Flatland optics. II. Basic experiments

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In "Flatland optics: fundamentals" [J. Opt. Soc. Am. A **17**, 1755 (2000)] we described the basic principles of two-dimensional (2D) optics and showed that a wavelength λ in three-dimensional (3D) space (x,y,z) may appear in Flatland (x,z) as a wave with another wavelength, $\Lambda = \lambda/\cos\alpha$. The tilt angle α can be modified by a 3D (Spaceland) individual who then is able to influence the 2D optics in a way that must appear to be magical to 2D Flatland individuals—in the spirit of E. A. Abbott's science fiction story [*Flatland, a Romance of Many Dimensions*, 6th ed. (Dover, New York, 1952)] of 1884. We now want to establish the reality or objectivity of the 2D wavelength Λ by some basic experiments similar to those that demonstrated roughly 200 years ago the wave nature of light. Specifically, we describe how to measure the 2D wavelength Λ by mean of five different arrangements that involve Young's biprism configuration, Talbot's self-imaging effect, measuring the focal length of a Fresnel zone plate, and letting light be diffracted by a double slit and by a grating. We also performed experiments with most of these arrangements. The results reveal that the theoretical wavelength, as predicted by our Flatland optics theory, does indeed coincide with the wavelength Λ as measured by Flatland experiments. Finally, we present an alternative way to understand Flatland optics in the spatial frequency domains of Flatland and Spaceland.

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1. INTRODUCTION

Some skeptical readers of this series on Flatland optics[1] might ask how "real" the Flatland optics really is. There are different answers to this hypothetical question. The first answer, provided earlier,[1] showed that the wave equation in Flatland is a special case of the standard three-dimensional (3D) wave equation, the Helmholtz equation. This fact may establish the reality of Flatland optics in the mind of a theoretician.

A more philosophical answer could be based on the paradigm: "Real" is what is observable. So in this paper we provide a second answer. It is designed primarily to convince an experimental physicist of the reality of Flatland optics by demonstrating that the classical experiments for establishing the wave nature of light in three dimensions approximately 200 years ago have their counterparts in Flatland. We considered five arrangements for measuring the Flatland wavelength Λ, which is different from the 3D wavelength λ. These five arrangements lead to basic experiments, denoted by Young, Talbot, Fresnel Zone Plate, Double Slit, and Grating. As we will show now, the Flatland wavelength Λ is indeed observable.

A third answer that may satisfy an applied physicist or an engineer will follow in the near future. We intend to show that Flatland optics is actually useful. Usefulness may be necessary and perhaps also sufficient for being "real." One can obtain an additional point of view on Flatland optics by looking at Flatland in the spatial frequency domain.

In this paper we are concerned mainly with diffraction as it is perceived in Flatland. The only parameter of the illumination that affects diffraction is the effective wavelength in the medium (not the temporal frequency). Thus we will treat here only the spatial aspects of the light (i.e., wavelength and spatial frequency) as they are measured in Flatland and postpone the treatment of temporal aspects (temporal frequency, propagation, the speed of light, and the dispersion relation in Flatland) to the next parts of this series.

2. YOUNG'S BIPRISM EXPERIMENT

In Young's biprism experiment one measures the angle between two plane waves and a fringe period. From these two parameters one can deduce the wavelength, using a basic interference equation. We will now describe such an experiment, performed in two-dimensional (2D) Flatland instead of in 3D spaceland where Thomas Young performed his famous experiment.

Figure 1 shows the generic setup for Flatland optics. A one-dimensional object $u_0(x)$ is illuminated by a plane wave, which is tilted by an angle α. The world of Flatland is a plane $y =$ constant, which might touch the lower end of the object. For performing Young's experiment in Flatland the object consists of a biprism, which creates two tilted waves that overlap in a region where one would expect to observe interference fringes, as shown in Fig. 2. As observed in Flatland, the fringe period is p and the deflection angle is β. With these two parameters it is possible to deduce the wavelength Λ by means of Young's formula as

$$2p \sin \beta = \Lambda. \tag{1}$$

This Λ is what a Flatland creature calls "wavelength." To verify the theory of Flatland optics[1] we checked how the experimental 2D result Λ is related to the 3D wavelength λ of the monochromatic point source and to the 3D tilt angle α. Indeed, we found that Λ, λ, and α obey

$$\Lambda = \lambda/\cos \alpha. \tag{2}$$

In our experiment we replaced the biprism with two vertical mirrors so as to be able to obtain two beams with a very small angle between them. This results in coarse

