Wigner formulation of optical processing with light of arbitrary coherence

Avi Pe'er, Dayong Wang, Adolf W. Lohmann, and Asher A. Friesem

A unified mathematical formulation for designing and analyzing even the most general optical processor is presented. It exploits the Wigner distribution function to characterize the illumination, the input, the inherent filter, and the output results. To characterize the propagation of the light through the optical processor setup, we exploit the Wigner matrix formalism, which is appealing because it allows simple geometric analysis. The Wigner distribution function was extended to include illumination of arbitrary coherence so that processors using either coherent light or partially coherent light can be designed and analyzed with the same Wigner formalism. The basic principles, design, and analysis of the imaging and Fourier-transform operations and use of the Wigner formalism to evaluate the performance and tolerances of optical processors are presented.

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1. Introduction

In general, the characterization of light depends on the degree of coherence. When light is coherent, it is characterized by the complex amplitude of the field as a function of space. When light is partially coherent, it is characterized by the two-point correlation of the complex amplitude of the field, namely, the mutual coherence function $\Gamma(x_1, x_2)$. When light is incoherent, no correlation exists between two different points, so it is characterized by the local light intensity. Specifically, when light is not coherent, it needs to be characterized by a second-order function of the field (two-point correlation or intensity) instead of the field itself. The degree of coherence of light is particularly important in optical processing, in which it affects how the filtering operation will be performed, the signal-to-noise ratio of the output, and the various tolerances on the positioning of the optical elements in an optical processor.

In this paper we propose use of the Wigner distribution function (WDF) for the mathematical characterization of light. The WDF is a second-order transformation of a wave field, which can be considered as a wave generalization of a phase-space probability density. The WDF allows analysis of the wave field in both space and spatial frequency (or time and temporal frequency) simultaneously. Wigner devised the WDF in 1932 in the context of quantum mechanics,[1] but it is appropriate in any research field in which nonstationary signal analysis is required. Recently the WDF has been used widely in many areas of physics and digital signal processing.[2–5]

The WDF was introduced into the field of optics in the late 1970's mostly by Bastiaans.[6–8] In optics, as in other fields, use of the WDF seems at first somewhat awkward and nonintuitive compared with a description in terms of the complex amplitude of the optical field. Yet the WDF has some appealing and convenient properties that make it a useful tool in optics and especially in optical processing. One important property is that free-space propagation and passage through a lens can be viewed, in the paraxial approximation, as simple unitary linear coordinate transformations applied on the position and frequency coordinates of the WDF.[6,9] Because of this property, the propagation of light through a rotationally symmetric uniaxial optical setup can be characterized by a unitary 2×2 coordinate transformation, which is essentially equivalent to the well-known canonical *ABCD* transform matrix.[9] One can calculate this transformation matrix by successively

The authors are with the Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel. D. Wang is also with the Department of Applied Physics, Beijing Polytechnic University, Beijing 100022, China. A. W. Lohmann is also with the Laboratory for Nachrichtentechnik, Erlangen-Nurenberg University, Erlangen 91058, Germany. The e-mail address for A. Pe'er is feavip@wisemail.weizmann.ac.il.

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multiplying the basic matrices of free space and lenses in the correct order. A second important property is that the WDF can be easily generalized to partially coherent fields, without affecting the mathematical matrix formalism of propagation through the optical setup.[6–8] A third important property is that the geometry of the coordinate transformations in two dimensions is relatively easy, providing insight and intuition for analyzing the performance and operational tolerances of the optical processor from simple geometric considerations.

In the following, we review briefly the formalism of Wigner matrices to determine the propagation of a light field through a first-order optical setup and use it to design an optical processor.[6–11] Then we apply the generalization of the WDF to partially coherent light to discuss the implications of the degree of coherence of the light on the WDF. Then we apply the Wigner formalism to analyze an optical processor, in which the illumination has an arbitrary degree of coherence, and perform a tolerance evaluation.

2. Propagation of the Wigner Distribution Function through a First-Order Optical Setup

We begin with the usual definition of the WDF $W(x, \nu)$ of some optical-field amplitude $U(x)$ given by

$$W(x, \nu) \equiv \int_{-\infty}^{\infty} \mathrm{d}x' U\left(x - \frac{x'}{2}\right) U^*\left(x + \frac{x'}{2}\right) \exp(2\pi i \nu x'), \tag{1}$$

where x, x' denote spatial coordinates and ν denotes spatial-frequency coordinates. The mathematical properties of this definition of the WDF have been discussed extensively.[1–8] Hence we merely note that the WDF is always real, but not necessarily positive, so it cannot be viewed as a probability density function in a narrow sense. Yet once it is smoothed by convolution with proper functions, the negative values disappear, and then the resulting WDF can be considered as a phase-space density distribution. The important relations between the rather abstract WDF to the measurable intensity $I(x)$ and power spectrum $P(\nu)$ are given by

$$I(x) = |U(x)|^2 = \int_{-\infty}^{\infty} \mathrm{d}\nu W(x, \nu),$$

$$P(\nu) = |\tilde{U}(\nu)|^2 = \int_{-\infty}^{\infty} \mathrm{d}x W(x, \nu), \tag{2}$$

where $\tilde{U}$ denotes the Fourier transform of the field amplitude U.

