Introduction to turbulence theory

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Contents

1	**Introduction to turbulence theory**	*page* 1
1.1	Introduction	1
1.2	Weak wave turbulence	3
1.3	Strong wave turbulence	8
1.4	Incompressible turbulence	12
1.5	Zero modes and anomalous scaling	18
1.6	Inverse cascades	24
1.7	Conclusion	30
References		40

1
Introduction to turbulence theory

The emphasis of this short course is on fundamental properties of developed turbulence, weak and strong. We shall be focused on the degree of universality and symmetries of the turbulent state. We shall see, in particular, which symmetries remain broken even when the symmetry-breaking factor goes to zero, and which symmetries, on the contrary, emerge in the state of developed turbulence.

1.1 Introduction

Turba is Latin for crowd and "turbulence" initially meant the disordered movements of large groups of people. Leonardo da Vinci was probably the first to apply the term to the random motion of fluids. In 20th century, the notion has been generalized to embrace far-from-equilibrium states in solids and plasma. We now define turbulence as a state of a physical system with many interacting degrees of freedom deviated far from equilibrium. This state is irregular both in time and in space and is accompanied by dissipation.

We consider here developed turbulence when the scale of the externally excited motions deviate substantially from the scales of the effectively dissipated ones. When fluid motion is excited on the scale L with the typical velocity V, developed turbulence takes place when the Reynolds number is large: $Re = VL/\nu \gg 1$. Here ν is the kinematic viscosity. At large Re, flow perturbations produced at the scale L have their viscous dissipation small compared to the nonlinear effects. Nonlinearity produces motions of smaller and smaller scales until viscous dissipation stops this at a scale much smaller than L so that there is a wide (so-called inertial) interval of scales where viscosity is negligible and nonlinearity dominates. Another example is the system of waves excited on a fluid surface by wind or moving bodies and in plasma and solids by external electromagnetic fields. The state of such system is called wave turbulence when the wavelength of the waves excited strongly differs from the wavelength of the waves that effectively dissipate. Nonlinear interaction excites waves in the interval of wavelengths (called transparency window or inertial interval) between the injection and dissipation scales.

Simultaneous existence of many modes calls for a statistical description based upon averaging either over regions of space or intervals of time. Here we focus on a single-time statistics of steady turbulence that is on the spatial structure of fluctuations in the inertial range. The basic question is that of universality: to what extent the statistics of such fluctuations is independent of the details of external forcing and internal friction and which features are common to different turbulent systems. This quest for universality is motivated by the hope of being able to distinguish general principles that govern far-from-equilibrium systems, similar in scope to the variational principles that govern thermal equilibrium.

Since we generally cannot solve the nonlinear equations that describe turbulence, we try to infer the general properties of turbulence statistics from symmetries or conservation laws. The conservation laws are broken by pumping and dissipation, but both factors do not act directly in the inertial interval. For example, in the incompressible turbulence, the kinetic energy is pumped by a (large-scale) external forcing and is dissipated by viscosity (at small scales). One may suggest that the kinetic energy is transferred from large to small scales in a cascade-like process i.e. the energy flows throughout the inertial interval of scales. The cascade idea (suggested by Richardson in 1921) explains the basic macroscopic manifestation of turbulence: the rate of dissipation of the dynamical integral of motion has a finite limit when the dissipation coefficient tends to zero. For example, the mean rate of the viscous energy dissipation does not depend on viscosity at large Reynolds numbers. Intuitively, one can imagine turbulence cascade as a pipe in wavenumber space that carries energy. As viscosity gets smaller the pipe gets longer but the flux it carries does not change. Formally, that means that the symmetry of the inviscid equation (here, time-reversal invariance) is broken by the presence of the viscous term, even though the latter might have been expected to become negligible in the limit $Re \to \infty$.

One can use the cascade idea to guess the scaling properties of turbulence. For incompressible fluid, the energy flux (per unit mass) ϵ through the given scale r can be estimated via the velocity difference δv measured at that scale as the energy $(\delta v)^2$ divided by the time $r/\delta v$. That gives $(\delta v)^3 \sim \epsilon r$. Of course, δv is a fluctuating quantity and we ought to make statements on its moments or probability distribution $\mathcal{P}(\delta v, r)$. Energy flux constancy fixes the third moment, $\langle(\delta v)^3\rangle \sim \epsilon r$. It is a natural wish to have turbulence scale invariant in the inertial interval so that $\mathcal{P}(\delta v, r) = (\delta v)^{-1} f[\delta v/(\epsilon r)^{1/3}]$ is expressed via the dimensionless function f of a single variable. Initially, Kolmogorov made even stronger wish for the function f to be universal (i.e. pumping independent). Nature is under no obligation to grant wishes of even great scientists, particularly when it is in a state of turbulence. After hearing Kolmogorov talk, Landau remarked that the moments different from third are nonlinear functions of the input rate and must be sensitive to the precise statistics of the pumping. As we show below, the cascade idea can indeed be turned into an exact relation for the simultaneous correlation function which expresses the flux (third or fourth-order moment depending on the degree of nonlinearity). The relation requires the mean flux of the respective integral of motion to be constant across the inertial interval of scales. We shall see that flux constancy determines the system completely only for a weakly nonlinear system (where the statistics is close to Gaussian i.e. not only scale invariant but also perfectly universal). To describe an entire turbulence statistics of strongly interacting systems, one has to solve problems on a case-by-case basis with most cases still unsolved. Particularly difficult (and interesting) are the cases when not only universality but also scale invariance is broken so that knowledge of the flux does not allow one to predict even the order of magnitude of high moments. We describe the new concept of statistical integrals of motion which allows for the description of system with broken scale invariance. We also describe situations when not only scale invariance is restored but a wider conformal invariance takes place in the inertial interval.

1.2 Weak wave turbulence

It is easiest to start from a weakly nonlinear system. Such is a system of small-amplitude waves. Examples include waves on the water surface, waves in plasma with and without magnetic field, spin waves in magnetics etc. We assume spatial homogeneity and denote a_k the amplitude of the wave with the wavevector $\mathbf{k}$. Considering for a moment wave system as closed (that is without external pumping and dissipation) one can describe it as a Hamiltonian system using wave amplitudes as normal canonical variables — see, for instance, the monograph Zakharov et al 1992. At small amplitudes, the Hamiltonian can be written as an expansion over a_k, where the second-order term describes non-interacting waves and high-order terms determine the interaction†:

$$H = \int \omega_k |a_k|^2 \, d\mathbf{k} \tag{1.1}$$
$$+ \int \Big(V_{123} a_1 a_2^* a_3^* + c.c. \Big) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 + \mathrm{O}(a^4) \, .$$

The dispersion law ω_k describes wave propagation, $V_{123} = V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is the interaction vertex and c.c. means complex conjugation. In the Hamiltonian expansion, we presume every subsequent term smaller than the previous one, in particular, $\xi_k = |V_{kkk} a_k| k^d / \omega_k \ll 1$ — wave turbulence that satisfies that condition is called weak turbulence. Here d is the space dimensionality.

The dynamic equation which accounts for pumping, damping, wave propagation and interaction has the following form:

$$\partial a_k / \partial t = -i \delta H / \delta a_k^* + f_k(t) - \gamma_k a_k \; . \tag{1.2}$$

Here γ_k is the decrement of linear damping and f_k describes pumping. For a linear system, a_k is different from zero only in the regions of $\mathbf{k}$-space where f_k is nonzero. Nonlinear interaction provides for wave excitation outside pumping regions.

It is likely that the statistics of the weak turbulence at $k \gg k_f$ is close to Gaussian for wide classes of pumping statistics. When the forcing $f_k(t)$ is Gaussian then the statistics of $a_k(t)$ is close to Gaussian as long as nonlinearity is weak. However, in most cases in nature and in the lab, the force is not Gaussian even though its amplitude can be small. It is an open problem to find out under what conditions the wave field is close to Gaussian with $\langle a_k(0) a_{k'}^*(t) \rangle = n_k \exp(-\imath \omega_k t) \delta(\mathbf{k} + \mathbf{k}')$. This problem actually breaks into two parts. The first one is to solve the linear equation for the waves in the spectral interval of pumping and formulate the criteria on the forcing that guarantee that the cumulants are small for $a_k(t) = \exp(-\imath \omega_k t - \gamma_k t) \int_0^t f_k(t') \exp(\imath \omega_k t + \gamma_k t) \, dt'$. The second part is more interesting: even when the pumping-related waves are non-Gaussian, it may well be that as we go in k-space away from pumping (into the inertial interval) the field $a_k(t)$ is getting more Gaussian. Unless we indeed show that, most of the applications of the weak turbulence theory described in this section are in doubt. See also Choi et al 2005.

We consider here and below a pumping by a Gaussian random force statistically isotropic and homogeneous in space, and white in time:

$$\langle f_k(t) f_{k'}^*(t') \rangle = F(k) \delta(\mathbf{k} + \mathbf{k}') \delta(t - t') \; . \tag{1.3}$$

† For example, for sound one expands the (kinetic plus internal) energy density $\rho v^2/2 + E(\rho)$ assuming $v \ll c$ and using $\mathbf{v}_k = \mathbf{k}(a_k - a_{-k}^*)(ck/2\rho_0)^{1/2}$, $\rho_k = k(a_k + a_{-k}^*)(\rho_0/2ck)^{1/2}$.

Angular brackets mean spatial average. We assume $\gamma_k \ll \omega_k$ (for waves to be well defined) and that $F(k)$ is nonzero only around some k_f.

As long as we assume the statistics of the wave system to be close to Gaussian, it is completely determined by the pair correlation function. Here we are interesting in the spatial structure which is described by the single-time pair correlation function $\langle a_k(t)a^*_{k'}(t)\rangle = n_k(t)\delta(\mathbf{k}+\mathbf{k}')$. Since the dynamic equation (1.2) contains a quadratic nonlinearity then the time derivative of the second moment, $\partial n_k/\partial t$, is expressed via the third one, the time derivative of the third moment ix expressed via the fourth one etc; that is the statistical description in terms of moments encounters the closure problem. Fortunately, weak turbulence in the inertial interval is expected to have the statistics close to Gaussian so one can express the fourth moment as the product of two second ones. As a result one gets a closed equation (see e.g. Zakharov *et al* 1992):

$$\frac{\partial n_k}{\partial t} = F_k - \gamma_k n_k + I^{(3)}_k\,, \qquad I^{(3)}_k = \int (U_{k12} - U_{1k2} - U_{2k1})\, d\mathbf{k}_1 d\mathbf{k}_2\,, \tag{1.4}$$

$$U_{123} = \pi[n_2 n_3 - n_1(n_2+n_3)]|V_{123}|^2\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_1-\omega_2-\omega_3)\ .$$

It is called kinetic equation for waves. The collision integral $I^{(3)}_k$ results from the cubic terms in the Hamiltonian i.e. from the quadratic terms in the equations for amplitudes. It can be *interpreted* as describing three-wave interactions: the first term in the integral (1.4) corresponds to a decay of a given wave while the second and third ones to a confluence with other wave.

The inverse time of nonlinear interaction at a given k can be estimated from (1.4) as $|V(k,k,k)|^2 n(k)k^d/\omega(k)$. We define the dissipation wavenumber k_d as such where this inverse time is comparable to $\gamma(k_d)$ and assume nonlinearity to dominate over dissipation at $k \ll k_d$. As has been noted, wave turbulence appears when there is a wide (inertial) interval of scales where both pumping and damping are negligible, which requires $k_d \gg k_f$, the condition analogous to $Re \gg 1$. This is schematically shown in Fig. 1.

