}{\partial t}+\Big\{\gamma(\boldsymbol{k})+i\Big[\omega_{\mathrm{NL}}(\boldsymbol{k},t)-\frac{\omega_{\mathrm{p}}}{2}\Big]\Big\}c(\boldsymbol{k},t)=-iP(\boldsymbol{k},t)c^*(\boldsymbol{k},t)\,, \tag{5.4.15}$$

$$\begin{aligned}\omega_{\mathrm{NL}}(\boldsymbol{k},t)&=\omega(\boldsymbol{k})+2\sum_{\boldsymbol{k}_1}T(\boldsymbol{k},\boldsymbol{k}_1)|c(\boldsymbol{k}_1,t)|^2\,,\\P(\boldsymbol{k},t)&=hV(\boldsymbol{k})+\sum_{\boldsymbol{k}_1}S(\boldsymbol{k},\boldsymbol{k}_1)c^2(\boldsymbol{k},t)\,.\end{aligned} \tag{5.4.16}$$

These equations can be obtained directly from the initial dynamic equations (5.2.2), if we assume $b(\boldsymbol{k})=b(-\boldsymbol{k})=c(\boldsymbol{k})$ and substitute the four-wave interaction Hamiltonian (1.1.32) for its diagonal part in pairs:

$$\tilde{\mathcal{H}}_{\mathrm{S}}=\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}T(\boldsymbol{k}_1,\boldsymbol{k}_2)|c_1|^2|c_2|^2+\frac{1}{2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}S(\boldsymbol{k}_1,\boldsymbol{k}_2)c_1^{*2}c_2^2\,. \tag{5.4.17}$$

This implies that the basic equations of the S-theory are in fact dynamic, although they are obtained through statistical averaging and correlation splitting by the (5.4.9,10) rule. Moreover, at $\gamma(\boldsymbol{k})=0$ the equations of the S-theory are Hamiltonian. In the variables c,c^* this fact is trivial, since the (5.4.15) coincide with (5.2.2). As to (5.4.13), obviously, they can be obtained by the variation of the Hamiltonian

$$\begin{aligned}\mathcal{H}_{\mathrm{S}}=&\sum_1\Big[\omega(\boldsymbol{k}_1)+\sum_2 T(\boldsymbol{k}_1,\boldsymbol{k}_2)n_2\Big]n_1+\frac{1}{2}\sum_1\Big[hV(\boldsymbol{k}_1)\cos\Psi_1\\&+\frac{1}{2}\sum_2 S(\boldsymbol{k}_1,\boldsymbol{k}_2)n_2\cos(\Psi_1-\Psi_2)\Big]\end{aligned} \tag{5.4.18}$$

by the rule

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t}+\gamma(\boldsymbol{k})n(\boldsymbol{k},t)=-\frac{\delta\mathcal{H}_{\mathrm{S}}}{\delta n(\boldsymbol{k},t)}\,,\qquad\frac{\partial\Psi(\boldsymbol{k},t)}{\partial t}=\frac{\delta\mathcal{H}_{\mathrm{S}}}{\delta n(\boldsymbol{k},t)}\,. \tag{5.4.19}$$

The Hamiltonian $\tilde{\mathcal{H}}_{\mathrm{S}}$ (5.4.17) or $\mathcal{H}_{\mathrm{S}}$ (5.4.18) is called the ***diagonal Hamiltonian of the S-theory***. In the Hamiltonians only such terms were retained that are either fully independent of the wave phases (the first sum in $\mathcal{H}_{\mathrm{S}}$), or depend only on the sum of the phases $\Psi(\boldsymbol{k})=\varphi(\boldsymbol{k})+\varphi(-\boldsymbol{k})$ in the pairs. All the other terms depending on the individual phases of waves (or, to be more exact, on the differences $\varphi(\boldsymbol{k})-\varphi(-\boldsymbol{k})$) are omitted. It is clear from the above that in first-order perturbation theory with respect to $\mathcal{H}_{\mathrm{int}}$ these terms are zero after averaging over wave phases of the Gaussian ensemble of the free wave field. The interaction itself, of course, leads to some correlations of the phases. Therefore, in the second (and higher) orders of perturbation theory with respect to $\mathcal{H}_{\mathrm{int}}$ additional terms will arise. Those terms will be proportional to $T^2(\mathbf{1},\mathbf{2};\mathbf{3},\mathbf{4})$, and will describe the four-wave scattering of particular parametric waves. This theory was named *ST^2-theory*; a short outline will be given in Sect. 11.2. The approximation of the S-theory will be shown to give a good description of the basic characteristics of parametric waves system up to the pumping amplitudes $h<h_{\mathrm{S}}$, where

$$h_{\mathrm{S}}V\simeq\sqrt{\gamma k\partial\omega(\boldsymbol{k})/\partial k}\,. \tag{5.4.20}$$

5.5 Ground State of System of Interacting Parametric Waves

5.5.1 Stationary States and Analysis of Instability

Now let us proceed to a discussion of the stationary states of the system of pairs in which all the numbers $n(\boldsymbol{k},t)$ and phases $\Psi(\boldsymbol{k},t)$ are time-independent. Taking in (5.4.13) derivatives with respect to time to be zero we immediately obtain for the point of $\boldsymbol{k}$-space, where $n(\boldsymbol{k}) \neq 0$, the condition

$$|P(\boldsymbol{k})|^2 = \gamma^2(\boldsymbol{k}) + [\omega_{\rm NL}(\boldsymbol{k}) - \omega_{\rm p}/2] \ . \qquad (5.5.1)$$