Earlier investigations on the propagation of the WDF of coherent light through first-order optical systems[8–10] yielded the following coordinate transformation matrices representing free-space propagation and a lens (multiplication by a quadratic phase):

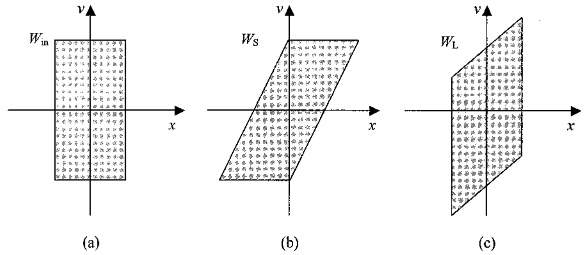

Fig. 1. Effect of lens and free space on WDF: (a) input WDF, (b) WDF x sheared after free-space propagation, and (c) WDF ν sheared after lens action.

$$S(z) = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}, \qquad L(f) = \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix}, \tag{3}$$

where the matrix S denotes the x shear imposed on the WDF by free-space propagation over a distance z and the matrix L denotes the ν shear imposed by a lens with focal length f. Both f and z may depend on the wavelength λ. The impact of free-space propagation and passage through a lens on the WDF is illustrated in Fig. 1. Figure 1(a) shows the initial WDF, Fig. 1(b) shows the shearing as a result of free-space propagation, and Fig. 1(c) shows the shearing as a result of passage through a lens.

The operation of a complete optical system can be represented by one total matrix that is obtained by successive multiplication, in the correct order, of the matrices of Eqs. (3). Although the matrices in the Wigner formulation are similar to the well-known *ABCD* matrices, they also contain the wavelength dependence of the transformations, so they are more suitable to use in the design and the analysis of optical processors that operate with polychromatic (i.e., temporally incoherent) light.

3. Wigner Matrices as a Tool for Analyzing Optical Imaging and Fourier Transformation

To use the Wigner formulation to design and analyze an optical setup it is necessary to identify the specific influence of each element in the Wigner matrix on the resulting amplitude distribution at the output plane. We begin with the Wigner formulation[9–11] that includes the basic Wigner matrix A with elements a, b, c, and d written as

$$\begin{bmatrix} x_{\text{out}} \\ \nu_{\text{out}} \end{bmatrix} = A \begin{bmatrix} x_{\text{in}} \\ \nu_{\text{in}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ \nu_{\text{in}} \end{bmatrix}, \tag{4}$$

where $(x_{\text{out}}, \nu_{\text{out}})$ are the phase-space coordinates at the output plane and $(x_{\text{in}}, \nu_{\text{in}})$ are the coordinates at the input plane. Note that elements b and c mix space and spatial frequency, thus having dimensions of $[\text{length}]^2$ and $[\text{length}]^{-2}$, respectively. We now consider optical imaging and optical Fourier transformation, the two most important operations in optical processing. Specifically, we determine how

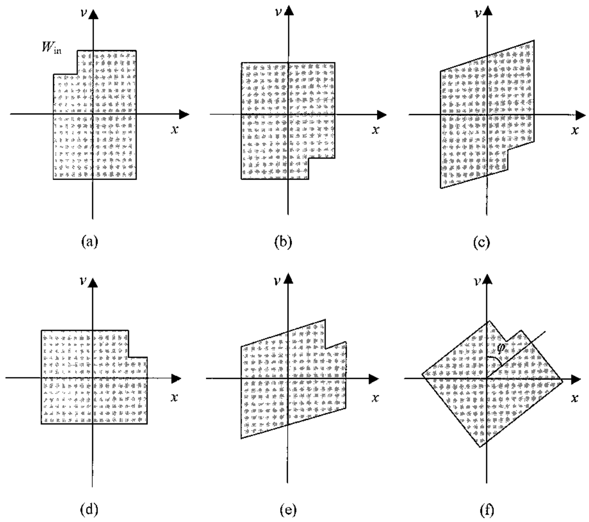

Fig. 2. Effect of imaging and Fourier transformation on the WDF. (a) Input WDF, (b) perfect imaging: a π-rad rotation, (c) imperfect imaging: a π-rad rotation and frequency shear, (d) perfect Fourier: a π/2-rad rotation, (e) imperfect Fourier: a π/2-rad rotation and frequency shear, and (f) perfect fractional Fourier transform of order p: a rotation by an angle $\varphi = p\pi/2$.

imaging and Fourier transformation affect the WDF. The results are shown in Fig. 2.

