Introduction to Fourier optics

Textbook: Goodman (chapters 2-4)

Overview:

Linear and invariant systems
The Fourier transform
Scalar diffraction
Fresnel and Fraunhoffer approximations.

1. Linear systems and Fourier transform tutorial (reminder)

A system connects an input g1 to an output g2 such that:

$$g_2(x_2, y_2) = S\{g_1(x_1, y_1)\}$$

It is said to be linear if it satisfies:

$$S\{\alpha g(x, y) + \beta h(x, y)\} = \alpha S\{g(x, y)\} + \beta S\{h(x, y)\}$$

The response of a linear system can be expressed by its response to a (basis) set of elementary functions. In particular, since

$$g_1(x_1, y_1) = \iint g_1(\xi, \eta)\delta(x - \xi, y - \eta)d\xi d\eta$$

and invoking linearity we find that:

$$g_2(x_2, y_2) = \iint g_1(\xi, \eta)S\{\delta(x_1 - \xi, y_1 - \eta)\}d\xi d\eta$$

This defines the impulse response (or point spread function) of the linear system:

$$h(x_2, y_2, \xi, \eta) = S\{\delta(x_1 - \xi, y_1 - \eta)\}$$

An important subset of linear systems are invariant systems, in which the impulse response is only dependent on the difference in coordinates:

$$h(x_2, y_2, \xi, \eta) = h(x_2 - \xi, y_2 - \eta)$$

In this case the system response is simply a convolution between the input and the transfer function:

$$g_2 = g_1 \otimes h$$

This implies that in Frequency domain, the output of an invariant linear system is merely a multiplication of the input by the transfer function H

$$G_2 = G_1 H$$

Where

$$H = \int_{-\infty}^{\infty}\int h(x, y)\exp\left[-2\pi i\left(f_x x + f_y y\right)\right]dx\,dy$$

To further discuss this let us remember some of the properties of the Fourier transform:

Definition of the 2D Fourier transform

$$F\{g\} = \int_{-\infty}^{\infty}\int g(x,y)\exp\left[-2\pi i\left(f_x x + f_y y\right)\right]dx\,dy$$

Similarly, the inverse transform is:

$$F^{-1}\{G\} = \int_{-\infty}^{\infty}\int G\left(f_x, f_y\right)\exp\left[2\pi i\left(f_x x + f_y y\right)\right]df_x\,df_y$$

The conditions for this being definite are that the function is "regular", which applies for all physical realizations in the optical realm.

The Fourier transform operation can be viewed as a decomposition of a function in a new basis of elementary functions of the form $\exp\left[2\pi i\left(f_x x + f_y y\right)\right]$, which are tilted plane waves with angle $\theta = a\tan\left(f_y / f_x\right)$ to the X-axis, and a spatial period $L = \left(f_x^2 + f_y^2\right)^{-1/2}$.

Some important properties of the FT (quite evident from the definition)

Are:

Linearity:

$$F\left(\alpha g + \beta h\right) = \alpha F\left(g\right) + \beta F\left(h\right)$$

Similarity:

$$F\left(g\left(ax, by\right)\right) = \frac{1}{|ab|}G\left(\frac{f_x}{a}, \frac{f_y}{b}\right)$$

Shift theorem (linear shift transform to a phase shift):

$$F\left(g\left(x-a, y-b\right)\right) = \exp\left[-2\pi i\left(f_x a + f_y b\right)\right]F\left(g\left(x,y\right)\right)$$

Parseval Theorem ("conservation of energy"):

$$\int_{-\infty}^{\infty}\int \left|g\left(x,y\right)\right|^2 dxdy = \int_{-\infty}^{\infty}\int \left|G\left(f_x, f_y\right)\right|^2 df_x df_y$$

Proof in 1D (2D proof follows directly the same procedure)

$$\int_{-\infty}^{\infty} |G(f_x)|^2 df_x = \iiint g(x) e^{-2\pi i f_x x} g^*(x') e^{2\pi i f_x x'} dx dx' df_x =$$
$$= \iint g(x) g^*(x') \int e^{-2\pi i f_x (x - x')} df_x dx dx' =$$
$$= \iint g(x) g^*(x') \delta(x - x') dx dx' = \int_{-\infty}^{\infty} |g(x)|^2 dx$$