Before analyzing this result we shall make two remarks of a general character. First, it is clear that the amplitudes of the pairs differ from zero only over a thin layer near the resonance surface $2\omega(\boldsymbol{k}) = \omega_{\rm p}$. Because of this it is convenient to use the following coordinates in $\boldsymbol{k}$-space: κ is the deflection from this surface in the normal direction and Ω is the coordinate on the surface. Second, the coefficients of (5.4.13) with the dimensionality of frequency:

$$\gamma(\boldsymbol{k}) \ , \quad hV(\boldsymbol{k}) \ , \quad \sum_1 T(\boldsymbol{k},\boldsymbol{k}_1)n_1 \ , \quad \sum_1 S(\boldsymbol{k},\boldsymbol{k}_1)n_1$$

are considerably less than the eigenfrequency $\omega(\boldsymbol{k})$. Therefore in theory it will be sufficient to allow dependence on κ only in the function $\omega(\boldsymbol{k}) - \omega_{\rm p}/2$ and in all other functions (namely $\gamma(\boldsymbol{k}), hV(\boldsymbol{k}), T(\boldsymbol{k},\boldsymbol{k}_1)$ and $S(\boldsymbol{k},\boldsymbol{k}_1)$) to substitute their values on the resonant surface ($\gamma(\Omega), hV(\Omega)$, etc). Making use of the above approximations one can easily obtain from (5.5.1) those κ for which $n(\kappa,\Omega)$ can be non-zero:

$$\omega_{\rm NL}(\kappa,\Omega) = \omega_{\rm p}/2 \pm \sqrt{|P(\Omega)|^2 - \gamma^2(\Omega)} \ . \qquad (5.5.2)$$

Thus, in the stationary state the distribution of the occupation numbers of the pairs is singular: $n(\kappa,\Omega)$ can differ from zero only on two surfaces (5.5.2). Note that there are numerous stationary states differing both in the function $P(\Omega)$ defining the surfaces (5.5.2) and in the distribution of $n(\Omega)$ over them. Indeed, one can arbitrarily set the direction of Ω where $n(\Omega)$ is equal to zero. In fact, of all the stationary states, only those states can be realized which are stable with respect to small perturbations. The requirement that the states should be stable considerably narrows the range of feasible stationary states. The study of stability of stationary states within the limits of the diagonal Hamiltonian can be shown to split into two independent problems: the investigation of *internal stability* with respect to the perturbation of amplitudes and the phases of pairs already existing and the study of the *external stability* with respect to the generation of new pairs.

The external stability is the easiest to treat. Let us write an equation for the wave pair of perturbations with the help of (5.4.15) $c(\boldsymbol{k},t)$ and $c^*(\boldsymbol{k},t)$ $\propto \exp[\nu(\boldsymbol{k}_1)t]$, analogous to (5.3.2):

$$\left\{\frac{\partial}{\partial t} + \gamma(\boldsymbol{k}_1) - i\left[\omega_{\rm NL}(\boldsymbol{k}_1) - \frac{\omega_{\rm p}}{2}\right]\right\}c(\boldsymbol{k}_1,t) + iP(\boldsymbol{k}_1)c^*(-\boldsymbol{k}_1,t) = 0 \ .$$

The expression for the external instability increment $\nu(\boldsymbol{k}_1)$ takes a form similar to (5.3.3) with the substitution $\gamma(\boldsymbol{k}) \to \gamma(\boldsymbol{k}_1)$, $\omega(\boldsymbol{k}) \to \omega_{\rm NL}(\boldsymbol{k}_1)$ and $hV(\boldsymbol{k}) \to P(\boldsymbol{k}_1)$:

$$\nu(\boldsymbol{k}) = -\gamma(\boldsymbol{k}_1) + \sqrt{|P(\boldsymbol{k}_1)|^2 - [\omega_{\rm NL}(\boldsymbol{k}_1) - \omega_{\rm p}/2]^2} \ .$$

The increment $\nu(\boldsymbol{k}_1)$ maximum with respect to k_1 (under fixed Ω_1) corresponds to $k_1 = k$, satisfying the equation $2\omega_{\rm NL}(\boldsymbol{k}) = \omega_{\rm p}$. This means that the most unstable waves have wave vectors $|\boldsymbol{k}_1| = k$ which are between the surfaces (5.5.2). The maximum $\nu(\Omega_1)$ of the increment $\nu(\boldsymbol{k}_1)$ equals:

$$\nu(\Omega_1) = \max_{\boldsymbol{k}_1} \nu(\boldsymbol{k}_1) = |P(\Omega_1)| - \gamma(\Omega_1) \ .$$

The condition of external stability $\nu(\Omega) < 0$ can therefore be written in the form:

$$|P(\Omega)| \le \gamma(\Omega) \ . \qquad (5.5.3)$$

On the other hand, it follows from (5.5.2) that $|P(\Omega)| \ge \gamma(\Omega)$ for the directions Ω where $n(\kappa,\Omega) \neq 0$. Consequently, for those directions both inequalities are compatible only in the case $|P(\Omega)| = \gamma(\Omega)$, when two surfaces (5.5.2) converge into one

$$2\omega_{\rm NL}(\boldsymbol{k}) = \omega_{\rm p} \ . \qquad (5.5.4)$$

Therefore the condition of external stability under the given angular distribution of the parametric waves completely eliminates the arbitrariness in choosing the surface on which $n(\boldsymbol{k}) \neq 0$. It will be called the *resonance surface* and the stationary state characterized by the external stability will be termed the *ground state*.