Imaging corresponds to the case of $b = 0$, so the space coordinate x at the output plane is just a scaled version of the space coordinate at the input plane ($x_{\text{out}} = ax_{\text{in}}$). Thus, according to Eqs. (2), the intensity distribution at the output is a magnified version of the intensity distribution at the input. In the Wigner formulation we obtain

$$\begin{bmatrix} x_{\text{out}} \\ \nu_{\text{out}} \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 1/a \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ \nu_{\text{in}} \end{bmatrix}. \tag{5}$$

In this case a is the magnification and c represents a ν shear, which, according to Eqs. (3), is equivalent to a quadratic phase at the output plane. When also $c = 0$, a perfect image of the input object without any quadratic phase is obtained at the output. The effect of imaging on the WDF is illustrated in Figs. 2(b) and 2(c). Typically, it is a 180-deg (or 360-deg) rotation followed by a ν shear.

With polychromatic illumination, ideal imaging will be obtained under the conditions of $b(\lambda) = 0$ and $a(\lambda) = a_0 = \text{const}$ independent of wavelength. In practice, however, such conditions cannot be achieved exactly because of chromatic aberrations. Thus both $b(\lambda)$ and $a(\lambda)$ have some residual dependence on wavelength whereby $b(\lambda) \neq 0$ can be regarded as a defocus error and $a(\lambda) \neq a_0$ as a magnification error. With such errors the image will be blurred with an effective resolution δx_{out} given by

$$\delta x_{\text{out}} = \overline{[a(\lambda) - a_0]} x_{\text{in}} + \bar{b} \Delta \nu_{\text{in}}, \tag{6}$$

where $\Delta\nu$ is the spatial-frequency bandwidth of the object and $\bar{b}$, $\overline{[a(\lambda) - a_0]}$ are some effective values of $b(\lambda)$ and $a(\lambda) - a_0$, respectively. The effective values can be determined, for example, as the maximal absolute values over the spectral range of interest, or some weighted mean-square average of $b(\lambda)$, $a(\lambda) - a_0$ over this spectral range. The effective resolution has two parts. The first part that includes $a(\lambda) - a_0$ is space variant and depends on the coordinate x. It causes nonuniform blurring of the image because the magnification varies with wavelength. This blurring degrades the edges of the image at the output plane but does not affect the image quality close to the optical axis. The second part that includes $b(\lambda)$ is space invariant, independent of the coordinate x. It causes uniform blurring over the image at the output plane because of a wavelength-dependent focus error. Therefore, to optimize an imaging setup, one should try to minimize both $\bar{b}$ and $\overline{[a(\lambda) - a_0]}$ over the spectral range of interest. This can be achieved when certain parameters, such as distances, focal powers, and dispersions, of the components in the imaging setup are controlled.

Optical Fourier transformation corresponds to the case $a = 0$, so the space coordinate at the output plane is equal to the spatial-frequency coordinate at the input plane ($x_{\text{out}} = b\nu_{\text{in}}$), and b is the scaling factor relating the two coordinates. Thus, according to Eqs. (2), the intensity distribution at the output will now be a scaled version of the power spectrum of the input object. In the Wigner formulation we obtain

$$\begin{bmatrix} x_{\text{out}} \\ \nu_{\text{out}} \end{bmatrix} = \begin{bmatrix} 0 & b \\ -1/b & d \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ \nu_{\text{in}} \end{bmatrix}. \tag{7}$$

If also $d = 0$ we obtain a perfect Fourier transform; otherwise, d denotes a quadratic phase at the Fourier plane. The effect of Fourier transformation on the WDF is illustrated in Figs. 2(d) and 2(e). Typically, it is a 90-deg (or 270-deg) rotation followed by a ν shear (given by the element d in the Wigner matrix).

Just as in imaging, the Fourier transformation can be considered as a rotation in Wigner space. Because a Fourier transformation is only a 90-deg rotation, whereas imaging is a 180-deg rotation, the roles of the matrix elements a and b are interchanged. Thus, with polychromatic light, an ideal Fourier transformation will be obtained under the conditions of $b(\lambda) = b_0 = \text{const}$ and $a(\lambda) = 0$, both independent of wavelength. In practice, however, such conditions cannot be achieved exactly. So here, similar to imaging, the effective resolution δx_{out} at the Fourier-transform plane will be

$$\delta x_{\text{out}} = \bar{a} \Delta x_{\text{in}} + \overline{[b(\lambda) - b_0]} \nu_{\text{in}}, \tag{8}$$

where Δx_{in} is the input object size and $\bar{a}$, $\overline{[b(\lambda) - b_0]}$ are again some effective values. Here again the effective resolution has two parts, but now the residual $a(\lambda)$ is responsible for uniform blurring over the Fourier-transform plane, whereas the residual $b(\lambda) - b_0$ causes nonuniform blurring because the scale of the Fourier transform depends on wavelength. It should be noted that with polychromatic illumination it is significantly more complicated to obtain high-quality Fourier transformation compared with imaging, requiring a combination of several diffractive and refractive lenses.[11–18]