The presence of frequency delta-function in $I^{(3)}_k$ means that in the first order of perturbation theory in wave interaction we account only for resonant processes which conserve the quadratic part of the energy $E = \int \omega_k n_k\, d\mathbf{k} = \int E_k dk$. For the cascade picture to be valid, the collision integral has to converge in the inertial interval which means that energy exchange is small between motions of vastly different scales, the property called interaction locality in k-space (see the exercise 1.1 below). Consider now a statistical steady state established under the action of pumping and dissipation. Let us multiply (1.4) by ω_k and integrate it over either interior or exterior of the ball with radius k. Taking $k_f \ll k \ll k_d$, one sees that the energy flux

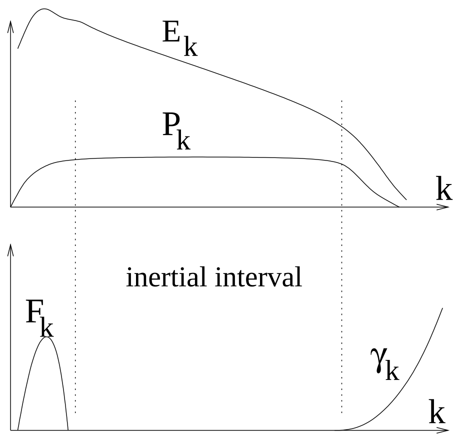

Fig. 1.1. A schematic picture of the cascade.

through any spherical surface (Ω is a solid angle), is constant in the inertial interval and is equal to the energy production/dissipation rate ϵ:

$$P_k = \int_0^k k^{d-1} dk \int d\Omega \, \omega_k I_k^{(3)} = \int \omega_k F_k \, d\mathbf{k} = \int \gamma_k E_k \, dk = \epsilon \; . \qquad (1.5)$$

That (integral) equation determines n_k. Let us assume now that the medium (characterized by the Hamiltonian coefficients) can be considered isotropic at the scales in the inertial interval. In addition, for scales much larger or much smaller than a typical scale (like Debye radius in plasma or the depth of the water) the medium is usually scale invariant: $\omega(k) = ck^\alpha$ and $|V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)|^2 = V_0^2 k^{2m} \chi(\mathbf{k}_1/k, \mathbf{k}_2/k)$ with $\chi \simeq 1$. Remind that we presumed statistically isotropic force. In this case, the pair correlation function that describes a steady cascade is also isotropic and scale invariant:

$$n_k \simeq \epsilon^{1/2} V_0^{-1} k^{-m-d} \; . \qquad (1.6)$$

One can show that (1.6), called Zakharov spectrum, turns $I_k^{(3)}$ into zero (see the exercise 1.1 below and Zakharov *et al* 1992).

If the dispersion relation $\omega(k)$ does not allow for the resonance condition $\omega(k_1) + \omega(k_2) = \omega(|\mathbf{k}_1 + \mathbf{k}_2|)$ then the three-wave collision integral is zero and one has to account for four-wave scattering which is always resonant, that is whatever $\omega(k)$ one can always find four wavevectors that satisfy $\omega(k_1) + \omega(k_2) = \omega(k_3) + \omega(k_4)$ and $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$. The collision integral that describes scattering,

$$I_k^{(4)} = \frac{\pi}{2}\int |T_{k123}|^2[n_2n_3(n_1+n_k) - n_1n_k(n_2+n_3)]\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$$
$$\times\delta(\omega_k+\omega_1-\omega_2-\omega_2)\,d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\ , \tag{1.7}$$

conserves the energy and the wave action $N = \int n_k\,d\mathbf{k}$ (the number of waves). Pumping generally provides for an input of both E and N. If there are two inertial intervals (at $k \gg k_f$ and $k \ll k_f$), then there should be two cascades. Indeed, if $\omega(k)$ grows with k then absorbing finite amount of E at $k_d \to \infty$ corresponds to an absorption of an infinitely small N. It is thus clear that the flux of N has to go in opposite direction that is to large scales. A so-called inverse cascade with the constant flux of N can thus be realized at $k \ll k_f$. A sink at small k can be provided by wall friction in the container or by long waves leaving the turbulent region in open spaces (like in sea storms). Two-cascade picture can be illustrated by a simple example with a wave source at $\omega = \omega_2$ generating N_2 waves per unit time and two sinks at $\omega = \omega_1$ and $\omega = \omega_3$ absorbing respectively N_1 and N_3. In a steady state, $N_2 = N_1 + N_3$ and $\omega_2 N_2 = \omega_1 N_1 + \omega_3 N_3$, which gives

$$N_1 = N_2\frac{\omega_3-\omega_2}{\omega_3-\omega_1}, \qquad N_3 = N_2\frac{\omega_2-\omega_1}{\omega_3-\omega_1}\ .$$

At a sufficiently large left inertial interval (when $\omega_1 \ll \omega_2 < \omega_3$), the whole energy is absorbed by the right sink: $\omega_2 N_2 \approx \omega_3 N_3$. Similarly, at $\omega_3 \gg \omega_2 > \omega_1$, we have $N_1 \approx N_2$, i.e. the wave action is absorbed at small ω.

The collision integral $I_k^{(3)}$ involved products of two n_k so that flux constancy required $E_k \propto \epsilon^{1/2}$ while for the four-wave case $I_k^{(4)} \propto n^3$ gives $E_k \propto \epsilon^{1/3}$. In many cases (when there is a complete self-similarity) that knowledge is sufficient to obtain the scaling of E_k from a dimensional reasoning without actually calculating V and T. For example, short waves on a deep water are characterized by the surface tension σ and density ρ so the dispersion relation must be $\omega_k \sim \sqrt{\sigma k^3/\rho}$ which allows for the three-wave resonance and thus $E_k \sim \epsilon^{1/2}(\rho\sigma)^{1/4}k^{-7/4}$. For long waves on a deep water, the surface-restoring force is dominated by gravity so that the gravity acceleration g replaces σ as a defining parameter and $\omega_k \sim \sqrt{gk}$. Such dispersion law does not allow for the three-wave resonance so that the dominant interaction is four-wave scattering which permits two cascades. The direct energy cascade corresponds to $E_k \sim \epsilon^{1/3}\rho^{2/3}g^{1/2}k^{-5/2}$ or $E_\omega = E_k dk/d\omega \sim \epsilon^{1/3}\rho^{2/3}g^2\omega^{-4}$. The inverse cascade carries the flux of N which we denote Q, it has the dimensionality $[Q] = [\epsilon]/[\omega_k]$ and corresponds to $E_k \sim Q^{1/3}\rho^{2/3}g^{2/3}k^{-7/3}$.

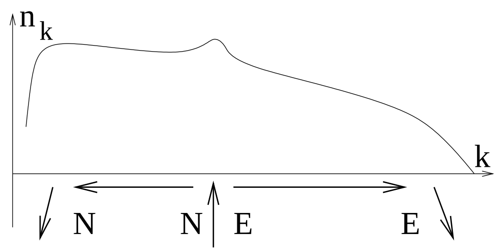

Fig. 1.2. Two cascades under four-wave interaction.

Under a weakly anisotropic pumping, stationary spectrum acquires a small stationary weakly anisotropic correction $\delta n(\mathbf{k})$ such that $\delta n(\mathbf{k})/\mathbf{n_0}(\mathbf{k}) \propto \omega(\mathbf{k})/\mathbf{k}$ (see exercise 2.2). The degree of anisotropy increases with k for waves with the decay dispersion law. That is the spectrum of the weak turbulence generated by weakly anisotropic pumping is getting more anisotropic as we go into the inertial interval of scales. We see that the conservation of the second integral (momentum) can lead to the non-restoration of symmetry (isotropy) in the inertial interval.

Since the statistics of weak turbulence is near Gaussian, it is completely determined by the pair correlation function, which is in turn determined by the respective flux (or fluxes). We thus conclude that weak turbulence is perfectly universal: deep in the inertial interval it "forgets" all the properties of pumping except the flux value.

1.3 Strong wave turbulence

Weak turbulence theory breaks down when the wave amplitudes are large enough (so that $\xi_k \geq 1$). We need special consideration also in the particular case of the linear (acoustic) dispersion relation $\omega(k) = ck$ for arbitrarily small amplitudes (as long as the Reynolds number remains large). Indeed, there is no dispersion of wave velocity for acoustic waves so that waves moving at the same direction interact strongly and produce shock waves when viscosity is small. Formally, there is a singularity due to coinciding arguments of delta-functions in (1.4) (and in the higher terms of perturbation expansion for $\partial n_k/\partial t$), which is thus invalid at however small amplitudes. Still, some features of the statistics of acoustic turbulence can be understood even without a closed description. We discuss that in a one-dimensional case which pertains, for instance, to sound propagating in long pipes. Since weak shocks are stable with respect to transversal perturbations (Landau and Lifshits 1987), quasi one-dimensional perturbations may propagate in 2d and 3d as well. In the reference moving with the sound velocity, the weakly compressible 1d flows ($u \ll c$) are described by the Burgers equation (Landau and Lifshits 1987, E et al 1997, Frisch and Bec 2001):

$$u_t + uu_x - \nu u_{xx} = 0 \ . \tag{1.8}$$

Burgers equation has a propagating shock-wave solution $u = 2v\{1+\exp[v(x-vt)/\nu]\}^{-1}$ with the energy dissipation rate $\nu \int u_x^2\, dx = 2v^3/3$ independent of ν. The shock width ν/v is a dissipative scale and we consider acoustic turbulence produced by a pumping correlated on much larger scales (for example, pumping a pipe from one end by frequencies much less than cv/ν). After some time, it will develop shocks at random positions. Here we consider the single-time statistics of the Galilean invariant velocity difference $\delta u(x,t) = u(x,t) - u(0,t)$. The moments of δu are called structure functions $S_n(x,t) = \langle [u(x,t) - u(0,t)]^n \rangle$. Quadratic nonlinearity relates the time derivative of the second moment to the third one:

$$\frac{\partial S_2}{\partial t} = -\frac{\partial S_3}{3\partial x} - 4\epsilon + \nu \frac{\partial^2 S_2}{\partial x^2} \ . \tag{1.9}$$

Here $\epsilon = \nu \langle u_x^2 \rangle$ is the mean energy dissipation rate. Equation (1.9) describes both a free decay (then ϵ depends on t) and the case of a permanently acting pumping which generates turbulence statistically steady at scales less than the pumping length. In the first case, $\partial S_2/\partial t \simeq S_2 u/L \ll \epsilon \simeq u^3/L$ (where L is a typical distance between shocks) while in the second case $\partial S_2/\partial t = 0$ so that $S_3 = 12\epsilon x + \nu \partial S_2/\partial x$.

Consider now limit $\nu \to 0$ at fixed x (and t for decaying turbulence). Shock dissipation provides for a finite limit of ϵ at $\nu \to 0$ then

$$S_3 = -12\epsilon x \ . \tag{1.10}$$

This formula is a direct analog of (1.5). Indeed, the Fourier transform of (1.9) describes the energy density $E_k = \langle |u_k|^2 \rangle / 2$ which satisfies the equation $(\partial_t - \nu k^2) E_k = -\partial P_k/\partial k$ where the k-space flux

$$P_k = \int_0^k dk' \int_{-\infty}^{\infty} dx S_3(x) k' \sin(k'x)/24 \ .$$

Note that the shock dissipation dissipation rate $2v^3/3$ gives the mean dissipation rate per unit length $\epsilon = 2v^3/3L$ so that (1.10) corresponds to $S_3 = \langle \delta u^3 \rangle = (2v)^3 x/L$.