The above result has a simple physical meaning. As it is clear from (5.3.2), at the linear stage the waves with zero frequency shift are most closely connected to the pumping. The value $\omega_{\rm NL}(\boldsymbol{k}) - \omega_{\rm p}/2$ is a frequency shift allowing for the nonlinear terms. If two surfaces (5.5.2) on which $n(\boldsymbol{k}) \neq 0$ do not converge, the wave pair whose values fall into a spherical layer between these surfaces prove to be more closely connected with the pumping than the waves already excited. As a consequence, the waves in that spherical layer will increase.

It must be noted that the above-described ambiguity of the solution of stationary equations and the elimination of this arbitrariness (complete or partial) via the condition of stability is specific not only for the S-theory but is a common property of the mean–field approximation in the theory of nonlinear waves. Another method for eliminating that arbitrariness (technically a little bit more complicated but more natural for many people) consists in introducing the thermal noise into the S-theory by substituting the difference $\gamma(\boldsymbol{k})[n(\boldsymbol{k},t)-n_0(\boldsymbol{k})]$ instead of $\gamma(\boldsymbol{k})n(\boldsymbol{k},t)$ into equation (5.4.13) for correlator $n(\boldsymbol{k},t)$. This substitution in the absence of pumping (at $h=0$) ensures that the relaxation $n(\boldsymbol{k},t)$ tends not towards zero, but to the thermodynamic equilibrium at temperature T value of $n_0(\boldsymbol{k})=T/\omega(\boldsymbol{k})$. On the other hand, under $hV(\boldsymbol{k})>\gamma(\boldsymbol{k})$ the term $\gamma(\boldsymbol{k})n_0(\boldsymbol{k})$ maintains the development of all instabilities and, as a consequence, ensures the uniqueness of the stationary solutions (5.4.2). This question will be treated in more detail in Sect. 6.4.

It would be also interesting to find out how the thermal noise $n_0(\boldsymbol{k})$ in the course of development of the parametric instability finally brings about the wave state $n(\boldsymbol{k})\simeq\delta[\omega_{\rm NL}(\boldsymbol{k})-\omega_{\rm p}/2]$ coherent in $\boldsymbol{k}$. This question will be discussed in Sect. 7.5. We shall give a theoretical and computer proof, in particular, of the fact that after a certain time after the pumping is turned on, the distribution of waves in $\boldsymbol{k}$ has a Gaussian form, with the distribution width asymptotically tending to zero as $1/\sqrt{t}$.

Let us now turn to the question of the pair distribution over the resonance surface. Let us introduce the distribution function $n(\Omega)$, "the number" of pairs per unit of the solid angle, defining it in the following way:

$$N=\sum_{\boldsymbol{k}}n(\boldsymbol{k})=\int n(\Omega)d\Omega\,. \tag{5.5.5}$$

The stationary equation for $n(\Omega)$ and $\Psi(\Omega)$ following from (5.4.13) and (5.5.4) will be rewritten as

$$\{P(\Omega)\exp[i\Psi(\Omega)]-i\gamma(\Omega)\}n(\Omega)=0\,,$$
$$P(\Omega)=hV(\Omega)+\int S(\Omega,\Omega_1)n(\Omega_1)\exp[-i\Psi(\Omega_1)]d\Omega_1\,. \tag{5.5.6}$$

These equations do not yet determine unambiguously the distribution $n(\Omega)$ and $\Psi(\Omega)$ since the areas on the surface where $n(\Omega)=0$ can be given arbitrarily. As will be shown in the following section, the condition of external stability with respect to the generation of new pairs on the resonance surface considerably reduces the class of possible solutions and as a result in some cases only one stable distribution remains.

It is useful geometrically to interpret the condition of external stability in the following way. The expressions $\gamma=\gamma(\Omega)$ and $|P|=|P(\Omega)|$ are the equations of a surface in $\boldsymbol{k}$-space. The condition (5.5.3) implies that the surface $|P(\Omega)|$ is wholly contained within the surface $\gamma(\Omega)$ and is tangent to it at the points $\tilde{\Omega}$ where the solution is concentrated, i.e. $n(\tilde{\Omega})\neq 0$. By virtue of the relations $V(\boldsymbol{k})=V(-\boldsymbol{k})$ and $S(\boldsymbol{k},\boldsymbol{k}_1)=S(-\boldsymbol{k},\boldsymbol{k}_1)$ both the surfaces have a symmetry center. The tangency of the surfaces $|P(\Omega)|$ and $\gamma(\Omega)$ can take place either over a discrete set of points or over a continuum, i.e. a line or even part of the surface. In the first case a finite number of monochromatic wave pairs is excited in the ground state; in the second case the distribution of $n(\Omega)$ is continuous. The situation can be intermediate when the surfaces are tangent at an isolated pair of points and or over some line. In this case a monochromatic pair and a continuous background are simultaneously present in the system. Note that the field of applicability of the S-theory in the case when there is a small number of discrete pairs requires special justification including the investigation of their stability within an exact Hamiltonian.