Convolution theorem:

$$F\left\{ \iint_{-\infty}^{\infty} g(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta \right\} = G(f_x, f_y) H(f_x, f_y)$$

Proof in 1D (2D proof follows directly the same procedure)

$$F\left\{ \int_{-\infty}^{\infty} g(\xi) h(x - \xi) d\xi \right\} = \iint g(\xi) h(x - \xi) d\xi e^{-2\pi i f_x x} dx =$$
$$= \iint g(\xi) e^{-2\pi i f_x \xi} d\xi h(x - \xi) e^{-2\pi i f_x (x - \xi)} d(x - \xi) =$$
$$= G(f_x) H(f_x)$$

Autocorrelation theorem:

$$F\left\{ \iint_{-\infty}^{\infty} g(\xi, \eta) g^*(\xi - x, \eta - y) d\xi d\eta \right\} = |G(f_x, f_y)|^2$$

Proof in 1D (2D proof follows directly the same procedure)

$$F\left\{ \int_{-\infty}^{\infty} g(\xi) g^*(\xi - x) d\xi \right\} = \iint g(\xi) g^*(\xi - x) d\xi e^{-2\pi i f_x x} dx =$$
$$= \iint g(\xi) e^{-2\pi i f_x \xi} d\xi g^*(\xi - x) e^{2\pi i f_x (\xi - x)} dx =$$
$$= G(f_x) G^*(f_x) = |G(f_x)|^2$$

Fourier integral theorem:

$$F\, F^{-1}\{g\} = F^{-1}\, F\{g\} = g$$

Fourier transform of separable functions:

If

$$g(x, y) = g_x(x) g_y(y)$$

then

$$F\{g(x,y)\} = F_x\{g_x(x)\}F_y\{g_y(y)\}$$

It is useful to consider some FT

$$F\{\exp(-\pi a^2 x^2)\} = \frac{1}{|a|}\exp\left(-\pi \frac{f_x^2}{a^2}\right)$$

$$F\{rect(ax)\} = \frac{1}{|a|}\sin c\left(\frac{f_x}{a}\right)$$

$$F\{\delta(ax)\} = \frac{1}{|a|}$$

$$F\{\exp(i\pi ax)\} = \delta\left(f_x - \frac{a}{2}\right)$$

As well as to consider FT in cylindrical coordinates (especially for functions which are only dependent on r).

<u>Derivation of the Fresnel diffraction formula</u>

Lets start again from Maxwell's equations:

$$\nabla x H - \frac{1}{c}\frac{\partial D}{\partial t} = 0 \qquad \nabla \cdot D = 0$$
$$\nabla x E - \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \cdot B = 0$$

Since μ is generally space invariant, and by using:

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$$

We get:

$$\nabla\times(\nabla\times E) = -\frac{1}{c}\left(\nabla\times\frac{\partial B}{\partial t}\right) = -\frac{1}{c}\left(\frac{\partial(\nabla\times B)}{\partial t}\right) = -\frac{\mu}{c}\left(\frac{\partial}{\partial t}\frac{1}{c}\frac{\partial \varepsilon E}{\partial t}\right) = -\frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\nabla\cdot E = \nabla\cdot\varepsilon E - E\cdot\nabla\varepsilon = -\frac{E\cdot\nabla\varepsilon}{\varepsilon} = -E\cdot\nabla(\ln\varepsilon) = -2E\cdot\nabla(\ln n)$$

From which we get, overall:

$$\nabla^2 E + 2\nabla(E\cdot\nabla\ln n) - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Again, for a homogeneous medium we get the scalar equation:

$$\nabla^2 E - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Which is valid anywhere except the boundaries

If the scalar field is assumed to be of the form (monochromatic):

$$u(P,t) = A(P)\cos(2\pi\nu t - \phi(P))$$

And the scalar Maxwell equation holds, we get for $U(P) = A(P)\exp(i\phi(P))$ the Helmholtz equation:

$$(\nabla^2 + k^2)U = 0$$

with $k = 2\pi n\frac{\nu}{c}$.