It is thus the flux constancy that fixes $S_3(x)$ which is universal that is determined solely by ϵ and depends neither on the initial statistics for decay nor on the pumping for steady turbulence. On the contrary, other structure functions $S_n(x)$ are not given by $(\epsilon x)^{n/3}$. Indeed, the scaling of the structure functions can be readily understood for any dilute set of shocks (that is when shocks do not cluster in space) which seems to be the case both for smooth initial conditions and large-scale pumping in Burgers turbulence. In this case, $S_n(x) \sim C_n|x|^n + C'_n|x|$ where the first term comes from the regular (smooth) parts of the velocity (the right x-interval in Fig. 1.3) while the second comes from $O(x)$ probability to have a shock in the interval x. The scaling exponents, $\xi_n = d\ln S_n/d\ln x$, thus behave as follows: $\xi_n = n$ for $n \leq 1$ and $\xi_n = 1$ for $n > 1$. That means that the probability density

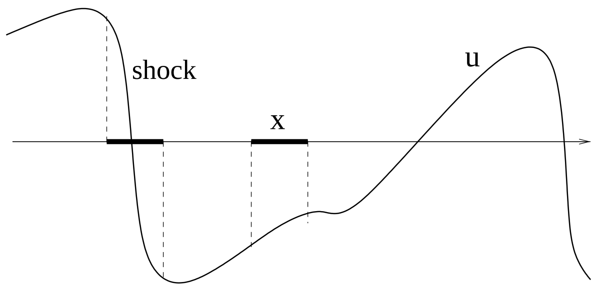

Fig. 1.3. Typical velocity profile in Burgers turbulence.

function (PDF) of the velocity difference in the inertial interval $P(\delta u, x)$ is not scale-invariant, that is the function of the re-scaled velocity difference $\delta u/x^a$ cannot be made scale-independent for any a. Simple bi-modal nature of Burgers turbulence (shocks and smooth parts) means that the PDF is actually determined by two (nonuniversal) functions, each depending of a single argument: $P(\delta u, x) = \delta u^{-1} f_1(\delta u/x) + x f_2(\delta u/u_{rms})$. Breakdown of scale invariance means that the low-order moments decrease faster than the high-order ones as one goes to smaller scales, i.e. the smaller the scale the more probable are large fluctuations. In other words, the level of fluctuations increases with the resolution. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when $x/L \to 0$.

As an alternative to the description in terms of structures (shocks), one can relate the anomalous scaling in Burgers turbulence to the additional integrals of motion. Indeed, the integrals $E_n = \int u^{2n}\, dx/2$ are all conserved by the inviscid Burgers equation. Any shock dissipates the finite amount of E_n at the limit $\nu \to 0$ so that similarly to (1.10) one denotes $\langle \dot{E}_n \rangle = \epsilon_n$ and obtains $S_{2n+1} = -4(2n+1)\epsilon_n x/(2n-1)$ for integer n. We thus conclude that the statistics of velocity differences in the inertial interval depends on the infinitely many pumping-related parameters, the fluxes of all dynamical integrals of motion.

Note that $S_2(x) \propto |x|$ corresponds to $E(k) \propto k^{-2}$, since every shock gives $u_k \propto 1/k$ at $k \ll v/\nu$, that is the energy spectrum is determined by the type of structures (shocks) rather than by energy flux constancy. That is Burgers turbulence demonstrates the universality of a different kind: the type of structures that dominate turbulence (here, shocks) is universal while the statistics of their amplitudes depends on pumping. Similar ideas were suggested for other types of strong wave turbulence assuming them to be dominated by different structures. Weak wave turbulence, being a set of weakly interacting plane waves, can be studied uniformly for different systems (Zakharov et al 1992). On the contrary, when nonlinearity is strong, different structures appear. Broadly, one distinguishes conservative structures (like solitons and vortices) from dissipative structures which usually appear as a result of finite-time singularity of the non-dissipative equations (like shocks, light self-focussing or wave collapse).

For example, an envelope of a spectrally narrow wave packets is described by the Nonlinear Schrödinger Equation ,

$$i\Psi_t + \Delta\Psi + T|\Psi|^2\Psi = 0 \ . \tag{1.11}$$

This equation also describes Bose-Einstein condensation (then it is usually called Gross-Pitaevsky equation). Weak turbulence is determined by $|T|^2$ and is the same both for $T < 0$ (wave repulsion) and $T > 0$ (wave attraction). Inverse cascade tends to produce a uniform condensate $\Psi(k=0) = A$. At high levels of nonlinearity, different signs of T correspond to dramatically different physics. At $T < 0$ the condensate is stable, it renormalizes the linear dispersion relation from $\omega_k = k^2$ to the Bogolyubov form $\omega_k^2 = k^4 - 2TA^2k^2$. That dispersion relation is close to acoustic at small k, it allows for three-wave interactions. The resulting over-condensate turbulence is a mixture of phonons, solitons, kinks and vortices, I shall comment briefly on its properties at the end of the course. On the contrary, the condensate and sufficiently long waves are unstable at $T > 0$; that instability leads to wave collapse at $d = 2, 3$ with the energy being fast transferred from large to small scales where it dissipates (Dyachenko et al 1992). No analytic theory is yet available for such strong turbulence.

Nonlinearity parameter $\xi(k)$ generally depends on k so that there may exist weakly turbulent cascade until some k_* where $\xi(k_*) \sim 1$ and strong turbulence beyond this wavenumber, that is weak and strong turbulence can coexist in the same system. Presuming that some mechanism (for instance, wave breaking) prevents appearance of wave amplitudes that correspond to $\xi_k \gg 1$, one may suggest that some cases of strong turbulence correspond to the balance between dispersion and nonlinearity local in k-space so that $\xi(k)$ =const throughout its domain in k-space. That would correspond to the spectrum $E_k \sim \omega_k^3 k^{-d}/|V_{kkk}|^2$ which is ultimately universal that is independent even of the flux (only the boundary k_* depends on the flux). For gravity waves on a water surface, this gives $E_k \sim \rho g k^{-3}$ or $E_\omega \sim \rho g^3 \omega^{-5}$. That spectrum was presumed to be due to wave profile having cusps (another type of dissipative structure leading to whitecaps in stormy sea, Phillips 1977). However, cusp passing through a point corresponds to the second time-derivative of surface elevation to be proportional to delta function which gives $E_\omega \propto \omega^{-4}$ (the same ω-dependence as for the weak turbulence obtained in Sec. 1.2), while $E_k \propto k^{-3}$ is a spectrum of long parallel cusps, in both cases, the magnitude of the spectrum is not universal but is determined by the density of cusps (Kuznetsov 2004). It is unclear if flux-independent spectra are realized.

1.4 Incompressible turbulence

Incompressible fluid flow is described by the Navier-Stokes equation

$$\partial_t \mathbf{v}(\mathbf{r},t) + \mathbf{v}(\mathbf{r},t)\cdot\nabla\mathbf{v}(\mathbf{r},t) - \nu\nabla^2\mathbf{v}(\mathbf{r},t) = -\nabla p(\mathbf{r},t)\,, \quad \mathrm{div}\,\mathbf{v} = 0\,. \tag{1.12}$$

See Lecture 1 of the Gawędzki course for more details on this equation. We are again interested in the structure functions $S_n(\mathbf{r},t) = \langle[(\mathbf{v}(\mathbf{r},t) - \mathbf{v}(0,t))\cdot\mathbf{r}/r]^n\rangle$ and consider distance r smaller than the force correlation scale for a steady case and smaller than the size of turbulent region for a decay case.

3d turbulence. We treat first the three-dimensional case. Similar to (1.9), one can derive the Karman-Howarth relation between S_2 and S_3 (see Landau and Lifshits 1987):

$$\frac{\partial S_2}{\partial t} = -\frac{1}{3r^4}\frac{\partial}{\partial r}(r^4 S_3) + \frac{4\epsilon}{3} + \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right)\,. \tag{1.13}$$

Here $\epsilon = \nu\langle(\nabla\mathbf{v})^2\rangle$ is the mean energy dissipation rate. Neglecting time derivative (which is zero in a steady state and small comparing to ϵ for decaying turbulence) one can multiply (1.13) by r^4 and integrate: $S_3(r) = -4\epsilon r/5 + 6\nu dS_2(r)/dr$. Kolmogorov considered the limit $\nu \to 0$ for fixed r and *assumed* nonzero limit for ϵ which gives the so-called 4/5 law (Kolmogorov 1941, Landau and Lifshits 1987, Frisch 1995):

$$S_3 = -\frac{4}{5}\,\epsilon\, r\ . \tag{1.14}$$

Similar to (1.5,1.10), this relation means that the kinetic energy has a constant flux in the inertial interval of scales (the viscous scale η is defined by $\nu S_2(\eta) \simeq \epsilon\eta^2$). Let us stress that this flux relation is built upon the assumption that the energy dissipation rate ϵ has a nonzero limit at vanishing viscosity. Since the input rate can be independent of viscosity, this is the assumption needed for an existence of a steady state at the limit: no matter how small the viscosity, or how high the Reynolds number, or how extensive the scale-range participating in the energy cascade, the energy flux is expected to remain equal to that injected at the stirring scale. Unlike compressible (Burgers) turbulence, here we do not know the form of the specific singular structures that are supposed to provide non-vanishing dissipation in the inviscid limit (as shocks waves do). Experimental data show, however, that the dissipation rate is indeed independent of the Reynolds number when $Re \gg 1$. Historically, persistence of the viscous dissipation in the inviscid limit (both in compressible and incompressible turbulence) is the first example of what is now called "anomaly" in theoretical physics: a symmetry of the equation (here, time-reversal invariance) remains broken even as the symmetry-breaking factor (viscosity) becomes vanishingly small (see e.g. Falkovich and Sreenivasan 2006). If one screens a movie of steady turbulence backwards, we can tell that something is indeed wrong!

The law (1.14) shows that the third-order moment is universal, i.e. it does not depend on the details of the turbulence production but is determined solely by the mean energy dissipation rate. The rest of the structure functions have never been derived. Kolmogorov (1941) and also Heisenberg, von Weizsacker and Onsager *presumed* the pair correlation function to be determined only by ϵ and r which would give $S_2(r) \sim (\epsilon r)^{2/3}$ and the energy spectrum $E_k \sim \epsilon^{2/3}k^{-5/3}$. Experiments suggest that $\zeta_n = d\ln S_n/d\ln r$ lie on a smooth concave curve sketched in Fig. 1.4. While ζ_2 is close to $2/3$ it has to be a bit larger because experiments show that the slope at zero $d\zeta_n/dn$ is larger than $1/3$ while $\zeta(3) = 1$ in agreement with (1.14). Like in Burgers, the PDF of velocity differences in the inertial interval is not scale invariant in the 3d incompressible turbulence. So far, nobody was able to find an explicit relation between the anomalous scaling for 3d Navier-Stokes turbulence and either structures or additional integrals of motion.

While not exact, the Kolomogorov's approximation $S_2(\eta) \simeq (\epsilon\eta)^{2/3}$ can be used to estimate the viscous scale: $\eta \simeq LRe^{-3/4}$. The number of degrees

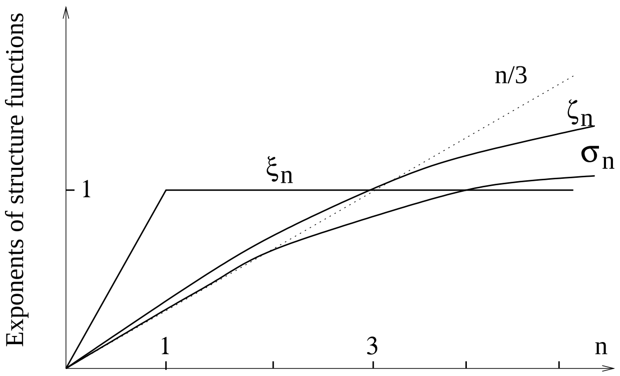

Fig. 1.4. The scaling exponents of the structure functions ξ_n for Burgers, ζ_n for 3d Navier-Stokes and σ_n for the passive scalar. The dotted straight line is $n/3$.

of freedom involved into 3d incompressible turbulence can thus be roughly estimated as $N \sim (L/\eta)^3 \sim Re^{9/4}$. That means, in particular, that detailed numerical simulation of water or oil pipe flows ($Re \sim 10^4 \div 10^7$) or turbulent cloud ($Re \sim 10^6 \div 10^9$) is beyond the reach of today (and possibly tomorrow) computers. To calculate correctly at least the large-scale part of the flow, it is desirable to have some theoretical model to parameterize the small-scale motions. Here, the main obstacle is our lack of qualitative understanding and quantitative description of how turbulence statistics changes with the scale. This breakdown of scale invariance in the inertial range is another example of anomaly (effect of pumping scale does not disappear even at the limit $r/L \to 0$). Such an anomalous (or multi-fractal) scaling, is an important feature of turbulence, and sets it apart from the usual critical phenomena: one needs to work out the behavior of moments of each order independently rather than get it from dimensional analysis. Anomalous scaling in turbulence is such that $\zeta_{2n} < n\zeta_2$ so that S_{2n}/S_2^n for $n > 2$ increases as $r \to 0$. The relative growth of high moments means that strong fluctuations become more probable as the scales become smaller. Its practical importance is that it limits our ability to produce realistic models for small-scale turbulence.

