Stratified Media

- A medium with stacked layers
- The main question here is how can we design such a stack to get a specific trans. spectrum?

- You saw in lectures
  \[ n_0, n_1, \quad E_{11} \quad \text{and} \quad \frac{dE_{11}}{dz} \]
  are continuous!

\[ \hat{H} = \nabla \times \vec{E} = \hat{k} \times \vec{E} \frac{dE}{dz} \]

Therefore, we define:

\[ \begin{pmatrix} E \\ F \end{pmatrix} = \begin{pmatrix} E' \\ \frac{dE}{dz} \end{pmatrix} \]

In a 1D case:

\[ E = E_+ e^{ikz} + E_- e^{-ikz} \]
\[ F = iKE_+ e^{ikz} - iKE_- e^{-ikz} \]
How does free prop. look?

At \( z = 0 \):

\[
E(0) = E_+ + E_- \quad \Rightarrow \quad 2i_k E_+ = i_k E_0 + F(0)
\]

\[
F(0) = i_k E_+ - i_k E_-
\]

Use in \( z \) to get:

\[
E(z) = \left( \frac{1}{2} E(0) + \frac{F(0)}{2i_k} \right) e^{ikz} + \left( \frac{1}{2} E(0) - \frac{F(0)}{2iK} \right) e^{-ikz}
\]

\[
= E(0) \cdot \cos(kz) + \frac{F_0}{k} \frac{\sin(kz)}{k}
\]

Similarly for \( F \):

\[
F = \cdot E(0) (-k \sin(kz)) + F_0 \cos(kz)
\]

In matrix form:

\[
\begin{pmatrix}
E(z) \\
F(z)
\end{pmatrix}
= \begin{pmatrix}
\cos(kz) & \frac{1}{k} \sin(kz) \\
-k \sin(kz) & \cos(kz)
\end{pmatrix}
\begin{pmatrix}
E(0) \\
F(0)
\end{pmatrix}
\]

\( N^{-1} \quad \text{Det} = 1 \)

\[
\begin{pmatrix}
E(0) \\
F(0)
\end{pmatrix}
= \begin{pmatrix}
\cos(kz) & -\frac{1}{k} \sin(kz) \\
K \sin(kz) & \cos(kz)
\end{pmatrix}
\begin{pmatrix}
E(z) \\
F(z)
\end{pmatrix}
\]
This is great since we can already propagate through the stack:

- The surfaces are cont.
- Each film adds another prop. matrix

Before we calculate such matrices, what can we do with it?

Let's take a general case

\[ E_0 e^{-ik_2 z} \quad \rightarrow \quad \begin{pmatrix} M \end{pmatrix} \quad \rightarrow \quad t e^{-ik_2 z} \]

\[ N_1 \quad \begin{pmatrix} M \end{pmatrix} \quad N_2 \]

How do we get \( r \)?

On the left: \( E = (1 + r) E_0 \quad \rightarrow \quad F = i k_1 (1 - 1) E_0 \)

\[ \begin{pmatrix} 1 + r \\ ik_1 (1 - 1) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} t \\ -i k_2 t \end{pmatrix} \]
And we can find expressions for \( r, t \) as a function of \( u, v \). We will do it soon for a particular case.

Going back to \( M \):

**Example:** single layer for anti-reflection

For destructive interference: \( kz = \frac{\pi}{2} \) \( (z = \frac{1}{2}) \)

\[
M = \begin{pmatrix} 0 & -\frac{1}{k} \\ k & 0 \end{pmatrix}
\]

For this case (or every case where \( n_{01} = n_{02} = 0 \))

\[
\Gamma = -\frac{1 - \frac{k_1 k_2}{k_2}}{1 + \frac{k_1 k_2}{k_2}}
\]

\[
k_1 = n_1 k_0 \quad k = n_2 k_0 \quad k_2 = n_2 k_0
\]

\[
\Gamma = -1 - \frac{n_1 n_2}{n_2^2} \frac{1}{1 + \frac{n_1 n_2}{n_2^2}}
\]

So for \( n = \sqrt{n_1 n_2} \) we get \( \Gamma = 0 \)
Two problems arise in this case:

1. $r = 0$ only for normal incidence and specific specific

2. Practically $n_1 = 1$, $n_2 \approx 1.5 \Rightarrow n \approx 1.2$
   But it's hard to come by such materials.
   
   $n_{\text{glass}} \approx 1.58$

This gets you from 4% (air $\rightarrow$ glass) to

$\approx 1.5\%$.