To generalize the Wigner formulation to include fractional Fourier transformations (FRT's)[19–21] is fairly simple because a perfect FRT of fraction p is equivalent to a rotation of the WDF by an angle $\varphi = p\pi/2$, to yield

$$\begin{bmatrix} x_{\text{out}} \\ \nu_{\text{out}} \end{bmatrix} = \begin{bmatrix} a_0 \cos\varphi & b_0 \sin\varphi \\ -(1/b_0)\sin\varphi & (1/a_0)\cos\varphi \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ \nu_{\text{in}} \end{bmatrix}, \tag{9}$$

where a_0 and b_0 are scaling factors. A perfect FRT is illustrated in Fig. 2(f). An imperfect FRT has an additional quadratic phase at the output and can be considered as a perfect FRT matrix multiplied by an arbitrary lens matrix. With polychromatic illumination, it is necessary to ensure that the fraction p and scaling factors a_0 and b_0 are independent of wavelength. Analysis of errors is similar to that of imaging and Fourier transformation.

4. Wigner Distribution Function with Partially Coherent Light

The basic WDF definition can be generalized to include partially coherent light.[6] This generalization is based on the fact that the WDF is the Fourier transform of $U(x - x'/2)U^*(x + x'/2)$ with respect to x'. This term is not constant in time if the light is not coherent; hence a conceivable generalization for the case of partially coherent light would be to replace this term by its ensemble average, which is equal to a time average for temporally stationary signals. This leads to the following definition for the generalized WDF with partially coherent light[6]:

$$\begin{aligned} W_{\text{pc}}(x, \nu) &\equiv \langle W_c(x, \nu)\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}x'\Gamma\left(x - \frac{x'}{2}, x + \frac{x'}{2}\right)\exp(2\pi i\nu x'), \end{aligned} \tag{10}$$

where W_{pc} is the WDF with partially coherent light, W_{c} is the WDF with coherent light as defined in Eq. (1), the brackets $\langle\ \rangle$ denote ensemble averaging, and Γ is the two-point correlation (mutual coherence function) $\Gamma(x - x'/2, x + x'/2) = \langle U(x - x'/2)U^*(x + x'/2)\rangle$.

In general, the incident illumination is modulated by the object, so it is important to understand how such modulation affects the WDF. If we denote the light field immediately before the object as $U_-(x)$, the field immediately after the object as $U_+(x)$, and the object as $f(x)$, then $W_+(x, \nu)$, the WDF of the light immediately after the object, will be

$$\begin{aligned} W_+(x, \nu) &= \left\langle \int_{-\infty}^{\infty} \mathrm{d}x' U_+\left(x - \frac{x'}{2}\right)U_+^*\left(x + \frac{x'}{2}\right) \times \exp(2\pi i\nu x')\right\rangle \\ &= \left\langle \int_{-\infty}^{\infty} \mathrm{d}x' \left[U_-\left(x - \frac{x'}{2}\right)U_-^*\left(x + \frac{x'}{2}\right)\right] \times \left[f\left(x - \frac{x'}{2}\right)f^*\left(x + \frac{x'}{2}\right)\right]\exp(2\pi i\nu x')\right\rangle. \end{aligned} \tag{11}$$

Usually the object properties are deterministic and can be taken out of the ensemble average, but even if not, as long as they are statistically independent of the light field, then

$$\begin{aligned} W_+(x, \nu) = \int_{-\infty}^{\infty} \mathrm{d}x' \left\langle U_-\left(x - \frac{x'}{2}\right)U_-^*\left(x + \frac{x'}{2}\right)\right\rangle \\ \times \left\langle f\left(x - \frac{x'}{2}\right)f^*\left(x + \frac{x'}{2}\right)\right\rangle \exp(2\pi i\nu x'). \end{aligned} \tag{12}$$

Equation (12) is just the Fourier transform with respect to x' of a multiplication of the two-point correlation of the field with the two-point correlation of the object. Hence the result is a convolution in frequency of the WDF of the incident light, and the WDF of the object itself, given by

$$\begin{aligned} W_+(x, \nu) &= \int_{-\infty}^{\infty} \mathrm{d}\nu' W_-(x, \nu')W_f(x, \nu - \nu') \\ &= W_-(x, \nu) \overset{*}{_{(\nu)}} W_f(x, \nu), \end{aligned} \tag{13}$$

where W_- is the WDF of the incident light, W_f is the WDF of the object, and $\overset{*}{_{(\nu)}}$ denotes convolution in frequency but not in space.