Since we know neither the structures nor the extra conservation laws that are responsible for an anomalous scaling in the 3d incompressible turbulence, then, to get some qualitative understanding of this very complicated problem, we now pass to another (no less complicated) problem of 2d turbulence. That latter problem will motivate us to consider passive scalar turbulence, which will, in particular, teach us a new concept of statistical conservation laws that will shed some light on 3d turbulence too.

2d Turbulence. Large-scale motions in shallow fluid can be approximately considered two-dimensional. When the velocities of such motions are much smaller than the velocities of the surface waves and the velocity of sound, such flows can be considered incompressible. Their description is important for understanding atmospheric and oceanic turbulence at the scales larger than the atmosphere height and the ocean depth. Vorticity $\omega = curl\, \mathbf{v}$ is a scalar in a two-dimensional flow. It is advected by the velocity field and dissipated by viscosity. Taking *curl* of the Navier-Stokes equation one gets

$$d\omega/dt = \partial_t \omega + (\mathbf{v} \cdot \nabla)\omega = \nu \nabla^2 \omega \ . \tag{1.15}$$

Two-dimensional incompressible inviscid flow just transports vorticity from place to place and thus conserves spatial averages of any function of vorticity, $\Omega_n \equiv \int \omega^n d\mathbf{r}$. In particular, we now have the second quadratic inviscid invariant (in addition to energy) which is called enstrophy: $\Omega_2 = \int \omega^2 \, d\mathbf{r}$. Since the spectral density of the energy is $|\mathbf{v}_k|^2/2$ while that of the enstrophy is $|\mathbf{k} \times \mathbf{v}_k|^2$ then (similarly to the cascades of E and N in wave turbulence under four-wave interaction) one expects that the direct cascade (towards large k) is that of enstrophy while the inverse cascade is that of energy, as was suggested by Kraichnan (1967). What about other Ω_n? The intuition developed so far might suggest that the infinity of dynamical conservation laws must bring about anomalous scaling. As we shall see, turbulence never fails to defy intuition.

Passive Scalar Turbulence. Before discussing vorticity statistics in two-dimensional turbulence, we describe a similar yet somewhat simpler problem of passive scalar turbulence which allows one to introduce the necessary notions of Lagrangian description of the fluid flow. Consider a scalar quantity $\theta(\mathbf{r}, t)$ which is subject to molecular diffusion and advection by the fluid flow but has no back influence on the velocity (i.e. passive):

$$d\theta/dt = \partial_t \theta + (\mathbf{v} \cdot \nabla)\theta = \kappa \nabla^2 \theta \ . \tag{1.16}$$

Here κ is molecular diffusivity. The examples of passive scalar are smoke in the air, salinity in the water and temperature when one can neglect thermal convection. Without viscosity and diffusion, ω and θ behave in the same way in the same 2d flow — they are both Lagrangian invariants satisfying $d\omega/dt = d\theta/dt = 0$. Note however that vorticity is related to velocity while the passive scalar is not.

Let us now consider passive scalar turbulence. For that we add random source of fluctuations φ:

$$\partial_t\theta + (\mathbf{v}\cdot\nabla)\theta = \kappa\nabla^2\theta + \varphi \ . \tag{1.17}$$

If the source φ produces the fluctuations of θ on some scale L then the inhomogeneous velocity field stretches, contracts and folds the field θ producing progressively smaller and smaller scales — this is the mechanism of the scalar cascade. If the rms velocity gradient is Λ then molecular diffusion is substantial at the scales less than the diffusion scale $r_d = \sqrt{\kappa/\Lambda}$. For scalar turbulence, the ratio $Pe = L/r_d$, called Peclet number, plays the role of the Reynolds number. When $Pe \gg 1$, there is an inertial interval with a constant flux of θ^2:

$$\langle(\mathbf{v}_1\cdot\nabla_1 + \mathbf{v}_2\cdot\nabla_2)\theta_1\theta_2\rangle = 2P \ , \tag{1.18}$$

where $P = \kappa\langle(\nabla\theta)^2\rangle = \langle\varphi\theta\rangle$ and subscripts denote the spatial points. In considering the passive scalar problem, the velocity statistics is presumed to be given. Still, the correlation function (1.18) mixes $\mathbf{v}$ and θ and does not generally allow one to make a statement on any correlation function of θ. The proper way to describe the correlation functions of the scalar at the scales much larger than the diffusion scale is to employ the Lagrangian description that is to follow fluid trajectories. Indeed, if we neglect diffusion, then the equation (1.17) can be solved along the characteristics $\mathbf{R}(t)$ which are called Lagrangian trajectories and satisfy $d\mathbf{R}/dt = \mathbf{v}(\mathbf{R}, t)$. Presuming zero initial conditions for θ at $t \to -\infty$ we write (see also Sect. 1.2.3 in the Gawędzki course)

$$\theta\Big(\mathbf{R}(t), t\Big) = \int_{-\infty}^{t} \varphi\Big(\mathbf{R}(t'), t'\Big)\, dt' \ . \tag{1.19}$$

In that way, the correlation functions of the scalar $F_n = \langle\theta(\mathbf{r}_1, t)\ldots\theta(\mathbf{r}_n, t)\rangle$ can be obtained by integrating the correlation functions of the pumping along the trajectories that satisfy the final conditions $\mathbf{R}_i(t) = \mathbf{r}_i$. We consider a pumping which is Gaussian, statistically homogeneous and isotropic in space and white in time:

$$\langle\varphi(\mathbf{r}_1, t_1)\varphi(\mathbf{r}_2, t_2)\rangle = \Phi(|\mathbf{r}_1 - \mathbf{r}_2|)\delta(t_1 - t_2)$$

where the function Φ is constant at $r \ll L$ and goes to zero at $r \gg L$. The pumping provides for symmetry $\theta \to -\theta$ which makes only even correlation functions F_{2n} nonzero. The pair correlation function is as follows:

$$F_2(r, t) = \int_{-\infty}^{t} \Phi\Big(R_{12}(t')\Big)\, dt' \ . \tag{1.20}$$

Here $R_{12}(t') = |\mathbf{R}_1(t') - \mathbf{R}_2(t')|$ is the distance between two trajectories and $R_{12}(t) = r$. The function Φ essentially restricts the integration to the time interval when the distance $R_{12}(t') \le L$. Simply speaking, the stationary pair correlation function of a tracer is $\Phi(0)$ (which is twice the injection rate of θ^2) times the average time $T_2(r, L)$ that two fluid particles spent within the correlation scale of the pumping. The larger r the less time it takes for the particles to separate from r to L and the less is $F_2(r)$. Of course, $T_{12}(r, L)$ depends on the properties of the velocity field. A general theory is available only when the velocity field is spatially smooth at the scale of scalar pumping L. This so-called Batchelor regime happens, in particular, when the scalar cascade occurs at the scales less than the viscous scale of fluid turbulence (Batchelor 1959, Kraichnan 1974, Falkovich et al 2001). This requires the Schmidt number ν/κ (called Prandtl number when θ is temperature) to be large, which is the case for very viscous liquids. In this case, one can approximate the velocity difference $\mathbf{v}(\mathbf{R}_1, t) - \mathbf{v}(\mathbf{R}_2, t) \approx \hat{\sigma}(t)\mathbf{R}_{12}(t)$ with the Lagrangian strain matrix $\sigma_{ij}(t) = \nabla_j v_i$. In this regime, the distance obeys the linear differential equation

$$\dot{\mathbf{R}}_{12}(t) = \hat{\sigma}(t)\mathbf{R}_{12}(t) \ . \tag{1.21}$$

The theory of such equations is well-developed and is related to what is called Lagrangian chaos and multiplicative large deviations theory described in detail in the course of K. Gawędzki. Fluid trajectories separate exponentially as typical for systems with dynamical chaos (see, e.g. Antonsen and Ott 1991, Falkovich et al 2001): At t much larger than the correlation time of the random process $\hat{\sigma}(t)$, all moments of R_{12} grow exponentially with time and $\langle \ln[R_{12}(t)/R_{12}(0)]\rangle = \lambda t$ where λ is called a senior Lyapunov exponent of the flow (remark that for the description of the scalar we need the flow taken backwards in time which is different from that taken forward because turbulence is irreversible). Dimensionally, $\lambda = \Lambda f(Re)$ where the limit of the function f at $Re \to \infty$ is unknown. We thus obtain:

$$F_2(r) = \Phi(0)\lambda^{-1}\ln(L/r) = 2P\lambda^{-1}\ln(L/r) \ . \tag{1.22}$$

In a similar way, one shows that for $n \ll \ln(L/r)$ all F_n are expressed via F_2 and the structure functions $S_{2n} = \langle[\theta(\mathbf{r}, t) - \theta(0, t)]^{2n}\rangle \simeq (P/\lambda)^n \ln^n(r/r_d)$ for $n \ll \ln(r/r_d)$. That can be generalized for an arbitrary statistics of pumping as long as it is finite-correlated in time (Balkovsky and Fouxon 1999, Falkovich et al 2001). Note that those F_{2n} anf S_{2n} are completely determined by $\Phi(0)$ which is the flux of θ^2, only sub-leading corrections depend on the fluxes of the high-order integrals.

2d Enstrophy cascade. Now, one can use the analogy between passive scalar and vorticity in 2d (Kraichnan 1967,Falkovich and Lebedev 1994). For the enstrophy cascade, one derives the flux relation analogous to (1.18):

$$\langle(\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2)\omega_1\omega_2\rangle = 2D \ , \tag{1.23}$$

where $D = \langle\nu(\nabla\omega)^2\rangle$. The flux relation along with $\omega = curl\,\mathbf{v}$ suggests the scaling $\delta v(r) \propto r$ that is velocity being close to spatially smooth (of course, it cannot be perfectly smooth to provide for a nonzero vorticity dissipation in the inviscid limit, but the possible singularities are indeed shown to be no stronger than logarithmic). That makes the vorticity cascade similar to the Batchelor regime of passive scalar cascade with a notable change in that the rate of stretching λ acting on a given scale is not a constant but is logarithmically growing when the scale decreases. Physically, for smaller blobs of vorticity there are more large-scale velocity gradients that are able to stretch them. Since λ scales as vorticity, the law of renormalization can be established from dimensional reasoning and one gets $\langle\omega(\mathbf{r},t)\omega(0,t)\rangle \sim [D\ln(L/r)]^{2/3}$ which corresponds to the energy spectrum $E_k \propto D^{2/3}k^{-3}\ln^{-1/3}(kL)$. High-order correlation functions of vorticity are also logarithmic, for instance, $\langle\omega^n(\mathbf{r},t)\omega^n(0,t)\rangle \sim [D\ln(L/r)]^{2n/3}$. Note that both passive scalar in the Batchelor regime and vorticity cascade in 2d are universal that is determined by the single flux (P and D respectively) despite the existence of high-order conserved quantities. Experimental data and numeric simulations support those conclusions (Falkovich et al 2001, Tabeling 2002).