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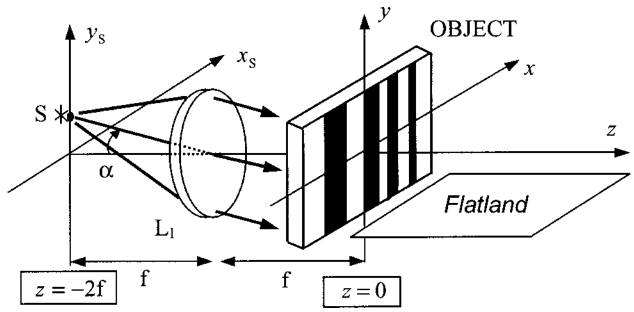

Fig. 1. Generic setup for Flatland optics. A tilted plane wave illuminates the object $u_0(x)$. The wave field $u(x,z)$ may be observed at a plane y = constant, at $z \geqslant 0$.

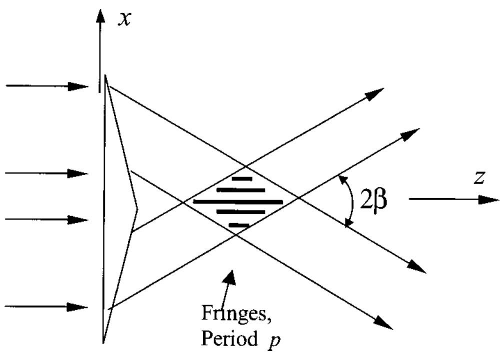

Fig. 2. Young's biprism experiment. Observed, p,β; deduced, $2p\sin\beta = \Lambda$.

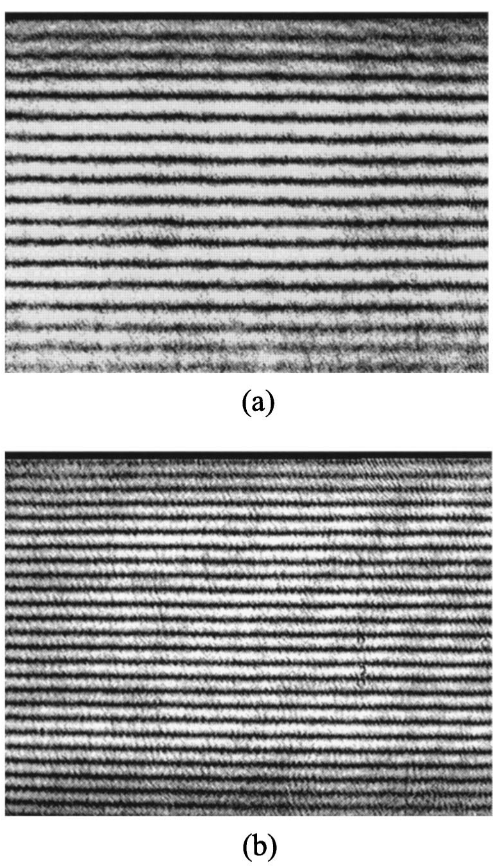

Fig. 3. Reflection-version Young's interference fringes at illumination angles (a) $\alpha = 50°$ and (b) $\alpha = 26.5°$.