Let us now consider how the degree of spatial coherence of the illumination influences the WDF of an object. When light is partially coherent, the mutual coherence function $\Gamma(x - x'/2, x + x'/2)$ has a finite width with respect to x', so the WDF of the light field W_-, which is the Fourier transform of Γ with respect to x', has a nonzero width with respect to the frequency ν. The convolution of W_- with the object WDF W_f will smear out some frequency details. The width of W_- will determine the resolution in frequency. In the extreme case of spatially incoherent illumination, the mutual coherence function tends to a Dirac delta function, as

$$\Gamma(x - x'/2, x + x'/2) = I(x)\delta(x'). \tag{14}$$

Accordingly, W_- is infinitely wide in frequency, and all frequency information in W_f is lost. We obtain a one-dimensional WDF:

$$W_+(x, \nu) = I_-(x)|f(x)|^2 = I_+(x), \tag{15}$$

where I_- and I_+ are the intensity distributions of the light in front of the object and after it, respectively.

It is evident then that the degree of spatial coherence of the light sets a fundamental limitation on the resolution in frequency. This effect is illustrated in Fig. 3 in which the WDF of a double rectangular

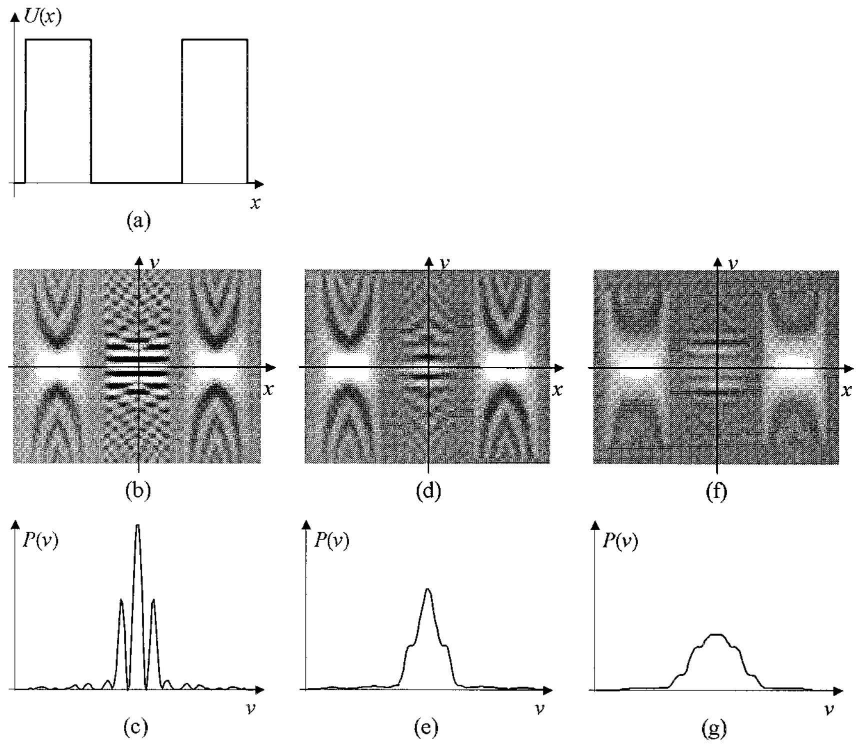

Fig. 3. Effect of degree of coherence on WDF and Fourier plane intensity distribution. (a) The double rectangular input function $U(x)$, (b) the coherent WDF and (c) the corresponding Fourier plane intensity $P(\nu)$, (d) the WDF with a high degree of coherence and (e) the corresponding Fourier plane intensity, and (f) WDF with a low degree of coherence and (g) the corresponding Fourier plane intensity.

function with illumination of various degrees of coherence is presented. The WDF of a double rectangular function is composed of three parts. Two are the WDF's of each single rectangle, and the third is an interference term located between them, which appears as a mixed term of the Wigner function, which is of second order in the complex amplitude. Figure 3(a) shows the double rectangle input function; Fig. 3(b) is the WDF with coherent illumination, and Fig. 3(c) is the corresponding Fourier plane intensity distribution; Fig. 3(d) is the WDF with illumination having a high degree of coherence, and Fig. 3(e) is the corresponding Fourier plane intensity; Fig. 3(f) is the WDF with illumination having a low degree of coherence, and Fig. 3(g) is the corresponding Fourier plane intensity. We obtained the results for the partially coherent WDF's by local averaging of the coherent WDF over a narrow and wide range in frequency. As evident from the results, the interference term is highly oscillatory in the frequency dimension. Thus it is more susceptible to the degradation in coherence.