1.5 Zero modes and anomalous scaling

How one builds the Lagrangian description when the velocity is not spatially smooth, for example, that of the energy cascades in the inertial interval? Again, the only exact relation one can derive for two fluid particles separated by a distance in the inertial interval is for the Lagrangian time derivative of the squared velocity difference (Falkovich et al 2001):

$$\left\langle \frac{d|\delta\mathbf{v}|^2}{dt} \right\rangle = 2\epsilon$$

— this is the Lagrangian counterpart to (1.5,1.10,1.14,1.29). One can *assume* that the statistics of the distances between particles is also determined by the energy flux. That assumption leads, in particular, to the Richardson law for the asymptotic growth of the inter-particle distance:

$$\langle R_{12}^2(t)\rangle \sim \epsilon t^3 \ , \tag{1.24}$$

first inferred from atmospheric observations (in 1926) and later from experimental data on the energy cascades both in 3d and in 2d. There is no consistent theoretical derivation of (1.24) and it is unclear whether it is exact (likely to be in 2d) or just approximate (possible in 3d). Semi-heuristic argument usually presented in textbooks is based on the mean-field estimate: $\dot{\mathbf{R}}_{12} = \delta\mathbf{v}(\mathbf{R}_{12}, t) \sim (\epsilon R_{12})^{1/3}$ which upon integration gives: $R_{12}^{2/3}(t) - R_{12}^{2/3}(0) \sim \epsilon^{1/3} t$. While this argument is at best a crude estimate in 3d (where there is no definite velocity scaling since every moment has its own exponent ζ_n) we use it to discuss implications for the passive scalar†.

For two trajectories, the Richardson law gives the separation time from r to L: $T_2(r, L) \sim \epsilon^{-1/3}[L^{2/3} - r^{2/3}]$. Note that $T_2(r, L)$ has a finite limit at $r \to 0$ — infinitesimally close trajectories separate in a finite time. That leads to non-uniqueness of Lagrangian trajectories (non-smoothness of the velocity field means that the equation $\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R})$ is non-Lipschitz). As discussed in much details elsewhere (see Falkovich et al 2001 and Lecture 4 of the Gawędzki course), that leads to a finite dissipation of a transported passive scalar even without any molecular diffusion (which corresponds to a dissipative anomaly and time irreversibility). Indeed, substituting $T_2(r, L)$ into (1.20), one gets the steady-state pair correlation function of the passive scalar: $F_2(r) \sim \Phi(0)\epsilon^{-1/3}[L^{2/3} - r^{2/3}]$ as suggested by Oboukhov (1949) and Corrsin (1952). The structure function is then $S_2(r) \sim \Phi(0)\epsilon^{-1/3} r^{2/3}$. Experiments measuring the scaling exponents $\sigma_n = d \ln S_n(r)/d \ln r$ generally give σ_2 close to 2/3 but higher exponents deviating from the straight line even stronger than the exponents of the velocity in 3d as seen in Fig. 1.4. Moreover, the scalar exponents σ_n are anomalous even when advecting velocity has a normal scaling like in 2d energy cascade (to be described in Sec. 1.6 below).

To explain the dependence $\sigma(n)$ and describe multi-point correlation functions or high-order structure functions one needs to study multi-particle statistics. Here an important question is what memory of the initial configuration remains when final distances far exceed initial ones. To answer this question one must analyze the conservation laws of turbulent diffusion. We now describe a general concept of conservation laws which, while conserved only on the average, still determine the statistical properties of strongly fluctuating systems. In a random system, it is always possible to find some fluctuating quantities which ensemble averages do not change. We now ask a more subtle question: is it possible to find quantities that are expected to change on the dimensional grounds but they stay constant (Falkovich et

† What matters here and below is that in a non-smooth flow $R_{12}^a(t) - R_{12}^a(0) \sim t$ with $a < 1$, not the precise value of a

al 2001, Falkovich and Sreenivasan 2006). Let us characterize n fluid particles in a random flow by inter-particle distances R_{ij} (between particles i and j) as in Figure 1.5. Consider homogeneous functions f of inter-particle distances with a nonzero degree ζ, i.e. $f(\lambda R_{ij}) = \lambda^\zeta f(R_{ij})$. When all the distances grow on the average, say according to $< R_{ij}^2 > \propto t^a$, then one expects that a generic function grows as $f \propto t^{a\zeta/2}$. How to build (specific) functions that are conserved on the average, and which ζ-s they have? As the particles move in a random flow, the n-particle cloud grows in size and the fluctuations in the shape of the cloud decrease in magnitude. Therefore, one may look for suitable functions of size and shape that are conserved because the growth of distances is compensated by the decrease of shape fluctuations.

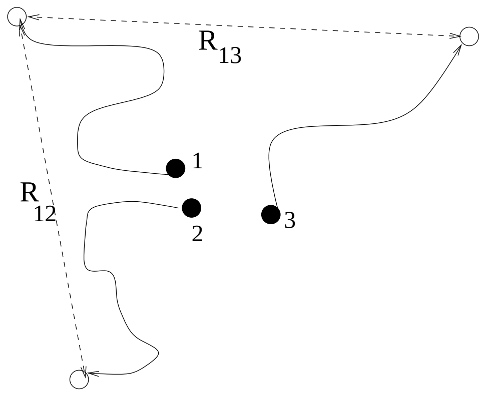

Fig. 1.5. Three fluid particles in a flow.

For the simplest case of Brownian random walk, inter-particle distances grow by the diffusion law: $\langle R_{ij}^2(t)\rangle = R_{ij}^2(0) + \kappa t$, $\langle R_{ij}^4(t)\rangle = R_{ij}^4(0) + 2(d+2)[R_{ij}^2(0)\kappa t + \kappa^2 t^2]/d$, etc. Here d is the space dimensionality. Two particles are characterized by a single distance. Any positive power of this distance grows on the average. For many particles, one can build conserved quantities by taking the differences where all powers of t cancel out: $f_2 = \langle R_{12}^2 - R_{34}^2\rangle$, $f_4 = \langle 2(d+2)R_{12}^2 R_{34}^2 - d(R_{12}^4 + R_{34}^4)\rangle$, etc. These polynomials are called harmonic since they are zero modes of the Laplacian in the $2d$-dimensional space of $\mathbf{R}_{12}$, $\mathbf{R}_{13}$. One can write the Laplacian as $\Delta = R^{1-2d}\partial_R R^{2d-1}\partial_R + \Delta_\theta$, where $R^2 = R_{12}^2 + R_{13}^2$ and Δ_θ is the angular Laplacian on $2d-1$-dimensional unit sphere. Introducing the angle, $\theta = \arcsin(R_{12}/R)$, which characterizes the shape of the triangle, we see that the conservation of both $f_2 = \langle R^2 \cos 2\theta\rangle$ and $f_4 = \langle R^4[(d+1)\cos^2 2\theta - 1]\rangle$ can be also described as due to cancellation between the growth of the radial part (as powers of t) and the decay of the angular part (as inverse powers of t). For n particles, the polynomial that involves all distances is proportional to R^{2n} (i.e. $\zeta_n = n$) and the respective shape fluctuations decay as t^{-n}.

The scaling exponents of the zero modes are thus determined by the laws that govern decrease of shape fluctuations. The zero modes, which are conserved statistically, exist for turbulent macroscopic diffusion as well. However, there is a major difference since the velocities of different particles are correlated in turbulence. Those mutual correlations make shape fluctuations decaying slower than t^{-n} so that the exponents of the zero modes, ζ_n, grow with n slower than linearly. This is very much like the total energy of the cloud of attracting particles does not grow linearly with the number of particles. Indeed, power-law correlations of the velocity field lead to super-diffusive behavior of inter-particle separations: the farther particles are, the faster they tend to move away from each other, as in Richardson's law of diffusion. That is the system behaves as if there was an attraction between particles that weakens with the distance, though, of course, there is no physical interaction among particles (but only mutual correlations because they are inside the correlation radius of the velocity field). Let us stress that while zero modes of multi-particle evolution exist for all velocity fields—from those that are smooth to those that are extremely rough as in Brownian motion—only those non-smooth velocity fields with power-law correlations provide for an anomalous scaling. Zero modes were discovered in Gawedzki and Kupiainen 1995, Shraiman and Siggia 1995, Chertkov et al 1995 and then described in Chertkov and Falkovich 1996, Bernard et al 1996, Balkovsky and Lebedev 1998.

The existence of multi-particle conservation laws indicates the presence of a long-time memory and is a reflection of the coupling among the particles due to the simple fact that they are all in the same velocity field.

We now ask: How does the existence of these statistical conservation laws (called martingales in the probability theory) lead to anomalous scaling of fields advected by turbulence? According to (1.19), the correlation functions of θ are proportional to the times spent by the particles within the correlation scales of the pumping. The structure functions of θ are differences of correlation functions with different initial particle configurations as, for instance, $S_3(r_{12}) \equiv \langle[\theta(\mathbf{r}_1) - \theta(\mathbf{r}_2)]^3\rangle = 3\langle\theta^2(\mathbf{r}_1)\theta(\mathbf{r}_2) - \theta(\mathbf{r}_1)\theta^2(\mathbf{r}_2)\rangle$. In calculating S_3, we are thus comparing two histories: the first one with two particles initially close to the position $\mathbf{r}_1$ and one particle at $\mathbf{r}_2$, and the second one with one particle at $\mathbf{r}_1$ and two particles at $\mathbf{r}_2$— see Fig 1.6. That is, S_3 is proportional to the time during which one can distinguish one history from another, or to the time needed for an elongated triangle to

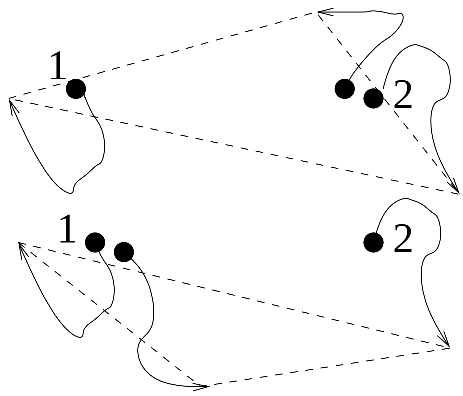

Fig. 1.6. Two configurations (upper and lower) whose difference determines the third structure function.

relax to the equilateral shape. That time grows with r_{12} (as it takes longer to forget more elongated triangle) by the law that can be inferred from the law of the decrease of the shape fluctuations of a triangle.

Quantitative details can be worked out for the white in time velocity (Kraichnan 1968). Profound insight of Kraichnan was that it is spatial rather than temporal non-smoothness of the velocity that is crucial for an anomalous scaling. The Kraichnan model is described in much detail in the course by Gawędzki, here we mention few salient points. The velocity ensemble is defined by the second moment:

$$\langle v^i(\mathbf{r},t)v^j(0,0)\rangle = \delta(t)\Big[D_0\delta_{ij} - d_{ij}(\mathbf{r})\Big]\,,$$
$$d_{ij} = D_1\, r^{\xi}\Big[(d-1+\xi)\,\delta^{ij} - \xi r^i r^j r^{-2}\Big]\,. \qquad (1.25)$$

Here the exponent $\xi \in [0,2]$ is a measure of the velocity non-smoothness with $\xi = 2$ corresponding to a smooth velocity while $\xi = 0$ to a velocity very rough in space (distributional). Richardson-Kolmogorov scaling of the energy cascade corresponds to $\xi = 4/3$. Lagrangian flow is a Markov random process for the Kraichnan ensemble (1.25). Every fluid particle undergoes a Brownian random walk with the so-called eddy diffusivity D_0. The PDF $P(r,t)$ for two particles to be separated by r after time t satisfies the diffusion equation (see e.g. Falkovich et al 2001)

$$\partial_t P = L_2 P\,, \quad L_2 = d_{ij}(\mathbf{r})\nabla^i\nabla^j = D_1(d-1)r^{1-d}\partial_r r^{d+\xi-1}\partial_r\,, \qquad (1.26)$$

with the scale-dependent diffusivity $D_1(d-1)r^{\xi}$. The asymptotic solution of (1.26) is $P(r,t) = r^{d-1}t^{d/(2-\xi)}\exp(-\mathrm{const}\, r^{2-\xi}/t)$, log-normal for $\xi = 2$. For $\xi = 4/3$, it reproduces, in particular, the Richardson law. Multi-particle probability distributions also satisfy diffusion equations in the Kraichnan model as well as all the correlation functions of θ. Multiplying (1.17) by $\theta_2 \dots \theta_{2n}$ and averaging over the Gaussian statistics of $\mathbf{v}$ and φ one derives

$$\partial_t F_{2n} = L_{2n}F_{2n} + \sum_{l,m} F_{2n-2}\Phi(\mathbf{r}_{lm})\,, \quad L_{2n} = \sum d_{ij}(\mathbf{r}_{lm})\nabla_l^i \nabla_m^j\,. \tag{1.27}$$

This equation enables one, in principle, to derive inductively all steady-state F_{2n} starting from F_2. The equation $\partial_t F_2(r,t) = L_2 F_2(r,t) + \Phi(r)$ has a steady solution $F_2(r) = 2[\Phi(0)/(2-\xi)d(d-1)D_1][dL^{2-\xi}/(d-2+\xi) - r^{2-\xi}]$, which has the Corrsin-Oboukhov form for $\xi = 4/3$. Further, F_4 contains the so-called forced solution having the normal scaling $2(2-\xi)$ but also, remarkably, a zero mode Z_4 of the operator L_4: $L_4 Z_4 = 0$. Such zero modes necessarily appear (to satisfy the boundary conditions at $r \simeq L$) for all $n > 1$ and the scaling exponents of Z_{2n} are generally different from $n\gamma$ that is anomalous. In calculating the scalar structure functions, all terms cancel out except a single zero mode (called irreducible because it involves all distances between $2n$ points). Analytically and numerical calculations of Z_n and their scaling exponents σ_n (described in detail in the course of K. Gawędzki and in the review Falkovich et al 2001) give σ_n lying on a convex curve (see Fig. 1.4) which saturates (Balkovsky and Lebedev 1998) to a constant at large n. Such saturation is a signature that most singular structures in a scalar field are shocks like in Burgers turbulence, the value σ_n at $n \to \infty$ is the fractal codimension of fronts in space (Celani et al 2001).