interference fringes whose period can be easily measured. We kept these two mirrors fixed and changed only the angle α of the illumination plane wave. We used a CCD camera to detect the output fringes. Figure 3 shows the interference fringes for two specific angles, $\alpha_1 = 50°$ and $\alpha_2 = 26.5°$. Then we measured the corresponding periods of the interference fringes. According to Eqs. (1) and (2), we verified the following relationship:

$$\frac{p_1}{p_2} = \frac{\Lambda_1}{\Lambda_2} = \frac{\cos\alpha_2}{\cos\alpha_1}, \tag{3}$$

where p_1 and p_2 are the periods of the fringes, and Λ_1 and Λ_2 are the 2D Flatland wavelengths with respect to the illumination angles α_1 and α_2, respectively. For example, when $\alpha_1 = 50°$, the period of the fringes is 28.11 pixels, and when $\alpha_2 = 26.5°$, the period of the fringes is 19.5 pixels. So $p_1/p_2 = 1.44$ and $\cos\alpha_2/\cos\alpha_1 = 1.392$. The experimental errors, in this example and with other measurements, were less than 8%.

Our experimental results indicate that the interference between two plane wave (in three dimensions) and that between two line waves (in two dimensions) is very similar, apart from the different wavelengths λ and Λ, respectively. The 3D wavelength λ depends only on the source. The 2D wavelength Λ depends also on a geometrical parameter α, which can be controlled by a 3D person but not by a 2D creature. Finally, we note that the dependence of 2D wavelength Λ on angle α is due to the fact that, when the illumination angle α in three dimensions is varied, the angle between the two interfering beams also changes.

3. TALBOT'S SELF-IMAGING EFFECT

That self-imaging effect[2] was discovered by Talbot 50 years before Abbott's invention of Flatland. Several improvements and applications of the Talbot effect did emerge in recent years.[3–5] In Flatland, when a grating with period p is illuminated by a "line-wave" (2D plane wave), it forms the pattern shown in Fig. 4. The Flatland creature observes a longitudinal period z_p. Knowing p and z_p allows him or her to determine the 2D wavelength Λ as

$$2p^2/z_p = \Lambda. \tag{4}$$

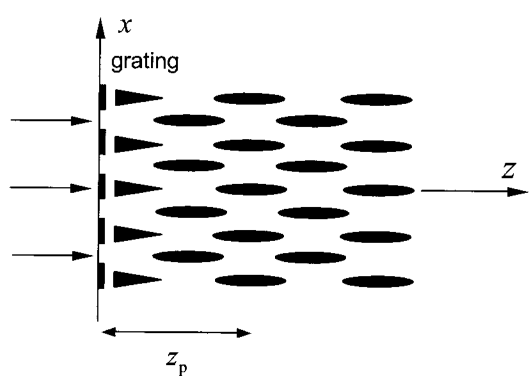

Fig. 4. Talbot effect. Known, grating period p; observed, longitudinal period z_p; deduced (based on Talbot's formula); $2p^2/z_p = \Lambda$.

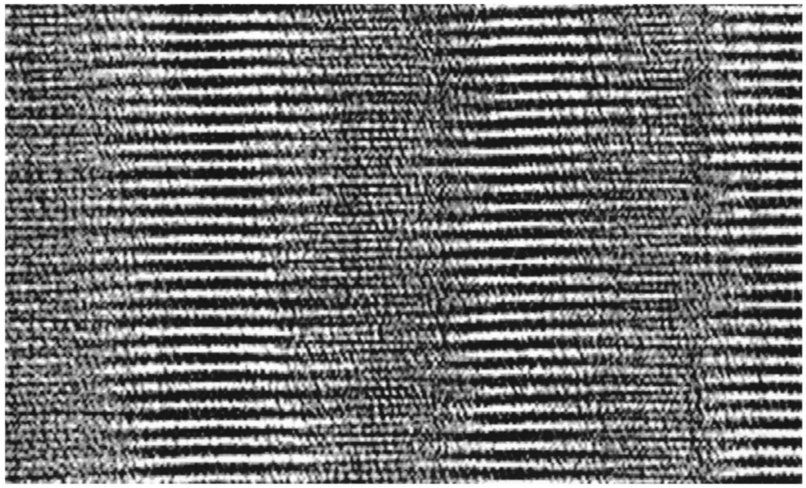

Fig. 5. Talbot's self-imaging effect in the Flatland, where angle α of the illumination plane wave is 80°.