5.5.2 Ground State Under Low Supercriticality

The simplest ground state in the S-theory refers to the case of spherical symmetry which is realized, for instance, in antiferromagnets (when the exceptionally small magnetic dipole interaction is neglected). At $V(\Omega)=V$, $\gamma(\Omega)=\gamma$ and $S(\Omega,\Omega_1)=S$ the equations (5.5.6) have an isotropic solution $n(\Omega)=N/4\pi$, under which

$$N=\sqrt{h^2V^2-\gamma^2}/|S|\,,\quad hV\sin\Psi=\gamma\,. \tag{5.5.7}$$

These equations have the same solution if $S(\Omega,\Omega_1)$ depends only on the angle of $\boldsymbol{k}$ with respect to $\boldsymbol{k}_1$. Then in (5.5.7) $4\pi S=\int S(\Omega,\Omega_1)d\Omega_1$. The case of axial symmetry, for which

$$V(\Omega)=V(\Theta,\varphi)=V(\Theta)\exp(im\varphi)\,,$$
$$S(\Omega,\Omega_1)=S(\Theta,\Theta_1,\varphi-\varphi_1)\,,\quad T(\Omega,\Omega_1)=T(\Theta,\Theta_1,\varphi-\varphi_1)\,.$$

is also of great interest.

By way of example, it should be recalled that under parametric excitation of spin waves in ferromagnets by parallel pumping (see (4.3.21)) $V(\Theta)=V\sin^2\Theta$, $m=2$, under parallel pumping (see (4.3.2)) $V(\Theta)\simeq V\sin 2\Theta$, $m=1$. Let us show that in the case of axial symmetry the dependence on the azimuthal angle φ can be excluded from the basic equations of the S-theory for the class of axially symmetric solutions $n(\Theta,\varphi)=2\pi n(\Theta)$ by substituting new variables. From (5.5.6) for $n(\Omega)$ it is clear that $P(\Omega)\exp[i\Psi(\Omega)]$ must not depend on φ. Then (5.5.6) for $P(\Omega)$ gives

$$\Psi(\Omega)=\Psi(\Theta,\varphi)=\Psi_{\rm inv}(\Theta)+m\varphi\,. \tag{5.5.8}$$

This is the equation for the *invariant phase* of the pairs $\Psi_{\rm inv}(\Theta)$ which is independent of φ. With the help of (5.5.8) Eqs. (5.5.6) are represented as:

$$P_{\rm inv}(\Theta)=hV(\Theta)+\int S_{\rm inv}(\Theta,\Theta_1)n(\Theta_1)\exp[-i\Psi_{\rm inv}(\Theta_1)]\sin\Theta_1 d\Theta_1\,,$$

$$\{P_{\text{inv}}(\Theta)\exp[i\Psi_{\text{inv}}(\Theta)] - i\gamma(\Theta)\}n(\Theta) = 0\,;$$
$$2\pi S_{\text{inv}}(\Theta,\Theta_1) = \int S(\Theta,\Theta_1,\varphi-\varphi_1)\exp[im(\varphi-\varphi_1)]d\Theta_1\,, \quad (5.5.9)$$
$$P_{\text{inv}}(\Theta) = P(\Theta,\varphi)\exp(-im\varphi)\,.$$

If desired, these equations can be additionally simplified by means of the following substitution after which the amplitudes of the wave interaction with the pumping will no longer depend on x ($x = \cos\Theta$):

$$N(x) = n(x)|f(x)|\,,\quad f(x) = V(x)/V_1\,,\quad V_1 = \max\{V(x)\}\,,$$
$$\tilde{P}(x) = P(x)/f(x)\,,\quad \Gamma(x) = \gamma(x)/|f(x)|\,, \quad (5.5.10)$$
$$\tilde{S}(x,x_1) = S(x,x_1)/f(x)f(x_1)\,.$$

For the functions $N(x)$, $\Psi_{\text{inv}}(x)$ the equations of motion retain the form (5.5.9) with the substitution $\gamma \to \Gamma$, $P_{\text{inv}} \to \tilde{P}$. In the expression for P the amplitudes of the wave-pumping interaction prove to be constant:

$$\tilde{P}(x) = hV_1 + \int \tilde{S}(x,x_1)N(x_1)\exp[-i\Psi_{\text{inv}}(x_1)]dx_1\,,$$
$$\{\tilde{P}(x)\exp[i\Psi_{\text{inv}}(x)] - i\Gamma(x)\}N(x) = 0\,. \quad (5.5.11)$$

To solve the problem of the distribution $N(x)$ under small supercriticality (when the pumping amplitude h is only a little above the threshold of the parametric excitation) we shall use the above-mentioned geometric interpretation of the external stability condition (5.5.3). Under very small supercriticality when the wave amplitudes are small, the line $|\tilde{P}(x)|$ differs insignificantly from a straight line $hV_1 = \text{const}$. The curvature of the line $|\tilde{P}(x)|$ is also small. It is clear that the lines $|\tilde{P}(x)|$ and $\Gamma(x)$ are tangent only at the point $x = x_1$, where the function $\Gamma(x)$ is minimum. This means that under small supercriticality the distribution $N(x)$ differs from zero only at $x = x_1$, and in this case for the total number of waves N and phase $\Psi_1 = \Psi_{\text{inv}}(x_1)$ we can readily obtain from (5.5.11)

$$N(x) = N_1\delta(x-x_1)\,,\quad N_1 = \sqrt{h^2V_1^2 - \Gamma_1^2}/|S_{11}|\,,$$
$$hV_1\sin\Psi_1 = \Gamma_1\,,\quad \Gamma_1 = \Gamma(x_1)\,,\quad S_{11} = \tilde{S}(x_1,x_1)\,. \quad (5.5.12)$$