The existence of statistical conserved quantities breaks the scale invariance of scalar statistics in the inertial interval and explains why scalar turbulence knows about pumping "more" than just the value of the flux. Here again the statistics in the inertial interval, apart from the flux of θ^2, depends on the infinity of pumping-related parameters. However, those parameters neither are fluxes of θ^n, nor we can interpret them as any other fluxes. At the present level of understanding, we thus describe an anomalous scaling in Burgers and in passive scalar in quite different terms. Of course, the qualitative appeal to structures (shocks) is similar but the nature of the conservation laws is different. The anomalies produced by dynamically conserved quantities (like anomalous scaling in Burgers and time irreversibility in all cases of turbulence) are qualitatively different from the anomalies produced by statistically conserved quantities (like breakdown of scale invariance in passive scalar turbulence). Indeed, dissipation is a singular perturbation which breaks conservation of dynamical integrals of motion and imposes (one or many) flux-constancy conditions, very much similar to quantum anomalies. On the contrary, there are no cascades of conserved quantity related to zero modes, nor their conservation is broken by dissipation. Anomalous scaling of zero modes is due to correlations between different fluid trajectories. On the other hand, the two types of anomalies are related intimately: the flux constancy requires a certain degree of velocity non-smoothness, which generally leads to an anomalous scaling of zero modes.

Both symmetries, one broken by pumping (scale invariance) and another by damping (time reversibility) are not restored even when $r/L \to 0$ and $r_d/r \to 0$.

For the vector field (like velocity or magnetic field in magnetohydrodynamics) the Lagrangian statistical integrals of motion may involve both the coordinate of the fluid particle and the vector it carries. Such integrals of motion were built explicitly and related to the anomalous scaling for the passively advected magnetic field in the Kraichnan ensemble of velocities (Falkovich et al 2001). Doing that for velocity that satisfies the 3d Navier-Stokes equation remains a task for the future.

1.6 Inverse cascades

Here we consider inverse cascades and discover that, while time reversibility remains broken, the scale invariance is restored in the inertial interval. Moreover, even wider symmetry of conformal invariance may appear there.

Passive scalar in a compressible flow. Similar to (1.20) one can derive from (1.19)

$$\begin{aligned}\langle\theta(t,\mathbf{r}_1)\dots\theta(t,\mathbf{r}_{2n})\rangle &= \int_0^t dt_1\dots dt_n\\ &\times\langle\Phi(R(t_1|T,\mathbf{r}_{12}))\dots\Phi(R(t_n|T,\mathbf{r}_{2n-1,2n}))\rangle+\dots\ , \qquad (1.28)\end{aligned}$$

The functions Φ in (1.28) restrict integration to the time intervals where $R_{ij} < L$. If the Lagrangian trajectories separate, the correlation functions reach at long times the stationary form for all r_{ij}. Such steady states correspond to a direct cascade of the tracer (i.e. from large to small scales) considered above. That generally takes place in incompressible and weakly compressible flows.

It is intuitively clear that in compressible flows the regions of compressions can trap fluid particles counteracting their tendency to separate. Indeed, one can show that particles cluster in flows with high enough compressibility (Chertkov et al 1998, Gawędzki and Vergassola 2000). In particular, the solution of the Problem 3 shows that all the Lyapunov exponents are negative when the compressibility degree of a short-correlated flow exceeds $d/4$ (Chertkov et al 1998). Even in the non-smooth flow with high enough compressibility, the trajectories are unique, particles that start from the same point will remain together throughout the evolution (Gawędzki and Vergassola 2000). That means that advection preserves all the single-point moments $\langle\theta^N\rangle(t)$. Note that the conservation laws are statistical: the moments are not dynamically conserved in every realization, but their average over the velocity ensemble are. In the presence of pumping, the moments are the same as for the equation $\partial_t\theta = \varphi$ in the limit $\kappa \to 0$ (nonsingular now). It follows that the single-point statistics is Gaussian, with $\langle\theta^2\rangle$ coinciding with the total injection $\Phi(0)t$ by the forcing. That growth is produced by the flux of scalar variance toward the large scales. In other words, the correlation functions acquire parts which are independent of r and grow proportional to time: when Lagrangian particles cluster rather than separate, tracer fluctuations grow at larger and larger scales — phenomenon that can be loosely called an inverse cascade of a passive tracer (Chertkov et al 1998, Gawędzki and Vergassola 2000). As is clear from (1.28), correlation functions at very large scales are related to the probability for initially distant particles to come close. In a strongly compressible flow, the trajectories are typically contracting, the particles tend to approach and the distances will reduce to the forcing correlation length L (and smaller) for long enough times. On a particle language, the larger the time the large the distance starting from which particle come within L. The correlations of the field θ at larger and larger scales are therefore established as time increases, signaling the inverse cascade process.

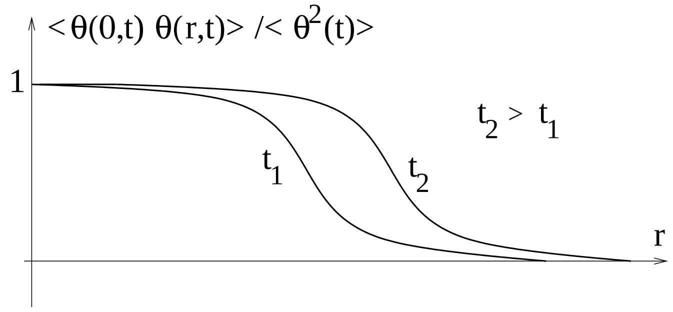

Fig. 1.7. Growth of large-scale correlations with time.

The uniqueness of the trajectories greatly simplifies the analysis of the PDF $\mathcal{P}(\delta\theta, r)$. Indeed, the structure functions involve initial configurations with just two groups of particles separated by a distance r. The particles explosively separate in the incompressible case and we are immediately back to the full N-particle problem. Conversely, the particles that are initially in the same group remain together if the trajectories are unique. The only relevant degrees of freedom are then given by the intergroup separation and we are reduced to a two-particle dynamics. It is therefore not surprising that the statistics of the passive tracer is scale invariant in the inverse cascade regime (Gawędzki and Vergassola 2000).

An example of strongly compressible flow is given by Burgers turbulence (1.8) where there is clustering (in shocks) for the majority of trajectories (full measure in the inviscid limit). Considering passive scalar in such a flow, $\theta_t + u\theta_x - \kappa\Delta\theta = \phi$, we conclude that it undergoes an inverse cascade. The statistics of θ is scale invariant at the scales exceeding the correlation scale of the pumping ϕ. While the limit $\kappa \to 0$ is regular (i.e. no dissipative anomaly), the statistics is time irreversible because of the flux towards large scales. It is instructive to compare u and θ which are both Lagrangian invariants (tracers) in the unforced undamped limit. Yet passive quantity θ (and all its powers) go to large scales under pumping while all powers of u cascade towards small scales and are absorbed by viscosity. Physically, the difference is evidently due to the fact that the trajectory depends on the value of u it carries, the larger the velocity the faster it ends in a shock and dissipates the energy and other integrals. Formally, for active tracers like u^n one cannot write a formula like (1.28) obtained by two independent averages over the force and over the trajectories.

Inverse energy cascade in two dimensions.

For the inverse energy cascade, there is no consistent theory except for the flux relation that can be derived similarly to (1.14):

$$S_3(r) = 4\epsilon r/3 \ . \tag{1.29}$$

This scaling one can also get from phenomenological dimensional arguments, though in two seemingly unrelated ways. Consider the velocity difference v_r at the distance r. On the one hand, one may require that the kinetic energy v_r^2 divided by the typical time r/v_r must be constant and equal to the energy flux, ϵ: $v_r^3 \sim \epsilon r$. On the other hand, it can be argued that vorticity, which cascades to small scales, must be in equipartition in the inverse cascade range. If this is the case, the enstrophy $r^d\omega_r^2$ accumulated in a volume of size r is proportional to the typical time r/v_r at such scale, i.e. $r^d\omega_r^2 \sim r/v_r$. Using $\omega_r \sim v_r/r$ we derive $v_r^3 \sim r^{3-d}$ which for $d = 2$ is exactly the requirement of constant energy flux. Amazingly, the requirements of vorticity equipartition (i.e. equilibrium) and energy flux (i.e. turbulence) give the same Kolmogorov-Kraichnan scaling in 2d. Let us stress that (1.29) means that time reversibility is broken in the inverse cascade. Experiments (Tabeling 2002, Kellay and Goldburg 2002, Chen et al 2006) and numerical simulations (Boffetta et al 2000), however, demonstrate a scale-invariant statistics with the vorticity having scaling dimension 2/3: $\omega_r \propto r^{-2/3}$. In particular, $S_2 \propto r^{2/3}$ which corresponds to $E_k \propto k^{-5/3}$. It is ironic that probably the most widely known statement on turbulence, the 5/3 spectrum suggested by Kolmogorov for 3d, is not correct in this case (even though the true scaling is close) while it is probably exact in the Kraichnan's inverse 2d cascade. Qualitatively, it is likely that the absence of anomalous scaling in the inverse cascade is associated with the growth of the typical turnover time (estimated, say, as $r/\sqrt{S_2}$) with the scale. As the inverse cascade proceeds, the fluctuations have enough time to get smoothed out as opposite to the direct cascade in 3d, where the turnover time decreases in the direction of the cascade. Note in passing that passive scalar undergoes direct cascade in the flow of the 2d inverse energy cascade, scalar statistics is not scale invariant since the velocity is non-smooth (compare with the relation between the Lagrangian invariants u and θ for Burgers turbulence).