The 3D physicist who looks over the shoulder of his 2D colleague can easily verify the validity of the basic relationship between 2D and 3D wavelengths as

$$\Lambda = \lambda/\cos \alpha. \tag{5}$$

The 2D creature, on the other hand, is not aware of this relationship [Eq. (5)]. Our experimental results showed that Eq. (5) holds. We performed a Talbot self-imaging experiment in Flatland. The object was a binary amplitude grating with a period of 72 μm, recorded on a photographic film with dimensions of 3.6 cm and 2.4 cm. A representative result, where the angle α was 80°, is shown in Fig. 5. Two Talbot images, one positive and one negative, are clearly shown. Another half of a positive Talbot image is also shown.

4. FRESNEL ZONE-PLATE LENS

A cylindrical Fresnel zone-plate lens consists of parallel bars at distances R_m from the center, as shown schematically in Fig. 6. When such a zone plate is used as the object in the generic Flatland setup, some of the incident lights will be focused at a distance f_α, as shown in Fig. 7. By measuring this f_α and knowing unit distance R_1 of the zone plate, it is possible for the 2D creature to deduce the 2D wavelength Λ, in accordance with the well-known law of zone-plate focal power, to obtain

$$R_1^2/2f_\alpha = \Lambda. \tag{6}$$

The focal power of the zone-plate lens in Flatland can be readily derived if $F(x)$ is described as quasi-Fourier series

$$F(x^2) = F(x^2 + mR_1^2), \quad m = 0, 1, 2,..., \tag{7}$$

$$F(x^2) = \sum_{(n)} A_n \exp[2\pi n(x/R_1)^2]. \tag{8}$$

When Eq. (8) is compared with the transmission function of a thin lens, of the form $\exp(i\pi x^2/\Lambda f_\alpha)$, then

$$R_1^2/2n = \Lambda f_\alpha. \tag{9}$$

Measuring Λ and knowing α and λ would lead to the verification of

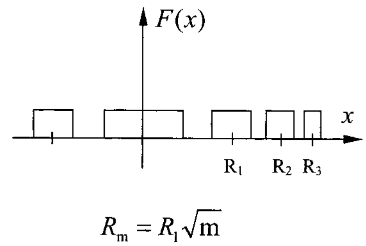

Fig. 6. One-dimensional Fresnel zone plate.

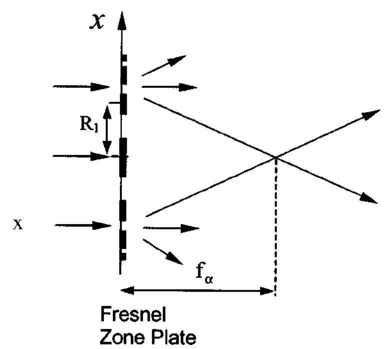

Fig. 7. Focal power of a Fresnel zone plate. Known, first radius R_1; observed, focal power $1/f_\alpha$; deduced, $R_1^2/2f_\alpha = \Lambda$.

$$\lambda/\Lambda = \cos \alpha. \tag{10}$$

So the 3D big brother again verifies the validity of Eq. (2).

5. DOUBLE-SLIT DIFFRACTION

Consider now the diffraction from a double slit that is illuminated by a line wave, as shown in Fig. 8. The double slit is schematically shown in Fig. 8(a). It is located at the front focal plane of a lens, for example, a Fresnel zone plate, as shown in Fig. 8(b). We know slit distance p and the focal length f_α in Flatland, and we measure the period D of the interference fringes behind the lens (the lens is not really necessary if the slit width is sufficiently small compared with the slit distance). The ratio p/f_α represents the angle between the two tilted plane waves emerging from the zone plate. Using these parameters, one can deduce Flatland wavelength Λ by means of the double-slit formula as

$$pD/f_\alpha = \Lambda. \tag{11}$$

The 3D scientist could have predicted the result of this Λ, since he or she knows how Λ depends on 3D wavelength λ and on tilting angle α.