To do better you need a bigger stack.
Drude model and reflection from a metallic surface

We have seen for normal incidence light

\[ \Gamma = \frac{1 - N_0}{1 + N_0} \]

\( N \neq 1 \quad N_0 \)

\[ \frac{\mathbf{E}}{\mathbf{E}_0} \]

But we have looked only at dielectrics where

\[ \{ \begin{align*} n & \text{ is real} \\ n & > 1 \end{align*} \]

A common thing is using a metallic reflector

The refractive index of a metal

There are some free electrons in a metal.

There EOF is given by

\[ \frac{d^2 x}{dt^2} - R \frac{dx}{dt} = \frac{eE_0}{m} e^{i \omega_0 t} \]

acceleration  \quad \text{friction}  \quad \text{force by electric field}

Use:

\[ x(t) = x(\omega_0) e^{i \omega_0 t} \]

And get

\[ x(t) = \frac{eE_0 m}{-\omega^2 + i \omega} e^{i \omega_0 t} \]
What is the conductivity?

\[ \sigma E = \mathbf{J} = N_e v = N_e \frac{d}{dt} = \frac{\text{Ne}^2 E_0}{\mu} e^{+i\omega t} \cdot (+i\omega) \]

In Maxwell's eq.

\[ \nabla \times \mathbf{H} = \mathbf{J} + \varepsilon_0 \frac{d\mathbf{E}}{dt} \]

Assuming no extra dipoles

Going through the usual steps: \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \)

\[ \nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} \]

\[ -\nabla^2 \mathbf{E} = \frac{\partial \mathbf{J}}{\partial t} + \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

Taking \( \mathbf{E}(r,t) = \mathbf{E}(r) e^{-i\omega t} \)

In dielectrics \( \mathbf{J} \to 0 \) but in a metal this becomes important

\[ -\nabla^2 \mathbf{E}(r) = (-i\omega \sigma + \omega^2 \varepsilon_0) \mathbf{E}(r) \]

We get the wave eq. with an important correction for a conductor. For a plane wave \( \mathbf{E} = E_0 \mathbf{e} \)

\[ \nabla \cdot \left( \frac{\mathbf{k}^2}{\omega^2} \mathbf{E} \right) = \left( \varepsilon_0 + \frac{i\sigma(\omega)}{\omega} \right) \mathbf{E} \]

(keep)

complex dielectric function
Going back to the Drude model:

\[ \varepsilon = \frac{\varepsilon_0}{1 - i\omega \tau} \]

\[ \tau = \frac{1}{\gamma} \]

\[ \sigma_0 = \frac{Ne^2}{m} \]

Divide to image and Re part the dielectric func:

\[ \varepsilon_r = \varepsilon_0 + \frac{i\sigma_0}{\omega - i\omega \tau} = \] 

\[ = \varepsilon_0 \left( 1 - \frac{\sigma_0 \tau}{1 + \omega^2 \tau^2} \right) + \frac{i\sigma_0}{\omega (1 + \omega^2 \tau^2)} \]

For optical freq. \( \omega \sim 10^{15} \) Hz

\[ \varepsilon_r = \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \quad \omega_p \equiv \frac{Ne^2}{\varepsilon_0 m} \]

\( \omega_p \) in silver is at UV.

So in visible \( \omega < \omega_p \)

\[ \varepsilon_r < 0 \]

Remember

\[ \frac{n^2}{c^2} \equiv \varepsilon \cdot \mu = \varepsilon_0 \mu_0 \cdot \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \]

\[ \sqrt{n} = \text{purely imaginary} \]
And what about reflections?

\[
R = \left| \frac{1 - n}{1 + n} \right|^2 = \left| \frac{1 - i\alpha}{1 + i\alpha} \right|^2 = \frac{4 + \alpha^2}{4 + \alpha^2} = 1
\]

And therefore a metal acts as a perfect reflector.

In practice:

Aluminum: Good for UV until 250nm
Not good for VIS

Silver: Most common
\sim 98\% in VIS and NIR

Gold: Good for NIR and FIR
Absorptive in most visible