This equation can be solved by means of the use of Green's theorem, relating the values of a function inside the volume to a surface integral.

Briefly, this states that for any two "reasonable" functions U,G:

$$\iiint dv\left(U\nabla^2 G - G\nabla^2 U\right) = \iint ds\left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)$$

I will not go through the details, but a long description can be found in Goodman. Intuitively, the Green's function chosen for the Helmholtz equation is a spherical wave:

$$G = \frac{e^{ikR}}{R}$$

For which:

$$\left(\nabla^2 + k^2\right)G = \delta(R)$$

Where R is the distance from an arbitrary point P1

Performing the integration on the left hand side we are left with the value of U at the origin (since the laplacian operator elsewhere is just k^2 – from Helmholtz equation). Thus, the integration is done over a surface comprising a plane and a hemisphere stretching to infinity. Under some conditions, the value of the latter vanishes, leaving only the integration over the plane.

This results in the Rayleigh-Sommerferld solution, which is really a mathematical formulation of the Huygens principle.

$$U(P_0) = \frac{1}{i\lambda} \iint_{aperture} U(P_1) \frac{e^{ikr_{01}}}{r_{01}} \cos\theta ds$$

Where the cosθ accounts for the exact choice of the Green's function, and in any case vanishes for small angles.

Relating this to what we said previously, this can be interpreted as the impulse response of propagation from the aperture, where:

$$h(P_0, P_1) = \frac{1}{i\lambda} \frac{e^{ikr_{01}}}{r_{01}} \cos\theta$$

Now let us consider the small angle approximations for this. We get the integral:

$$U(P_0) = \frac{1}{i\lambda} \iint_{aperture} U(x_1, y_1) \frac{e^{ik\left[z_0^2 + (x_1 - x_0)^2 + (y_1 - y_0)^2\right]^{1/2}}}{\left[z_0^2 + (x_1 - x_0)^2 + (y_1 - y_0)^2\right]^{1/2}} \cos\theta dx_1 dy_1$$

The dependence on the integration coordinates appears both in the numerator (phase term) and in the denominator (amplitude term). For large z, obviously the amplitude contribution is small. The cosine term is also an amplitude term which has a small effect for large z. Phase terms can have, of course, a much higher effect, so we will keep them. We thus get:

$$U(P_0) \approx \frac{1}{i\lambda z} \iint_{aperture} U(x_1, y_1) e^{ik\left[z_0^2 + (x_1 - x_0)^2 + (y_1 - y_0)^2\right]^{1/2}} dx_1 dy_1$$

<u>Fresnel diffraction:</u>

Let us now consider two more restrictive solutions to the problem of diffraction from an aperture. The first is effectively a small angle solution, retaining only the first order (in x,y) terms of:

$$r_{01} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2} \approx z\left\{1 + \frac{1}{2}\left[\left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2\right] + \frac{1}{8}\left[\left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2\right]^2\right\}$$

Since we want to nerglect the higher order terms, we require that

$$\frac{2\pi}{\lambda} \cdot \frac{z}{8}\left[\left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2\right]^2 << 1$$

or:

$$z^3 >> \frac{\pi}{4\lambda} \cdot \left[(x-\xi)^2 + (y-\eta)^2\right]^2 \approx \frac{\pi}{4\lambda} R^4$$

Where R is a typical dimension of observation in both planes.