Experimentally, instead of using a diffractive cylindrical lens we used a refractive cylindrical lens, which was a plano–convex glass lens. We have to take into account that the focal length f of the refractive cylindrical lens along the beam depends on the tilting angle α (the details are given in Appendix A). So, for each tilting angle α, we measured the back focal length of the cylindrical lens and then placed the double slits at the front focal plane with the same focal length. Then we measured the period of the interference fringes. According to the geometry of our experimental arrangement, the focal length f_α in the 2D Flatland is described by $f_\alpha = f(\alpha)\cos\alpha$. When the illumination is at normal incidence ($\alpha = 0°$), $pD_0/f_0 = \lambda$, where λ is the wavelength of the illumination in 3D space, D_0 is the period of the interference fringes, and f_0 is the focal length. Combining this with Eq. (11), we obtain

$$\Lambda = \frac{\lambda}{\cos\alpha}\frac{D}{D_0}\frac{f_0}{f(\alpha)}. \tag{12}$$

Now we need only to verify whether the term $(D/D_0) \times (f_0/f)$ is equal to 1.

In our experiment we used a cylindrical lens with a focal length f_0 of 15 cm and a 0.5 mm distance between the two slits, p. Thus D_0 was 20 pixels on our CCD camera that was used for detecting the output fringes. Table 1 presents the measured periods D (in pixels) and the focal length f for four tilting angles α. Using these measurements and the known D_0 and f_0, we calculated the ratio Df_0/D_0f given in the rightmost column of the table. As is evident, this ratio is approximately 1, with a maximum error of 5.1%, thus again verifying that $\Lambda = \lambda/\cos\alpha$.

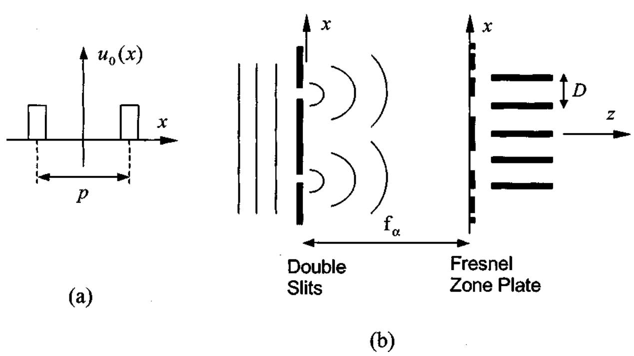

Fig. 8. Double-slit diffraction. A double slit at the front focal plane of the Fresenl zone-plate lens. Known, double-slit separation p and focal length f_α; observed, fringe period D; deduced $pD/f_\alpha = \Lambda$.

Table 1. Measured Results of the Double-Slit Interference Experiment[a]

Angle α(°)	Period D (pixel)	Focal Length f (cm)	$\frac{D}{D_0}\frac{f_0}{f}$
15	19.33	14.2	1.021
30	17.80	13.2	1.011
45	16.11	11.5	1.051
60	13.50	9.8	1.033

[a] D_0 = 20 pixels; f_0 = 15 cm.

6. GRATING DIFFRACTION

Consider now the diffraction from a grating with period p that is illuminated by a line wave, as shown in Fig. 9. Shortly after the grating is a cylindrical Fresnel zone plate lens. At the rear focal plane of the zone plate, one observes diffraction orders separated by distances S. The grating period p is known, and the deflection angle β is measured indirectly by the spacing S. Hence one may deduce the 2D wavelength Λ with the help of the grating formula

$$p\sin\beta = \Lambda. \tag{13}$$

Actually, as long as the angle α is ~85° or less, the spacing S is essentially a constant for a specific grating and a specific diffractive cylindrical lens.[6,7] The only variable parameter is the focal length f_α of the diffractive cylindrical lens in the Flatland. Again, $f_\alpha = f_z\cos\alpha$, where f_z is the focal length of the Fresnel zone-plate lens in 3D space. Using the small-angle approximation yields

$$S/f_\alpha = \tan\beta \approx \beta \approx \sin\beta = \Lambda/p. \tag{14}$$

Using Eq. (14) and $f_\alpha = f_z\cos\alpha$ finally leads to our desired verification:

$$\Lambda = p\frac{S}{f_\alpha} = p\frac{S}{f}\frac{1}{\cos\alpha} = \frac{\lambda}{\cos\alpha}, \tag{15}$$

where $\lambda = pS/f_z$ when $\alpha = 0°$.