Remarkably, there are indications that scale invariance of the vorticity can be extended to conformal invariance at least for its isolines (Bernard et al 2006). Under conformal transformations the lengths are re-scaled non-uniformly yet the angles between vectors are left unchanged (a useful property in navigation cartography where it is often more important to aim in the right direction than to know the distance). Conformal invariance has been discovered by analyzing the large-scale statistics of the boundaries of vorticity clusters, i.e. large-scale zero-vorticity (nodal) lines. In equilibrium critical phenomena, cluster boundaries in the continuous limit of vanishingly small lattice size were recently found to belong to a remarkable class of curves that can be mapped into Brownian walk. That class is called Schramm-Loewner Evolution or SLE curves (Schramm 2000, Gruzberg and Kadanoff 2004, Lawler 2005, Cardy 2005, Bauer and Bernard 2006). Namely, consider a curve $\gamma(t)$ that starts at a point on the boundary of the half-plane H (by conformal invariance any planar domain is equivalent to the upper half plane). One can map the half-plane H minus the curve $\gamma(t)$ back onto H by an analytic function $g_t(z)$ which is unique upon imposing the condition $g_t(z) \sim z + 2t/z + \mathrm{O}(1/z^2)$ at infinity. The growing tip of the curve is mapped into a real point $\xi(t)$. Loewner found in 1923 that the conformal map $g_t(z)$ and the curve $\gamma(t)$ are fully parametrized by the driving function $\xi(t)$. Almost eighty years later, Schramm (2000) considered random curves in planar domains and showed that their statistics is conformal invariant if $\xi(t)$ is a Brownian walk, i.e. its increments are identically and independently distributed and $\langle(\xi(t)-\xi(0))^2\rangle = \kappa t$. In simple words, the locality in time of the Brownian walk translates into the local scale-invariance of SLE curves, i.e. conformal invariance. SLE_κ provide a natural classification (by the value of the diffusivity κ) of boundaries of clusters of 2d critical phenomena described by conformal field theories (see Gruzberg and Kadanoff 2004, Lawler 2005, Cardy 2005, Bauer and Bernard 2006 for a review).

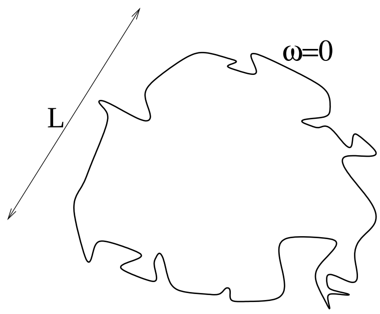

Fig. 1.8. Vorticity nodal line with the gyration radius L.

The fractal dimension of SLE_κ curves is known to be $D_\kappa = 1 + \kappa/8$ for $\kappa < 8$. To establish possible link between turbulence and critical phenomena, let us try to relate the Kolmogorov-Kraichnan phenomenology to the fractal dimension of the boundaries of vorticity clusters. Note that one ought to distinguish between the dimensionality 2 of the full vorticity level set (which is space-filling) and a single zero-vorticity line that encloses a large-scale cluster. Consider the vorticity cluster of gyration radius L which has the "outer boundary" of perimeter P (that boundary is the part of the zero-vorticity line accessible from outside, see Fig. 1.8 for an illustration). The vorticity flux through the cluster, $\int \omega dS \sim \omega_L L^2$, must be equal to the velocity circulation along the boundary, $\Gamma = \oint \mathbf{v} \cdot d\ell$. The Kolmogorov-Kraichnan scaling is $\omega_L \sim \epsilon^{1/3} L^{-2/3}$ (coarse-grained vorticity decreases with scale because contributions with opposite signs partially cancel) so that the flux is $\propto L^{4/3}$. As for circulation, since the boundary turns every time it meets a vortex, such a contour is irregular on scales larger than the pumping scale. Therefore, only the velocity at the pumping scale L_f is expected to contribute to the circulation, such velocity can be estimated as $(\epsilon L_f)^{1/3}$ and it is independent of L. Hence, circulation should be proportional to the perimeter, $\Gamma \propto P$, which gives $P \propto L^{4/3}$, i.e. the fractal dimension of the exterior of the vorticity cluster is expected to be 4/3. This remarkable dimension correspond to a self-avoiding random walk (SLE curve) which is also known to be an exterior boundary (without self-intersections) of percolation cluster (yet another SLE curve). Data analysis of the zero-vorticity lines have shown that indeed within an experimental accuracy their statistics is indistinguishable from percolation clusters while that of their exterior boundary from the statistics of self-avoiding random walk (Bernard et al 2006). Whether the statistics of the zero-vorticity isolines indeed falls into the simplest universality class of critical phenomena (that of percolation) deserves more study.

Let us briefly discuss wave turbulence from the viewpoint of conformal invariance. Gaussian scalar field in 2d is conformal invariant if its correlation function is logarithmic i.e. the spectral density decays as k^{-2}. Such is the case, for instance, for the fluid height in gravitational-capillary weak wave turbulence on a shallow water (see Zakharov et al 1992, Sect. 5.1.2). It is interesting if deviations from Gaussianity due to wave interaction destroy conformal invariance. Another interesting example is the inverse cascade of 2d strong optical turbulence described by the Nonlinear Schrödinger Equation. Numerics hint that in the case of a stable growing condensate, the statistics of the finite-scale fluctuations approach Gaussian with a logarithmic correlation function (Dyachenko and Falkovich 1996).

1.7 Conclusion

We reiterate the conclusions on the status of symmetries in turbulence.
Turbulence statistics is always time-irreversible.
Weak turbulence is scale invariant and universal (determined solely by the flux value). It is generally not conformal invariant.
Strong turbulence: Direct cascades often have symmetries broken by pumping (scale invariance, isotropy) non-restored in the inertial interval. In other words, statistics at however small scales is sensitive to other characteristics of pumping besides the flux. That can be alternatively explained in terms of either structures or statistical conservation laws (zero modes). Anomalous scaling in a direct cascade may well be a general rule apart from some degenerate cases like passive scalar in the Batchelor case (where all the zero modes have the same scaling exponent, zero, as the pair correlation function). Inverse cascades in systems with strong interaction may be not only scale invariant but also conformal invariant. It is an example of emerging or restored symmetry.
For Lagrangian invariants, we explain the difference between direct and inverse cascades in terms of separation or clustering of fluid particles. Generally, it seems natural that the statistics within the pumping correlation scale (direct cascade) is more sensitive to the details of the pumping statistics than the statistics at much larger scales (inverse cascade).

Exercises

1.1 Show that $n_k \propto k^{-s}$ with $s = m + d$ turns the collision integral $I_k^{(3)}$ into zero and corresponds to a constant energy flux, which sign is given by the derivative $-dI_k^{(3)}(s)/ds$.

1.2 A general equilibrium solution of $I_k^{(3)} = 0$ depends on the energy and the momentum of the wave system: $n(\mathbf{k}, T, \mathbf{u}) = T[\omega_k - (\mathbf{k} \cdot \mathbf{u})]^{-1}$ (Doppler-shifted Rayleigh-Jeans distribution). A general non-equilibrium solution depends on the fluxes P and $\mathbf{R}$ of the energy and momentum respectively. Find the form of the weakly anisotropic correction to the isotropic turbulence spectrum.

1.3 For the two-particle distance evolving in a spatially smooth random flow according to (1.21), $\dot{\mathbf{R}}(t) = \hat{\sigma}(t)\mathbf{R}(t)$, consider the Jacobi matrix defined by $R_i(t) = W_{ij}(t)R_j(t)$ — see also Sects. 1.2.3 and 2.2 in the Gawędzki course. An initial infinitesimal sphere evolves into an elongated ellipsoid with the inertia tensor $I(t) = W(t)W^T(t)$. Lyapunov exponents are the asymptotic in time eigenvalues of $W^T(t)W(t)$

which stabilizes in every realization. They can be also found from the tensor I, which on the contrary, rotates all the time. Decompose $I = O^T \Lambda O$ where O is an orthogonal matrix composed of the eigenvectors of I and Λ is a diagonal matrix with the eigenvalues $e^{2\rho_1}, \ldots, e^{2\rho_d}$. Derive the equation for $I(t)$ and $\rho_i(t)$ and find the Lyapunov exponents as $\lambda_i = \lim_{t\to\infty} \rho_i(t)/t$ for the short-correlated isotropic Gaussian strain $\langle \sigma_{ij}(0)\sigma_{kl}(t)\rangle = C_{ijkl}\delta(t)$.

Solutions

Exercise 1.1 We write the collision integral as follows

$$I_k^{(3)} = \int (U_{k12} - U_{1k2} - U_{2k1})\, d\mathbf{k}_1 d\mathbf{k}_2$$

with $U_{123} = \pi\left[n_2 n_3 - n_1(n_2+n_3)\right] |V_{123}|^2 \delta\left(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3\right)\delta\left(\omega_1 - \omega_2 - \omega_3\right)$. Here and below i stands for $\mathbf{k}_i$. To evaluate $I_k^{(3)}$ on $n_k \propto k^{-m-d}$ we first integrate over the directions of $\mathbf{k}_i$. In an isotropic medium, the interaction coefficient V_{k12} depends only on the scalar products of $\mathbf{k}$ and $\mathbf{k}_i$. Using the additional condition $\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 = 0$ one can express V_{k12}, like the frequency $\omega(k)$, as a function of wavenumbers. Then, only the δ-function of wave vectors is to be integrated over the angles in the $\mathbf{k}_1, \mathbf{k}_2$ space. The result of the angular integration is non-zero only if one can form a triangle out of $\mathbf{k}$, $\mathbf{k}_1$ and $\mathbf{k}_2$. Denote $\Theta(k_1, k_2, k_3)$ the product of step functions $\theta(k_1+k_2-k)\theta(k+k_2-k_1)\theta(k+k_1-k_2)$ that ensures the triangular inequalities $k < k_1 + k_2$, $k_1 < k + k_2$ and $k_2 < k + k_1$. We introduce $\hat{k}_i = \mathbf{k}_i/k_i$ and $d\mathbf{k}_i = k_i^{d-1} dk_i d\hat{k}_i$. Applying the formula $\int \delta\left(\mathbf{f} - g\hat{g}\right) d\hat{g} = 2\delta(f^2 - g^2)/g^{d-2}$ to the integration over $\hat{k}_2$ we obtain

$$H_d \equiv \int \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\hat{k}_1 d\hat{k}_2 = \int d\hat{k}_1 2k_2^{2-d} \delta\left(k^2 + k_1^2 - k_2^2 - 2\mathbf{k}\cdot\mathbf{k}_1\right).$$

We confine ourselves to physical dimensions $d = 2$ and $d = 3$ though one can easily generalize the following to arbitrary d. In $d = 2$ we have

$$H_2 = 4\int_0^\pi \delta\left(k^2 + k_1^2 - k_2^2 - 2kk_1\cos\theta\right) d\theta = \frac{2\Theta(k, k_1, k_2)}{kk_1 \sin\theta^0} = \frac{\Theta(k, k_1, k_2)}{\Delta(k, k_1, k_2)},$$

where $\cos\theta^0 = (k^2+k_1^2-k_2^2)/(2kk_1)$ and $\Delta(k, k_1, k_2) = (1/2)\Big[2(k^2k_1^2+k^2k_2^2+k_1^2k_2^2) - k^4 - k_1^4 - k_2^4\Big]^{1/2}$ is the area of the triangle formed by the vectors $\mathbf{k}, \mathbf{k}_1$ and $\mathbf{k}_2$. We have used the fact that the triangular inequalities are equivalent to one condition $|(k^2 + k_1^2 - k_2^2)/(2kk_1)| \le 1$. Analogous calculation in $d = 3$

produces

$$H_3 = \frac{4\pi}{k_2} \int_0^\pi \delta\left(k^2 + k_1^2 - k_2^2 - 2kk_1\cos\theta\right)\sin\theta d\theta = \frac{2\pi\Theta(k,k_1,k_2)}{kk_1k_2}\,.$$

Observe that H_d depends only on the wavenumbers k, k_1 and k_2, invariant with respect to their permutations and is non-zero only if triangular inequalities are satisfied:

$$I_k^{(3)} = \int_0^\infty k_1^{d-1}dk_1 \int_0^\infty k_2^{d-1}dk_2 H(k,k_1,k_2)\left(\tilde{U}_{k12} - \tilde{U}_{1k2} - \tilde{U}_{2k1}\right)\,, \tag{1.1}$$
$$\tilde{U}_{123} = \pi\left[n_2n_3 - n_1(n_2+n_3)\right]|V_{123}|^2\delta\left(\omega_1 - \omega_2 - \omega_3\right)\,.$$