By way of a third example of solving the basic equations of the S-theory under small supercriticality, consider the case when the function $\Gamma(\Omega) = \gamma(\Omega)V_1/|V(\Omega)|$ has a maximum at one pair of points. This is the case in particular with spin waves in ferromagnets parametrically excited by "oblique pumping" (see (4.3.23)) or under the second-order Suhl instability. In the second case the problem has axial symmetry; the amplitude of the spin wave interaction with the pumping is maximum at the pole of resonance surface at $\Theta = 0$ (in this case in (5.5.9) $m = 0$). Clearly, the solution (5.5.12) holds true also in this case, only by S_{11} we should understand $S(\Omega,\Omega_1)$ at $\Theta = \Theta_1 = 0$.

Thus, we consider three cases of distributions of parametric waves with different dimensionality d. In the first case (under spherical symmetry) the function $n(\Omega)$ is nonzero on the hole resonance surface, and, consequently, $d = 2$. In the second case (when the symmetry is axial) $n(\Omega)$ differs from zero on the lines (on the two parallels of the resonance surface $\Theta = \Theta_1$ and $\Theta = \pi - \Theta_1$ or on the equator $\Theta_1 = \pi/2$) and therefore $d = 1$. In the third case (when, for instance, there is no symmetry) $n(\Omega)$ is non-zero at one or several pairs of points and $d = 0$. It is essential that in all these cases expressions (5.5.7, 8) for the total number of parametric waves N_1 and invariant phase Ψ_1 (which is the same for all the excited parametric waves) can be similarly represented

$$N_1 = \int n(\Omega)d\Omega = \frac{\sqrt{h^2V_1^2 - \gamma_1^2}}{|S_{11}|}\,,\quad hV_1\sin\Psi_1 = \gamma_1\,. \quad (5.5.13a)$$

Here V_1 and γ_1 are the amplitudes of the interaction with the pumping and damping of parametric waves in the area where $n(\Omega) \neq 0$ (it shall be recalled that in this area $|V|$ and γ do not depend on Ω),

$$S_{11} = \int S(\Omega,\Omega')n(\Omega)n(\Omega')d\Omega d\Omega'/N_1^2 \quad (5.5.13b)$$

is the average value $S(\Omega,\Omega')$ in the area where $n(\Omega) \neq 0$. Therefore the total number of excited waves does not drastically depend on the dimensionality of their distribution d. This fact is typical not only of the S-theory; it is characteristic of many different versions of mean–field theories. Further the character and magnitude of corrections to the S-theory due to the influence of the thermal bath (Sect. 6.4), scattering of parametric waves by each other (Sect. 10.2), etc. depend cardinally on the dimensionality of the ground state and of the medium. This circumstance is also typical of mean–field theories. One of the best known examples of this is the physics of second-order phase transitions.

Finally we shall account for the physical cause of the limiting of the amplitude of parametric waves in this version of the S-theory. It is very simple. Each wave pair "does not know" anything about the external pumping hV and "feels" only the total pumping $P = hV + SN_1\exp(-i\Psi_1)$. "Trying" to receive as much energy from the external pumping as possible it "turns" its phase Ψ_1 at a right angle from the phase of total pumping $\Psi_{\text{p}} = \arg\{P\}$ (note that this energy flux is proportional to $\sin(\Psi_1 - \Psi_{\text{p}})$. Therefore, the difference between the phase of pair Ψ_1 and the phase of the external pumping $\arg\{hV\}$ (which we assumed to be equal to zero) falls off from the optimum value $\pi/2$, which results in a decreased energy flux from the external pumping $W_+ = \omega_{\text{p}}hV_1N_1\sin\Psi_1$ and, consequently, in the limitation on the total number of parametric waves N_1. To calculate N_1 and Ψ_1, remember the expression for the dissipation rate $W_- = \omega_p \gamma_1 N_1$. Therefore, the condition of energy balance $W_+ = W_-$ has the form $hV_1 = \gamma \sin \Psi_1$. This relation coincides with (5.5.13b) and specifies the dependence of the phases of pairs on the supercriticality. On the other hand, the energy flux from total pumping

$$W_+ = \omega_p |P| N_1 \sin(\Psi_1 - \Psi_p) = \omega_p |P| N_1 \ .$$

This results in another relation $|P| = \gamma_1$ (compare with (5.5.3)). Thus, we have a triangle a three vectors hV_1, $SN_1 \exp(i\Psi_1)$ and

$$P = hV + SN_1 \exp(i\Psi_1) = \gamma_1 \exp(i\Psi_p) \ ,$$

with the two last vectors being perpendicular. By Pythagoras theorem $(SN_1)^2 + \gamma_1^2 = (hV_1)^2$, which coincides with (5.5.13a) and specifies the dependence of N on the supercriticality.

Finally it must be noted that the fundamentally important conclusion of the S-theory about the dependence of the phase of the pair Ψ_1 on the supercriticality (5.5.13b) has been verified by direct experiments measuring the phase of the pair Ψ_1 under parametric excitation of magnons in an antiferromagnet $MnCO_3$ and in a ferromagnet YIG. These experiments were carried out by *Prozorova* and *Smirnov* [5.61] and then by *Melkov* and *Krutsenko* [5.52] in YIG. These will be described in Chap. 9. They corroborate the conclusion of the S-theory about the essential features of the above-threshold behavior of the parametric spin waves.