7. FLATLAND OPTICS IN THE SPATIAL FREQUENCY DOMAIN

The aspect of Flatland optics in the spatial frequency domain was touched on by an anonymous reviewer, to whom we are grateful. The basic concept here is that every statement in true space has an equivalent counterpart statement in the spatial frequency domain. This holds for the three dimensions of Spaceland (x,y,z), the two dimensions of Flatland (x,z), and the single dimension (y). The Helmholtz equation

$$\Delta_3 V(\mathbf{x}) + (2\pi/\lambda)^2 V(\mathbf{x}) = 0 \tag{16}$$

is equivalent to the statement that the 3D Fourier transform $\tilde{V}(\mathbf{v})$ of the wave field $V(\mathbf{x})$ can be nonzero only on the Ewald sphere, i.e., when

$${v_x}^2 + {v_y}^2 + {v_z}^2 = 1/\lambda^2. \tag{17}$$

If in addition $V(\mathbf{x})$ obeys

$$\Delta_{xz} V(\mathbf{x}) + \left(\frac{2\pi\cos\alpha}{\lambda}\right)^2 V(\mathbf{x}) = 0, \tag{18}$$

then $\tilde{V}(\mathbf{x})$ is confined to a cylinder, according to

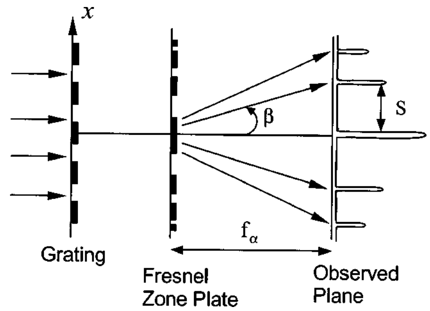

Fig. 9. Grating diffraction. Known, grating period p and focal length f_α; observed, β from $S = f_\alpha \tan\beta$; deduced, $p \sin\beta = \Lambda$.

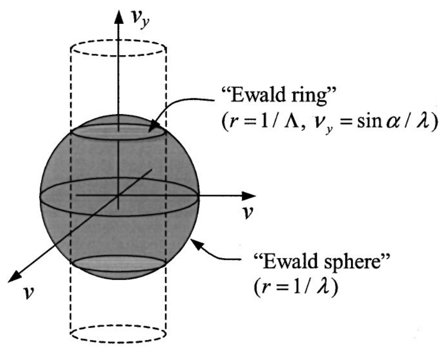

Fig. 10. Ewald rings in Flatland.

$$v_x^{\,2} + v_z^{\,2}\left(\frac{\cos\alpha}{\lambda}\right)^2. \tag{19}$$

Equation (19) represents a cylinder about the v_y axis. Since both Eq. (7) and Eq. (19) must be valid, we obtain, in geometrical terms, the intersection of a cylinder with the Ewald sphere: two rings of radii $\cos\alpha/\lambda$, at two planes of $v_y = \ \pm\sin\alpha/\lambda$. The Flatland people might call these rings "Ewald rings." The rings are illustrated in Fig. 10. The plus or minus ambiguity reflects the fact that the tilt angle α might be replaced by $-\alpha$. In other words, Flatland would then be illuminated from below. The Flatlanders would not know whether such replacement occurred, since the categories "above" and "below" are unknown to them.

8. CONCLUSIONS

We have described five experiments for measuring 2D wavelength Λ for the purpose of demonstrating the "reality" or "objectivity" of wave optics in Flatland. These five experiments were modeled in accordance with the history of 3D wave optics. Common to these experiments is that some parameters (i.e., the grating period) are known, that some parameters (i.e., the fringe period) are measured, and that some basic laws (i.e., $\sin\beta = \lambda/p$) are accepted. Using all these, one may infer the 2D wavelength Λ in five different ways, four of which were experimentally verified, proving that Λ genuinely exists and is properly related to the 3D wavelength λ.