For R=1cm and λ=1μm, this amounts to:

$$z^3 >> 10^4 cm^3 \rightarrow z >> 20cm$$

In this case, free space propagation can be described by:

$$U(x,y) \approx \frac{e^{ikz}}{i\lambda z} \iint_{aperture} U(\xi,\eta) e^{\frac{ik}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]} d\xi d\eta$$

<u>Fraunhoffer diffraction:</u>

This is a more restrictive approximation. Let us first expand the Fresnel formula. We get:

$$U(x,y) \approx \frac{e^{ikz}}{i\lambda z} \iint_{aperture} U(\xi,\eta) e^{\frac{ik}{2z}\left[\xi^2 - 2x\xi + x^2 + \eta^2 - 2y\eta + y^2\right]} d\xi d\eta$$

If z is so large that:

$$z >> \frac{\pi}{\lambda}\left(\xi^2 + \eta^2\right)$$

Than the quadratic phase terms on ξ and η can be neglected and we are left with:

$$U(x,y) = \frac{e^{ikz} e^{ik\frac{(x^2+y^2)}{2z}}}{i\lambda z} \iint_{aperture} U(\xi,\eta) e^{-\frac{ik}{z}(x\xi + y\eta)} d\xi d\eta$$

Which is just a (scaled) Fourier transform of the aperture field

For R=1cm and λ=1μm, this amounts to:

$$z >> 10^4 cm \to z >> 100m$$

Simple Fraunhoffer diffraction patterns:

Rectangular and circular apertures

For transmission through a rectangular aperture we get:

$$t_a(\xi,\eta) = rect\left(\frac{\xi}{2w_x}\right) rect\left(\frac{\eta}{2w_y}\right)$$

This being a separable function, the Fourier transform is just a multiplication of the two individual Fourier transforms, giving:

$$U(x,y) = \frac{Ae^{ikz}e^{ik\left(x^2+y^2\right)}}{i\lambda z} \sin c\left(\frac{2w_x x}{\lambda z}\right) \sin c\left(\frac{2w_y y}{\lambda z}\right)$$

and for the intensity $I = |U|^2$

$$I(x,y) = \frac{A^2}{\lambda^2 z^2} \sin c^2\left(\frac{2w_x x}{\lambda z}\right) \sin c^2\left(\frac{2w_y y}{\lambda z}\right)$$

The width of the central lobe in each direction is: $d = \dfrac{\lambda z}{w}$

For transmission through a circular aperture we get:

$$t_a(q) = circ\left(\frac{q}{w}\right)$$

For which the Fourier transform is a Jinc function

$$U(r) = 2\frac{Ae^{ikz}e^{ikr^2}}{i\lambda z} \frac{J_1(kwr/z)}{kwr/z}$$

With the corresponding intensity distribution:

$$I(r) = 4\frac{A^2}{\lambda^2 z^2} \frac{J_1{}^2(kwr/z)}{(kwr/z)^2}$$

This is called an Airy pattern. The width of the central lobe in this case is $d = 1.22\dfrac{\lambda z}{w}$

Amplitude grating

Consider a rectangular containing an amplitude grating:

$$t_a(\xi,\eta) = rect\left(\frac{\xi}{2w_x}\right) rect\left(\frac{\eta}{2w_y}\right)\left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right]$$

In the y direction this leads to the "standard" sinc dependence. In the x direction, we can apply the convolution theorem to get the response, using the Fourier transform of the grating:

$$F\left\{\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right\} = \frac{1}{2}\delta(f_x) + \frac{m}{4}\delta(f_x - f_0) + \frac{m}{4}\delta(f_x + f_0)$$

To get:

$$U(x,y) = \frac{Ae^{ikz}e^{ik(x^2+y^2)}}{2i\lambda z}\sin c\left(\frac{2w_y y}{\lambda z}\right)\left[\sin c\left(\frac{2w_x x}{\lambda z}\right) + \frac{m}{2}\sin c\left(\frac{2w_x(x + f_0\lambda z)}{\lambda z}\right) + \frac{m}{2}\sin c\left(\frac{2w_x(x - f_0\lambda z)}{\lambda z}\right)\right]$$

If $w_x f_0 >> 1$, corresponding to many grating lines in the aperture, the peaks are well separated, and the intensity distribution is approximately the sum of the squared amplitudes:

$$I(x,y) = \frac{A^2}{4\lambda^2 z^2}\sin c^2\left(\frac{2w_y y}{\lambda z}\right)\left[\sin c^2\left(\frac{2w_x x}{\lambda z}\right) + \frac{m^2}{4}\sin c^2\left(\frac{2w_x(x + f_0\lambda z)}{\lambda z}\right) + \frac{m^2}{4}\sin c^2\left(\frac{2w_x(x - f_0\lambda z)}{\lambda z}\right)\right]$$

Thus, the forward transmission is 1/4, and each of the sidelobes (termed +1 and -1 orders) have a maximal efficiency of $m^2/16$, or a maximum of ~6%.