5.5.3 Threshold of Generation of Second Group of Pairs

Let us proceed to the study of the parametric wave distribution $n(\Omega)$ over the resonance surface under an increasing amplitude of the external field h. Note first that geometrical considerations employed in 5.4.3 can easily be generalized to the case of arbitrary dependence $V(\Omega)$. A general theorem can be proven [5.6] on the fact that under sufficiently small supercriticality, $N(\Omega)$ is nonzero only at those points of the resonance surface where $|V(\Omega)|/\gamma(\Omega)$ is maximum. For spherical symmetry these are all the points of the surface, under axial symmetry these are points of one (at $\Theta_1 = \pi/2$) or two lines. Under lower symmetry it is one or several equivalent pairs of points.

Let us return to the case of axial symmetry and assume for simplicity that the first group of pairs has been generated at the equator: $x_1 = \cos\Theta_1 = 0$. This is so, for instance, when the spin waves are parametrically excited in ferromagnets by parallel pumping (see 4.3.2). Let us show that the distribution of the pairs (5.5.12) concentrated at the equator remains stable with respect to the generation of pairs at other latitudes up to sufficiently great supercriticality. To this end, consider the function $|\tilde{P}(x)|$. From (5.5.10-12) we obtain

$$|\tilde{P}(x)|^2 = N_1^2[S_{11} - \tilde{S}(x, x_1)]^2 + \Gamma_1^2 \ . \tag{5.5.14}$$

It is evident that the state (5.5.12) will remain stable until $|\tilde{P}(x)| < \Gamma(x)$ for all x, except $x = x_1 = 0$. The *second threshold* $p_2 = h_2^2/h_{th}^2$ corresponds to the minimum value of $p = h^2/h_{th}^2$, under which the lines $|\tilde{P}(x)|$ and $\Gamma(x)$ are tangent under a given $x = x_2 \neq x_1$. The value p_2 is determined from the condition $p_2 = \min\{p(x)\}$, where

$$\begin{aligned} p(x) =& 1 + S_{11}^2[\Gamma(x) - \Gamma_1]/\Gamma_1^2[S_{11} - \tilde{S}(x, x_1)]^2 \\ =& 1 + \frac{S_{11}^2[\gamma^2(x)V_1^2 - \gamma_1^2 V^2(x)]}{\gamma_1^2[S_{11}V(x) - S(x, x_1)V_1]^2} \ . \end{aligned} \tag{5.5.15}$$

Let us make the simplest assumption on the functions in this formula $\gamma(x) = \gamma$, $\tilde{S}(x, x_1) = S_{11}$ and take $V(x) = V_1[1 - x^2]$. Then expression for $p(x)$ assumes the form:

$$p(x) = 1 + (2 - x^2)/x^2 \ . \tag{5.5.16}$$

The minimum of this function is attained at $x = x_2 = 1$ and equals 2. Therefore in this simplest case the second pair of waves is generated at the poles under the supercriticality 3 dB ($p_2 = 2$).

Bearing in mind the experimental situation in ferromagnets consider a more meaningful example making the dependence of $S(x, x_1)$ on x more complicated:

$$\tilde{S}(x, x_1) = S_{11}(1 - x^2)(1 + bx^2) \tag{5.5.17}$$

and the same x-dependences for $V(x) = V[1 - x^2]$ and $\gamma(x) = \gamma$. In this case the expression (5.5.15) for $p(x)$ is transformed to:

$$p(x) = 1 + (2 - x^2)/b^2x^2(1 - x^2)^2 \ . \tag{5.5.18}$$

The minimum of this function p is realized at $x = x_2$, where

$$x_2 = (3 - \sqrt{5})/2 \simeq 0.38 \ , \tag{5.5.19}$$

$$p_2 = 1 + (11 + 5\sqrt{5})/2b^2 \simeq 1 + 11.1/b^2 \ . \tag{5.5.20}$$

The large numeric factor before b^{-2} in (5.5.20) is quite remarkable. If, for instance, $b = 1$, then $p_2 = 12$, which makes the stability of the first group of pairs (and accordingly, the applicability of (5.5.12)) equal to 13 dB. If $b < 1$, the region of the first-group stability increases. Qualitatively, we can conclude from the example that in the S-theory the applicability of the simple formulae (5.5.7), (5.5.11) and (5.5.13) describing the state with a single group of pairs and initially obtained at small supercriticalities appears to be anomalously great $p < 10$. This theoretical conclusion has been verified by a number of experiments described in Chap. 9.

Note again that in the first example the generation angle of the second group pairs is given by $\Theta_2 = 0$. In the second example (see (5.5.20a)),

$\Theta_2 \simeq 51°$. Consideration of more complicated cases shows that the angle Θ_2 can be arbitrary and can also be close to Θ_1 (see 6.1.2).

The analytical solution of the basic equations of the S-theory for two or more groups of pairs is very cumbersome and is performed elsewhere. Some results of this kind have been obtained in [5.76]. The problem is considerably simpler in the region of high supercriticalities. Now we shall proceed to this task.