5. Performance Analysis of an Optical Processor

In optical processing one typically correlates an input object with a reference pattern that exists indirectly as a holographic filter. The filter modulates the light at some specific plane within the optical processor. In this section we develop an expression for the WDF at the output plane of the processor as a function of the WDF of the input and of the holographic filter. In most cases, the correlation output plane is optically conjugated to the input plane, i.e., in the absence of a filter the input would be imaged onto this plane. Thus the Wigner matrix T of the entire processor is an imaging matrix given by

$$T = \begin{bmatrix} m & 0 \\ c & 1/m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/m & 1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & 1/m \end{bmatrix} = LM, \quad (16)$$

where c denotes the quadratic phase at the output plane and m is the magnification. This result is conveniently decomposed into the multiplication of two matrices L and M, such that L is a lens matrix representing the quadratic phase and M is a pure magnification matrix. When a filter is inserted into the processor, the WDF of the light is modified; thus it is reasonable to decompose the matrix T into two matrices as

$$T = BA, \quad (17)$$

where A and B represent the propagation matrices up to the filter and from the filter on, respectively. Using Eqs. (16) and (17), we obtain the relation between the A and B matrices as

$$B = LMA^{-1}. \quad (18)$$

We proceed to obtain an expression for the WDF at the output plane as a function of the WDF at the input plane and the WDF of the filter. According to

Eq. (13), the WDF immediately after the filter W_+ is given by

$$W_+\begin{bmatrix} x \\ \nu \end{bmatrix} = \int_{-\infty}^{\infty} d\nu' W_-\begin{bmatrix} x \\ \nu' \end{bmatrix} W_F\begin{bmatrix} x \\ \nu - \nu' \end{bmatrix}, \tag{19}$$

where W_- is the WDF immediately before the filter and W_F is the WDF of the filter transmission. Yet W_- is a propagation forward of the WDF at the input W_{in} according to A, and W_+ is a propagation backward of the WDF at the output W_{out} according to $B = LMA^{-1}$. Thus we obtain

$$W_{out}\left[B\begin{pmatrix} x \\ \nu \end{pmatrix}\right] = \int_{-\infty}^{\infty} d\nu' W_{in}\left[A^{-1}\begin{pmatrix} x \\ \nu' \end{pmatrix}\right] W_F\begin{bmatrix} x \\ \nu - \nu' \end{bmatrix}. \tag{20}$$

Because the quadratic phase at the output plane does not affect the intensity of the light and because the magnification rescales only the output results, let us ignore these factors for now and assume perfect one-to-one imagery; i.e., $B = A^{-1}$. For a general 2×2 unimodular matrix A, such as in Eq. (4), we obtain

$$W_{out}\begin{bmatrix} dx - b\nu \\ -cx + a\nu \end{bmatrix} = \int_{-\infty}^{\infty} d\nu' W_{in}\begin{bmatrix} dx - b\nu' \\ -cx + a\nu' \end{bmatrix} W_F\begin{bmatrix} x \\ \nu - \nu' \end{bmatrix}. \tag{21}$$

To make Eq. (21) more tractable, assuming $b \neq 0$, we change the variables according to

$$\begin{bmatrix} x_{out} \\ \nu_{out} \end{bmatrix} = A^{-1}\begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} dx - b\nu \\ -cx + a\nu \end{bmatrix},$$
$$x_{in} = (dx - b\nu') - x_{out}, \tag{22}$$

leading to

$$W_{out}\begin{bmatrix} x_{out} \\ \nu_{out} \end{bmatrix} = -\frac{1}{b}\int_{-\infty}^{\infty} dx_{in} W_{in}\begin{bmatrix} x_{in} - x_{out} \\ \frac{a}{b}x_{in} + \nu_{out} \end{bmatrix} W_F\begin{bmatrix} ax_{out} + b\nu_{out} \\ \frac{x_{in}}{b} \end{bmatrix}. \tag{23}$$

Although Eqs. (21) and (23) may seem not so friendly, they are easily interpreted geometrically. The matrix A corresponds usually to a simple geometric transformation of coordinates, such as rotation. The operation of processing can be conceived as rotating the WDF of the input, convolving it in frequency with the filter WDF and then rotating it back. We can see from Eq. (23) that, even though the unitary transformation A may in general have three free parameters, only two parameters a and b influence the output. The reason for this additional freedom is that the operation of convolution in frequency is invariant to translations in frequency; thus the result is insensitive to the value of the parameter c, which corresponds to a space-dependent shift of the frequency coordinate.