To become more efficient either phase gratings or reflectance gratings have to be used. These will be discussed more elaborately in the next tutorial.

<u>Angular spectrum of Waves</u>

An alternative solution to the problem of diffraction is presented by considering free-space propagation as a linear transformation of the solutions of the free-space Helmholtz equation:

$$(\nabla^2 + k^2)U = 0$$

The solutions of which are plane waves.

Let us consider the field distribution at z=0 as a summation of gratings of different spatial frequencies (a Fourier transform in real space):

Then:

$$A(f_x, f_y, 0) = \int\int dxdy U(x,y,0) e^{-2\pi i(f_x x + f_y y)}$$

where the inverse transform is:

$$A(x,y,0)=\iint df_x df_y U(f_x, f_y,0)e^{2\pi i(f_x x+f_y y)}$$

Physically we can interpret this as a sum of plane waves, since a plane wave of the form:

$$p(x,y,z,t)=e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

with:

$$\vec{k}=\frac{2\pi}{\lambda}(\alpha\hat{x}+\beta\hat{y}+\gamma\hat{z})$$

has an intensity pattern which is periodic in the plane, with the period which increases as $\gamma=\sqrt{1-\alpha^2-\beta^2}$ becomes smaller.

Thus, using:

$$\alpha=\lambda f_x;\beta=\lambda f_y;\gamma=\sqrt{1-(\lambda f_x)^2-(\lambda f_y)^2}$$

we see that the Fourier transform is just a decomposition to plane waves.

Let us now consider the propagation of these in space.

Since:

$$U(x,y,z)=\iint d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)e^{2\pi i\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}$$

Taking the Laplacian of U, we get:

$$\nabla^2U(x,y,z)=\left(\frac{2\pi}{\lambda}\right)^2(-\alpha^2-\beta^2)\iint d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)e^{2\pi i\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}+$$
$$\iint d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}\frac{d^2}{dz^2}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)e^{2\pi i\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}$$

which, using the Helmholtz equation becomes:

$$\left(\frac{2\pi}{\lambda}\right)^2(1-\alpha^2-\beta^2)\iint d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)e^{2\pi i\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}+$$
$$\iint d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}\frac{d^2}{dz^2}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)e^{2\pi i\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}=0$$

Since this integral equation is fulfilled for any distribution over the integration variables, the equation must also hold for the integrand, not just the integral, leaving:

$$\frac{d^2}{dz^2}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)+\left(\frac{2\pi}{\lambda}\right)^2\left(1-\alpha^2-\beta^2\right)A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)=0$$

Whose solution is:

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},0\right)e^{\frac{2\pi i}{\lambda}\sqrt{1-\alpha^2-\beta^2}z}$$

Physically this means that free space propagation just introduces a relative phase shift between interferences of various plane wave components.

The Fresnel approximation corresponds to the approximation

$$\sqrt{1-(\lambda f_x)^2-(\lambda f_y)^2}\approx 1-\frac{(\lambda f_x)^2}{2}-\frac{(\lambda f_y)^2}{2}$$

or to lowest order small angle diffraction.

Evanescent waves

Clearly, something is wrong with this picture when the spatial frequency is greater than $2\pi/\lambda$. In this case $\alpha^2+\beta^2>1$, which means that γ is imaginary. The solution in that case is not of a propagating wave, but rather of an exponentially decaying wave:

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},0\right)e^{-\frac{2\pi i}{\lambda}z\sqrt{\alpha^2+\beta^2-1}}$$

These waves do not propagate, and decay over a length scale of the inverse spatial frequency. All the information on these, which represents the fine detail of the aperture, is lost in the far field.