5.5.4 Ground State Under High Supercriticality

The behavior of a parametric wave system with a continuous distribution can be described most conveniently in the limiting case of very strong pumping ($hV \gg \gamma$). In the zeroth approximation in the parameter γ/hV, $P(\Omega) = 0$ whence according to (5.5.6) at real $S(\Omega, \Omega')$ it follows that

$$hV(\Omega) + \int S(\Omega, \Omega')X(\Omega')d\Omega' = 0 \,, \tag{5.5.21a}$$

$$\int S(\Omega, \Omega')Y(\Omega')d\Omega' = 0 \,, \tag{5.5.21b}$$

$$X(\Omega) = n(\Omega)\cos\Psi(\Omega) \,, \quad Y(\Omega) = n(\Omega)\sin\Psi(\Omega) \,. \tag{5.5.21c}$$

The problem is therefore reduced to the solution of first-order linear Fredholm equations. For simplicity assume that in the absence of pumping the medium is isotropic, i.e. $V(\Omega)$ depends only on the polar angle Θ and $S(\Omega, \Omega')$ depends only on the angle α of the direction Ω with the direction Ω'. Then, expanding all the functions in (5.5.21a, b) in terms of Legendre polynomials $P_n(\cos\alpha)$ and making use of the theorem of addition for $P_n(\cos\alpha)$, we obtain after elementary integration over Θ' and Ψ' the formal solution

$$4\pi X(\Theta) = -\sum_{n=0}(2n+1)hV_nP_n(\cos\Theta)/S_n \,,$$

$$S(\cos\alpha) = \sum_{n=0} S_nP_n(\cos\alpha) \,. \tag{5.5.22}$$

Here V_n and S_n are the expansion factors of the functions $V(\Theta)$ and $S(\alpha)$ into the series of Legendre polynomials.

A general consideration of the solutions of (5.5.21) includes three possible cases :

1. Equation (5.5.21b) has nontrivial solutions not orthogonal to $V(\Omega)$. Then (5.5.21a) has no solutions, and the theory proceeding from the diagonal Hamiltonian imposes no limitation on the amplitude. This case will be called *singular*. For the isotropic model the singular case occurs, for example, if $S_n = 0$ and $V_n \neq 0$ for some particular n. It has been shown in the previous section that under $V \neq$ const and not too large a supercriticality a solution in the form of one or more pairs exists also in the singular case. Thus, there is a certain critical amplitude h_c of the pumping and when it is attained the stationary state is disturbed. Above this amplitude the limitation takes place due to weaker nonlinear mechanisms. It follows from (5.5.22) that in most singular case when $S_0 = 0$ and $V =$ const, the amplitude is not limited, under any h, i.e. $h_c = h_{th}$. In less singular cases $h_c > h_{th}$.

Consider in detail the case when $S =$ const and $V(\Theta)$ decreases monotonically from V_1 to V_2. In this case the surface $|P(\Theta)|$ has not more than two maxima. Therefore not more than two pairs can be in a stationary state. Seeking the solution of (5.5.6) we obtain the following expression for the total number of waves:

$$N_1 + N_2 = \frac{\gamma}{|S|}\frac{h(V_1+V_2)}{\sqrt{4\gamma^2 - h^2(V_1-V_2)^2}} \,. \tag{5.5.23}$$

It is clear from the above that $h_c = 2\gamma/(V_1 - V_2)$ at $S =$ const.

2. Equations (5.5.21b) have non-trivial solutions orthogonal to $V(\Omega)$. Then (5.5.21a) has solutions calculated to an accuracy of the solutions of (5.5.21b) and the theory based on the diagonal Hamiltonian gives no unambiguous definition of the stationary state. For the isotropic models the indeterminate case is realized if $S_m = 0$ and $V_m = 0$ for one m at least. Under $V \neq$ const and $h - h_{th} \ll h_{th}$ there is the unique solution (describing the set of similar pairs) as in the singular case. Therefore there are some critical pumping amplitudes h_u such that under $h > h_u$ the solution loses its uniqueness. The indeterminacy in the solution increases with increasing number of harmonics with $S_m = V_m = 0$. If $S_m = V_m = 0$ for any m except $m = 0$ (i.e. $S =$ const and $V =$ const) the most indeterminate case applies. Only the 0-th harmonic of the $n(\Omega)$-distribution is determined by the basic equation of the S-theory (e.g., (5.4.15)) in this case, while all the other harmonics can be chosen at will in the S-theory approximation. For the 0-th harmonic n_0 the following expression can easily be obtained

$$4\pi n_0 = \int n(\Omega)d\Omega = \sqrt{h^2V^2 - \gamma^2}/|S| \,. \tag{5.5.24}$$

coinciding with (5.5.7).

3. Equation (5.5.21b) has no non-trivial solution. This case will be called *regular*. In the regular case all $S_n = 0$. Then (5.5.21a) has the unique solution (5.5.22) which, however, is physically meaningful only under some limitations. First, there must be a fast enough convergence of the sequence V_n/S_n to zero under $n \to \infty$. This condition, however, is not sufficient. Indeed, one must allow for the finiteness of the damping γ in (5.5.21a, b). To this end make use of (5.5.6)

$$-\gamma\sin\Psi(\Omega) = hV(\Omega) + \int S(\Omega, \Omega_1)n(\Omega_1)\cos\Psi(\Omega_1)d\Omega_1 \,,$$

$$-\gamma\cos\Psi(\Omega) = \int S(\Omega,\Omega_1)n(\Omega_1)\sin\Psi(\Omega_1)d\Omega_1 \qquad (5.5.25)$$

for the isotropic model. To obtain the solution (5.5.22) it was assumed that $\gamma/h\nu$ equals zero. To obtain corrections to that solution we linearize (5.5.6) to get for small $\gamma/h\nu$

$$\begin{aligned} -\gamma\delta\Psi(\Omega) &= \int S(\cos\alpha)\delta n(\Omega_1)d\Omega_1 \ , \\ -\gamma &= \int S(\cos\alpha)n(\Omega_1)\delta\Psi(\Omega_1)d\Omega_1 \ , \end{aligned} \qquad (5.5.26)$$

whence

$$\begin{aligned} &\delta\Psi(\Theta) = -\gamma/4\pi S_0 N(\Theta) \ , \quad N(\Theta) = \sum_m N_m P_m(\cos\Theta) \ , \\ &N_m = -\gamma(2m+1)\delta\Psi/4\pi S_m \ . \end{aligned} \qquad (5.5.27)$$