Equations (20)–(23) contain all the information required to analyze the performance of an optical processor with arbitrary coherence of the illumination. These equations allow calculation of the WDF at the output plane given the input object, the filter, and the illumination mutual coherence function. Let us now illustrate how Eq. (23) can be exploited to analyze two optical processors—one using spatially coherent illumination and the other using spatially incoherent illumination. In the case of optical processing with spatially coherent light, the filter should be placed at a Fourier plane of the input, so the matrix A must be a Fourier-transform matrix; i.e., $a = 0$. This results in

$$W_{out}\begin{bmatrix} x_{out} \\ \nu_{out} \end{bmatrix} = \frac{1}{b}\int_{-\infty}^{\infty} dx_{in} W_{in}\begin{bmatrix} x_{in} - x_{out} \\ \nu_{out} \end{bmatrix} W_F\begin{bmatrix} b\nu_{out} \\ \frac{x_{in}}{b} \end{bmatrix}. \tag{24}$$

Equation (24) is exactly the expected result for a VanderLugt correlator because the space coordinate of the input is correlated with the frequency coordinate of the filter. In the case of optical processing with spatially incoherent light, according to Eq. (15) the WDF at the input plane is one-dimensional $W_{in}(x, \nu) = I_{in}(x)$. Thus the output intensity I_{out} is

$$I_{out}(x_{out}) = \int_{-\infty}^{\infty} d\nu_{out} W_{out}\begin{bmatrix} x_{out} \\ \nu_{out} \end{bmatrix}$$
$$= -\frac{1}{b}\int_{-\infty}^{\infty} dx_{in} I_{in}(x_{out} - x_{in}) \tilde{I}_F\left(\frac{x_{in}}{b}\right), \tag{25}$$

where $\tilde{I}_F$ is the power spectrum of the filter. Again, this is the expected intensity correlation obtained with spatially incoherent light. It should be noted that in this case the only parameter of the transformation A affecting the output intensity is b. The value of a has no effect on the resulting output intensity because the input WDF is only one dimensional in this case.

6. Tolerance Analysis

We now use the Wigner formulation to analyze the tolerances of an optical processor for errors in the longitudinal and transverse position of the filter and the dependence of these tolerances on the degree of coherence of the illumination. Because this is mostly a qualitative analysis, numerical factors of the order of one are ignored throughout this section. An error in the longitudinal position of the filter will cause an error in the matrix A. In general, the filter should be positioned at the Fourier plane of the input; i.e., $a = 0$. If a longitudinal position error occurs, then a will not be identically zero. This will be manifested by a defocus error at the filter plane, resulting in resolution degradation. In addition, a longitudinal misalignment may cause a scale mismatch between the filter and the input, which will degrade the filter operation for the higher frequencies of the input. Following the procedure that led to Eq. (8), we find the resolution δx_F at the filter plane to be

$$\delta x_F = |a|\Delta x_{in}, \tag{26}$$

where Δx_{in} is the input object size. Yet the resolution at the filter plane defines also the tolerance for transverse misalignment, so we see that an error in the longitudinal position of the filter degrades the resolution, as well as increases the tolerance for transverse misalignments.

As determined in Section 4, when the illumination is partially coherent, the frequency resolution at the input plane $\delta\nu_{in}$ is equal to the frequency width of the WDF of the illumination, as dictated by the degree of coherence. The spatial resolution in the exact Fourier plane δx_F, which is also the tolerance for error in the lateral position of the filter, will be

$$\delta x_F = |b|\delta\nu_{in}. \tag{27}$$

We can also deduce from this fundamental resolution the tolerance for longitudinal position errors. Specifically, because the performance of the processor remains essentially the same as long as the resolution degradation that is due to longitudinal misalignment is small compared with the fundamental resolution δx_F, then

$$|a|\Delta x_{in} < |b|\delta\nu_{in}. \tag{28}$$

Assuming $b \approx b_0$, where b_0 is the scaling factor used to design the filter, we obtain the following limit on the defocus error a as

$$|a| < \frac{|b_0|\delta\nu_{in}}{\Delta x_{in}}. \tag{29}$$

We can see that the defocus tolerance depends on the degree of coherence and on the lateral size of the input object. The defocus tolerance can be translated to a misalignment tolerance for each specific processor. In the extreme case of totally incoherent illumination, $\delta\nu \to \infty$, and the processor is not affected by either longitudinal or transverse misalignment.

The tolerance for error in the scaling factor b can be determined similarly. At the filter plane, the WDF of the light is convolved with the WDF of the filter. For this convolution to yield a good peak, the scale of these two WDF's in frequency should match, i.e., the largest scale mismatch error should be small compared with the frequency resolution at the filter plane. The frequency resolution at the filter plane $\delta\nu_F$ is equivalent to the spatial resolution at the input plane according to

$$\delta\nu_F = \frac{\delta x_{in}}{|b|}. \tag{30}$$

Thus the largest mismatch acceptable error at the filter plane $E\nu_F$ becomes

$$E\nu_F = \left|\frac{\Delta x_{in}}{b_0} - \frac{\Delta x_{in}}{b}\right| < \frac{\delta x_{in}}{|b|}. \tag{31}$$