In fact, free space propagation in the far-field can be considered as a circular filter in spatial frequency space.

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},0\right)circ\left(\sqrt{\alpha^2+\beta^2}\right)e^{\frac{2\pi i}{\lambda}\sqrt{1-\alpha^2-\beta^2}}$$

The effect of a diffracting aperture on the ASPW

For a diffracting aperture, the transmitted signal is:

$$U_t(x,y)=U_0(x,y)t_a(x,y)$$

Since multiplication is real space is convolution in Fourier space, then:

$$A_t\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)=A_i\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\otimes T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)$$

$$T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)=\int\int t_a(x,y)e^{-2\pi i\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}dxdy$$

For an incoming plane wave, the outgoing angular spectrum is just the Fourier transform of the aperture. Thus, the smaller the aperture, the broader the angular distribution (wider angle diffraction).

<u>Fresnel diffraction of an amplitude grating (Talbot images).</u>

Let us consider again an amplitude grating, but now only in the Fresnel approximation, and using the ASPW approach. For simplicity lets consider an infinite grating in 1D.

We start with

$$t_a(\xi)=\left[\frac{1}{2}+\frac{m}{2}\cos(2\pi f_0\xi)\right]$$

The transfer function for any given component of the ASPW is:

$$H(\alpha,\beta)=circ\left(\sqrt{\alpha^2+\beta^2}\right)e^{\frac{2\pi i}{\lambda}\sqrt{1-\alpha^2-\beta^2}}$$

Such that for L=1/2πf0 we get:

$$H\left(\pm\frac{1}{L},0\right)\approx e^{-\frac{\pi i\lambda z}{L^2}}$$

Since at z=0 the Fourier transform is:

$$F\left\{\frac{1}{2}+\frac{m}{2}\cos(2\pi f_0\xi)\right\}=\frac{1}{2}\delta(f_x)+\frac{m}{4}\delta(f_x-f_0)+\frac{m}{4}\delta(f_x+f_0)$$

The at any z we get:

$$F\{U(x,y,z)\}=\frac{1}{2}\delta(f_x)+\frac{m}{4}\delta(f_x-f_0)e^{-j\frac{\pi\lambda z}{L^2}}+\frac{m}{4}\delta(f_x+f_0)e^{-j\frac{\pi\lambda z}{L^2}}$$

Which is simplified to:

$$U(x,y,z)=\frac{1}{2}+\frac{m}{2}\cos\left(\frac{2\pi x}{L}\right)e^{-j\frac{\pi\lambda z}{L^2}}$$

To yield the intensity distribution:

$$I(x,y,z)=\frac{1}{4}\left[1+2m\cos\left(\frac{2\pi x}{L}\right)\cos\left(\frac{\pi\lambda z}{L^2}\right)+m^2\cos^2\left(\frac{2\pi x}{L}\right)\right]$$

Now there are three special cases, where this expression get further simplified.

For $\frac{\pi\lambda z}{L^2}=2\pi n$:

$$I(x,y,z)=\frac{1}{4}\left[1+m\cos\left(\frac{2\pi x}{L}\right)\right]^2$$

Which is an exact image of the one at z=0.

For $\frac{\pi\lambda z}{L^2}=2\pi n+\pi$:

$$I(x,y,z)=\frac{1}{4}\left[1-m\cos\left(\frac{2\pi x}{L}\right)\right]^2$$

Which is an inverted image of the one at z=0.

For $\frac{\pi\lambda z}{L^2}=\pi n+\pi/2$:

$$I(x,y,z)=\frac{1}{4}\left[1+\frac{m^2}{2}+\frac{m^2}{2}\cos\left(\frac{4\pi x}{L}\right)\right]^2$$

Which is an amplitude grating at the doubled frequency.

This phenomenon is related to any revival in a periodic system. For the case of a general periodic system it holds only in the Fresnel approximation, since then the various frequency components propagate in integer multiples of the same phase factor.