Therefore, for the region of applicability of (5.5.22) to exist the distribution $n(\Omega)$ must never become zero. Recall that under V = const an accurate solution of equations with damping can be obtained for any pumping amplitude. This solution has the form (5.5.24). In the regular case, moreover, N = const, i.e. $n(\Omega) = N/4\pi$.

Qualitatively we can conclude from the above that the distribution of pairs $n(\Omega)$ under large amplitude h is highly sensitive to the fine structure of the functions $V(\Omega)$ and $S(\Omega,\Omega_1)$: in some cases continuous distributions of pairs over the resonance surface are established, in other cases the limitations are violated and the picture becomes essentially nonstationary.

5.5.5 Nonlinear Susceptibilities of Parametric Waves

One of the methods most commonly used for the experimental study of the parametric excitation of waves is based on measuring the energy flux W_+ from the pumping into the system of parametric waves. In statistical physics [5.77] the value W_+ is expressed in terms of the generalized susceptibility $\chi(\omega)$:

$$W_+ = \frac{1}{2}\text{Im}\{\chi(\omega)\}|f(\omega)|^2 \ . \qquad (5.5.28)$$

In the general case $f(\omega)$ is the ω-Fourier transform of the generalized force $f(t)$, $\chi(\omega)$ is the proportionality coefficient of the generalized coordinate $x(\omega)$ to $f(\omega)$:

$$x(\omega) = \chi(\omega)f(\omega) \ . \qquad (5.5.29)$$

The generalized coordinate $x(t)$ is selected so that the interaction energy of the external force with the system has the form

$$\mathcal{H}_{\text{int}} = -x(t)f(t) \ . \qquad (5.5.30)$$

For magnetodielectrics $f(t)$ is the magnetic field of the pumping $h(t)$, x is the magnetization m, the energy (5.5.30) is the Zeeman energy. Consequently, in magnetodielectrics χ is the magnetic susceptibility. Within the general theory of the parametric excitation of waves (not specifying their nature) proceeding from the "standard" Hamiltonian of the pumping (5.2.1) $\text{Im}\{\chi(\omega)\}$ can be calculated most readily if we compare (5.5.28) and (5.3.4) for W_+:

$$\chi'' = \text{Im}\{\chi(\omega_p)\} = -\frac{2}{h}\sum_{\boldsymbol{k}}\text{Im}\{V^*(\boldsymbol{k})\sigma(\boldsymbol{k})\} \ . \qquad (5.5.31a)$$

A similar expression can also be obtained for the real part of this susceptibility:

$$\chi' = \text{Re}\ \{\chi(\omega_p)\} = -\frac{2}{h}\sum_{\boldsymbol{k}}\text{Re}\ \{V^*(\boldsymbol{k})\sigma(\boldsymbol{k})\} \ . \qquad (5.5.31b)$$

These formulae can readily be combined

$$\chi = -\frac{2}{h}\sum_{\boldsymbol{k}} V^*(\boldsymbol{k})\sigma(\boldsymbol{k}) \ . \qquad (5.5.32)$$

By means of the angular distribution (5.5.5) the latter formula can be represented in a more convenient form:

$$\chi = -\frac{2}{h}\int V^*(\Omega)n(\Omega)\exp[-i\Psi(\Omega)]d\Omega \ . \qquad (5.5.33)$$

The behavior of susceptibilities χ' and χ'' above the threshold is largely determined by the mechanism limiting the amplitude. Thus, it follows for the mechanism of nonlinear damping from (5.5.33) and (5.3.8) that

$$\chi' = 0 \ , \quad \chi'' = 2V^2(h-h_{\text{th}})/\eta h \ , \qquad (5.5.34)$$

and for the phase limiting mechanism in the isotropic case it follows from (5.5.7):

$$\chi' = 2V^2(h^2-h_{\text{th}}^2)/Sh^2 \ , \quad \chi'' = 2V^2 h_{\text{th}}\sqrt{h^2-h_{\text{th}}}/|S|h^2 \ . \qquad (5.5.35)$$

Obviously, these formulae hold true also in the absence of spherical symmetry in a narrower range of supercriticalities below the generation threshold of the second group of pairs: ($h_{\text{th}} < h < h_2$). Equations (5.5.34, 35) show that the fundamental difference of the dissipative and phase mechanisms manifests itself in the behavior of the value χ'; $\chi' = 0$ for the dissipative and $\chi' \simeq \chi''$ for the phase mechanisms. Experimental results to be discussed in detail in Chap. 9 indicate that the real susceptibility χ' differs from zero and may be of the order of or even greater than χ''. Those facts also provide evidence of the phase mechanism for limiting of the parametric wave amplitudes.