APPENDIX A

In this appendix we derive the focal length $f(\alpha)$ of a refractive plano–convex cylindrical glass lens along the illumination beam as a function of tilting angle α. The lens and the illuminating geometry are shown in Fig. 11. The illumination plane wave has an angle α with respect to our Flatland (xz plane), as shown in Fig. 11(a). First, we consider one slice of the lens in an arbitrary plane parallel to the yz plane, as shown in Fig. 11(b). This slice has a thickness $H(x)$. By using Snell's law and triangular geometry, and after some algebraic manipulations, we obtain the total phase shift of the wave after it has passed through the slice along the direction of the beam propagation:

$$\begin{aligned}\Delta\phi &= \frac{2\pi}{\lambda}[(nAC + CE) - AD] \\ &= \frac{2\pi}{\lambda}(\sqrt{n^2 - \sin^2\alpha} - \cos\alpha)H(x),\end{aligned} \tag{A1}$$

where n is the refractive index of the lens material, λ is the 3D wavelength in air, and

$$H(x) = \Delta_0 + R_1^{\,2} - \sqrt{R_1^{\,2} - x^2}, \tag{A2}$$

where Δ_0 is the maximum thickness of the lens (on its axis) and R_1 is the radius of curvature of the convex surface of the lens.

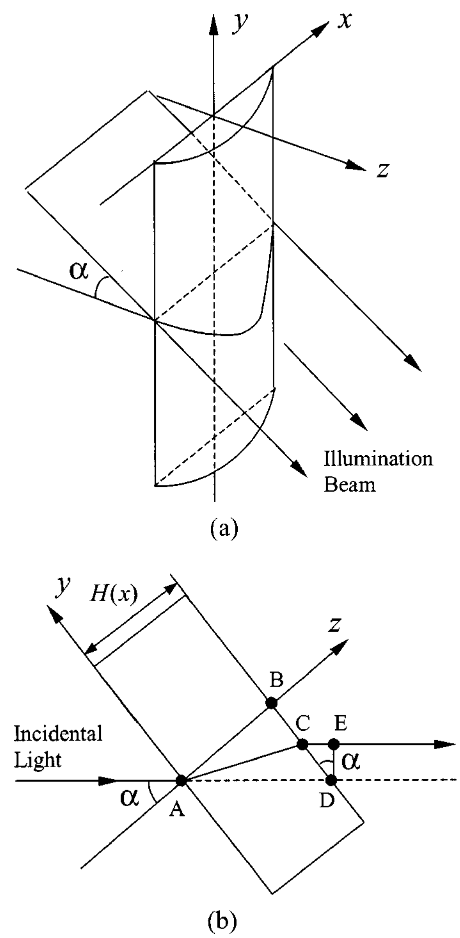

Fig. 11. Plano–convex refractive cylindrical lens used in Flatland optics. (a) 3D view of the lens, (b) one slice of the lens.

Using the paraxial approximation and neglecting phase shifts that are independent of x, we obtain the multiplicative phase transformation of the lens:

$$t(x) = \exp(i\Delta\phi) = \exp\left[-\frac{\pi i}{\lambda R_1}(\sqrt{n^2 - \sin^2\alpha} - \cos\alpha)x^2\right]. \quad \text{(A3)}$$

Compared with the usual thin lens formula, focal length $f(\alpha)$ of the refractive cylindrical lens along the beam may be written as

$$\frac{1}{f(\alpha)} = (\sqrt{n^2 - \sin^2\alpha} - \cos\alpha)\frac{1}{R_1}. \quad \text{(A4)}$$

Substituting into Eq. (A4) the refractive index of the lenses used in the experiments ($n = 1.52$), we obtain good agreement between Eq. (A4) and the data of Table 1 (as much as 2% error).

In the extreme case, when $\alpha = 0°$, $1/f_0 = (n - 1)/R_1$ as expected for normal incidence. In the other extreme case, when $\alpha \rightarrow 90°$, the optical power tends to the limit of $1/f \rightarrow 1/f_{90} = \sqrt{n^2 - 1}(1/R_1)$.Because $n > 1$, we obtain $f_{90} < f_0$, and it is easily shown that the focal length decreases monotonically between these limiting values.

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