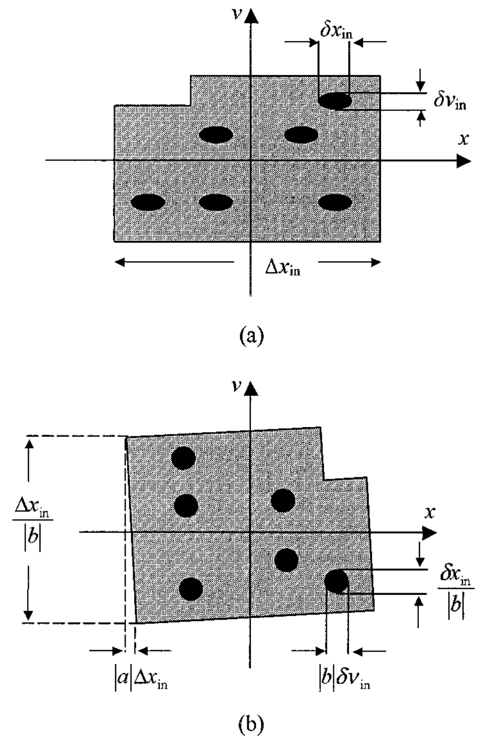

Fig. 4. Geometric analysis of tolerances: (a) schematic WDF at the input plane and (b) schematic WDF at the filter plane.

Now the relative scaling mismatch error ϵ can be defined as

$$\epsilon = \frac{b - b_0}{b_0}. \tag{32}$$

Accordingly, so as not to degrade the correlation output, ϵ should be bound by

$$|\epsilon| < \frac{\delta x_{in}}{\Delta x_{in}}. \tag{33}$$

The tolerance analysis above can also be explained from simple geometric considerations, with the aid of Fig. 4. In an optical processor, the input WDF is first rotated by 90 deg, where the initial frequency coordinate becomes the new space coordinate. This rotation inherently includes rescaling according to the scaling factor b. Then the rotated input WDF is convolved in frequency with the filter WDF, and the resulting WDF is rotated back. The required accuracy of rotation and scaling is dictated by the need for the rotated input WDF to match the WDF of the filter in both space and frequency dimensions. Good matching means that the maximal error in either frequency or space should be less than the resolution in that dimension. Because at the filter plane the input WDF is rotated, the spatial resolution at the filter plane δx_F is equivalent to the frequency resolution at the input plane $\delta\nu_{in}$, which is dictated by the degree of coherence, and the frequency resolution $\delta\nu_F$ at the filter plane is equivalent to the spatial resolution δx_{in} at the input plane.

As mentioned above, the Wigner matrices are usually wavelength dependent. Thus, for polychromatic illumination, we can ascertain that the performance of the processor is not degraded if the tolerance requirements above apply to all the wavelengths within the spectrum of illumination. Specifically,

$$|a(\lambda)| < |b_0| \frac{\delta \nu_{\text{in}}}{\Delta x_{\text{in}}}, \qquad |\epsilon(\lambda)| < \frac{\delta x_{\text{in}}}{\Delta x_{\text{in}}}. \tag{34}$$

The conditions in inequalities (34) may sometimes be too severe, and it may be sufficient to require that these conditions are met on average, in the sense of a mean-square-root value, such as

$$\sqrt{\langle a^2(\lambda)\rangle} < |b_0| \frac{\delta \nu_{\text{in}}}{\Delta x_{\text{in}}}, \qquad \sqrt{\langle \epsilon^2(\lambda)\rangle} < \frac{\delta x_{\text{in}}}{\Delta x_{\text{in}}}, \tag{35}$$

where $\langle\ \rangle$ denote weighted averaging over some wavelength range, with the power spectrum of the illumination $S(\lambda)$ as the weighting function. This yields

$$\langle a^2(\lambda)\rangle = \frac{\int \mathrm{d}\lambda a^2(\lambda) S(\lambda)}{\int \mathrm{d}\lambda S(\lambda)}. \tag{36}$$

7. Concluding Remarks

We demonstrated how the Wigner formulation can be exploited as a tool for designing and analyzing optical processors. The WDF describes the light field in phase space, which is a reasonable description when both space and frequency need to be discussed simultaneously, as is the case with optical processing. The dynamics of the WDF through first-order optical configurations can be described as a linear phase-space coordinates transformation, which allows rather simple geometric analysis to determine performance and tolerance for errors of optical processors. The Wigner formulation allows a unified mathematical description of optical processing with illumination having an arbitrary degree of coherence, both spatially and temporally. The degree of spatial coherence is treated by the generalization of the WDF for partially coherent light, and the degree of temporal coherence is treated by investigation of the dependence of the WDF on wavelength as a parameter. We hope that this paper will help to promote use of the Wigner formalism in the various optical configurations of physical optics.

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