We change the integration variables from k_1, k_2 to t $\omega(k_1) = \omega_1$, $\omega(k_2) = \omega_2$. Using $\omega(k) \propto k^\alpha$ we obtain for $\pi(2k)^{d-1}I(k)/v(k) \equiv I(\omega)$

$$I(\omega) = \int_0^\infty \int_0^\infty d\omega_1 d\omega_2 \Big[R(\omega,\omega_1,\omega_2) - R(\omega_1,\omega,\omega_2) - R(\omega_2,\omega,\omega_1)\Big] \tag{1.2}$$
$$= \int_0^\omega Q(\omega,\omega_1,\omega-\omega_1)[n(\omega_1)n(\omega-\omega_1) - n(\omega)(n(\omega_1)+n(\omega-\omega_1))]d\omega_1$$
$$-2\int_\omega^\infty Q(\omega_1,\omega,\omega_1-\omega)[n(\omega)n(\omega_1-\omega) - n(\omega_1)(n(\omega)+n(\omega_1-\omega))]\,d\omega_1.$$
$$R(\omega,\omega_1,\omega_2) = C|V(\omega,\omega_1,\omega_2)|^2 H_d(\omega,\omega_1,\omega_2)(\omega\omega_1\omega_2)^{-1+d/\alpha}\delta(\omega-\omega_1-\omega_2)$$
$$\times[n_1n_2 - n_\omega(n_1+n_2)] \equiv Q(\omega,\omega_1,\omega_2)\delta(\omega-\omega_1-\omega_2)[n_1n_2 - n_\omega(n_1+n_2)].$$

Here $v(k) = d\omega(k)/dk$. Homogeneity properties

$$V(\lambda\mathbf{k},\lambda\mathbf{k}_1,\lambda\mathbf{k}_2) = \lambda^m V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\,,\quad H_d(\lambda k,\lambda k_1,\lambda k_2) = \lambda^{-d}H_d(k,k_1,k_2)$$

allow us to write $Q(\omega,\omega_1,\omega-\omega_1) = \omega^\gamma f(\omega/\omega_1)$, where $\gamma = 2(m+d)/\alpha - 3$. All the information about the interactions is contained in one number γ and in one function $f(x)$. One solution that turns the collision integral into zero is the equilibrium Rayleigh-Jeans distribution ,$n(\omega) \propto \omega^{-1}$. Let us search for other power-law solutions $n(\omega_k) = k^{-s} = \omega^{-s/\alpha}$. Since integrals over a power-law function generally diverge either at zero or infinite frequency we first check for which s collision integral converges. Physically, convergence means locality of interactions in the frequency space as it signifies that $n(\omega)$ changes only due the waves with the frequencies of order ω.

We have in (1.2) a sum of integrals of power functions, of which every one separately diverges either at $\omega_1 \to 0$ or at $\omega_1 \to \infty$. Cancellations of leading divergencies (by one power at infinity and by two powers at zero) may provide for convergence. Assume $|V(k,k_1,k_2)|^2 \propto k_1^{m_1}k^{2m-m_1}$ at $k_1 \ll k$. Consider $\omega_1 \to \infty$ when $n(\omega_1-\omega) - n(\omega_1) \approx \omega\partial n_1/\partial\omega_1 \propto n_1\omega/\omega_1$, $|V(\omega_1,\omega,\omega_1-\omega)|^2 \propto \omega_1^{(2m-m_1)/\alpha}$ and $Q(\omega_1,\omega,\omega_1-\omega) \propto \omega_1^{-2+(2m-m_1+d+1)/\alpha}$.

The most dangerous (second) term in (1.2) converges when $s > s_2 = 2m - m_1 + d + 1 - 2\alpha$.

At small frequencies, one should take into account not only the contribution of small ω_1 and $\omega - \omega_1$ in the first term but also of small $\omega_1 - \omega$ in the second term of (1.2):

$$\begin{aligned}
&\left(\int_0^\epsilon + \int_{\omega-\epsilon}^{\omega}\right) Q(\omega,\omega_1,\omega-\omega_1)\{n_1 n(\omega-\omega_1) - n(\omega)[n_1 + n(\omega-\omega_1)]\}d\omega_1 \\
&-2\int_\omega^{\omega+\epsilon} d\omega_1 Q(\omega_1,\omega,\omega_1-\omega)[n(\omega)n(\omega_1-\omega) - n(\omega_1)(n(\omega)+n(\omega_1-\omega))] \\
&= 2\int_0^\epsilon \omega^\gamma f\left(\frac{\omega}{\omega_1}\right)[n(\omega_1)n(\omega-\omega_1) - n(\omega)(n(\omega_1)+n(\omega-\omega_1))]d\omega_1 \\
&-2\int_0^\epsilon (\omega_1+\omega)^\gamma f\left(\frac{\omega_1+\omega}{\omega_1}\right)[n(\omega)n(\omega_1) - n(\omega_1+\omega)(n(\omega)+n(\omega_1))]d\omega_1 \\
&= 2\int_0^\epsilon \Big[n(\omega_1)\Big(n(\omega-\omega_1)+n(\omega+\omega_1)-2n(\omega)\Big) + n(\omega)\Big(n(\omega+\omega_1) \\
&-n(\omega-\omega_1)\Big)\Big]\omega^\gamma f\left(\frac{\omega}{\omega_1}\right)d\omega_1 + 2\int_0^\epsilon\left[(\omega_1+\omega)^\gamma f\left(\frac{\omega_1+\omega}{\omega_1}\right) - \omega^\gamma f\left(\frac{\omega}{\omega_1}\right)\right] \\
&\times\Big[n(\omega_1+\omega)\left(n(\omega)+n(\omega_1)\right) - n(\omega)n(\omega_1)\Big]d\omega_1. \qquad (1.3)
\end{aligned}$$

The integrals converge if $s < s_1 = m_1 + d - 1 + 2\alpha$. Thus, if

$$s_1 > s_2, \quad 2m_1 > 2m + 2 - 4\alpha, \qquad (1.4)$$

then there exists an interval of exponents $2m - m_1 + d + 1 - 2\alpha < s < m_1 + d - 1 + 2\alpha$ such that on $n(\omega) \propto \omega^{-s/\alpha}$ the collision integral converges. The cancellations (one at infinity and two at zero) that provide for the "locality interval" are property of the kinetic equation, they generally do not take place for higher nonlinear corrections, at least, nobody was able so far to make resummations of the perturbation series to have such a locality order-by-order. The exponent of Zakharov distribution $s_0 = m + d$ lies exactly in the middle of the "locality interval": $s_0 = (s_1 + s_2)/2$ when it exists. That means that for the constant-flux distribution the contributions to interactions of all scales, from small to large ones, are counterbalanced and the collision integral in fact vanishes. To show this we use the transformation invented independently by Zakharov and Kraichnan. Let us substitute $n(\omega) = A\omega^{-s/\alpha}$ into (1.2) and make the change of variables $\omega_1 = \omega\omega/\omega_1'$, $\omega_2 = \omega_2'\omega/\omega_1'$ in the second term and do the same, interchanging $1 \leftrightarrow 2$, in the third term:

$$I(\omega) = \int\int_0^\infty d\omega_1 d\omega_2 \Big[1 - (\omega/\omega_1)^{\vartheta-1} - (\omega/\omega_2)^{\vartheta-1}\Big] R(\omega,\omega_1,\omega_2)$$

$$= A^2\omega^\gamma \int_0^\omega \left[1 - \left(\frac{\omega}{\omega_1}\right)^{\vartheta-1} - \left(\frac{\omega}{\omega-\omega_1}\right)^{\vartheta-1}\right]\left[1 - \left(\frac{\omega}{\omega_1}\right)^{-s/\alpha} - \left(\frac{\omega}{\omega-\omega_1}\right)^{-s/\alpha}\right]$$

$$\times\, [\omega_1(\omega-\omega_1)]^{-s/\alpha} f\left(\frac{\omega}{\omega_1}\right) d\omega_1, \tag{1.5}$$

where $\vartheta = 2(m + d - s)/\alpha$. The transformation interchange 0 and ∞ so they are legitimate only for converging integrals. Since $f(x)$ is a positive function, the integrals become zero only at $s_0 = \alpha$ and $s_1 = m + d$. Therefore, the Rayleigh-Jeans and Kolmogorov-Zakharov distributions are the only universal power-law stationary solutions of the kinetic equation. Each of these solutions is a one-parameter solution in the isotropic case. Rayleigh-Jeans equipartition describes equilibrium which takes place for the closed system of waves, that is at zero forcing and dissipation: $I_k^3\{k^{-s_0}\} = 0$. On the contrary, Kolmogorov-Zakharov solution has singularity at $\omega = 0$ (which corresponds to a source): $I_k^3\{k^{-s_1}\} \propto \delta(\omega)$. Let us show that indeed Kolmogorov-Zakharov solution provides a constant flux of energy in k-space: $P_k = -\int_{k'<k} d\mathbf{k}'\omega_{k'} I_{k'}^{(3)} = -\int_0^k \pi(2k')^{d-1}\omega_{k'} I_{k'}^{(3)} dk'$. Using (1.2) we find that $P(\omega) \equiv P_{k(\omega)}$ is given by $P(\omega) = -\int_0^\omega \omega' I(\omega') d\omega'$. For the exponents s from to the locality interval one can consider the collision integral as a function of s. Passing in (1.5) to $y = \omega_1/\omega$ one finds $I(\omega) \propto \omega^{\vartheta-2} A^2 I(s)$ and

$$I(s) = \int_0^1 dy \left[1 - y^{1-\vartheta} - (1-y)^{1-\vartheta}\right]\left[1 - y^{s/\alpha} - (1-y)^{s/\alpha}\right][y(1-y)]^{-s/\alpha} f(1/y).$$

Hence, the flux on power-law distributions is as follows: $P = -\omega^\vartheta A^2 I(s)/\vartheta$. At $s = s_0 = m+d$ there is an indeterminacy of the form 0/0 since $I(m+d) = 0$ and $\vartheta(s_0) = 0$. Using the L'Hospital's rule, we obtain an expression where the energy flux is proportional to the derivative of the collision integral with respect to the exponent: $P = -2A^2 I'(s_0)/\alpha$

$$P = -A^2 \int_0^1 [y\ln y + (1-y)\ln(1-y)]\left[y^{s_0/\alpha} + (1-y)^{s_0/\alpha} - 1\right]\frac{f\left(y^{-1}\right) dy}{[y(1-y)]^{s_0/\alpha}}.$$

The sign of P coincides with that of $1 - y^{s_0/\alpha} - (1-y)^{s_0/\alpha}$, which is that of $s_0/\alpha - 1$, i.e. the equilibrium exponent $s_0 = \alpha$ provides the boundary between the positive and the negative fluxes to large k. Indeed, flux must flow trying to restore equipartition. We conclude that the Kolmogorov-Zakharov spectrum describes physical turbulent state provided it decays faster than the equilibrium. In the opposite case, $s_0 < \alpha$, a flux solution is not physical, it is actually unstable (Zakharov et al 1992).

Exercise 1.2 Consider an anisotropic forcing that produces non-vanishing rate of injection of momentum into the system: $\int \mathbf{k} F_k d\mathbf{k} \equiv \mathbf{R}$. Using dimensional analysis we can write for a steady state (assuming it exists)

$$n(\mathbf{k}, P, \mathbf{R}) = \lambda P^{1/2} k^{-m-d} f(\xi), \quad \xi = \frac{(\mathbf{R}\cdot\mathbf{k})\omega(k)}{Pk^2}\ , \tag{1.6}$$

where λ is the dimensional Kolmogorov constant; the medium is assumed to be isotropic, therefore the solution depends on $(\mathbf{R}\cdot\mathbf{k})$. The dimensionless function $f(\xi)$ has been found analytically only for sound with positive dispersion (Zakharov et al 2001). In a general case, one can *assume* that $f(\xi)$ is analytical at zero; then, expanding (1.6), one obtains a stationary anisotropic correction to the isotropic solution

$$\begin{aligned} n(\mathbf{k}, \mathbf{P}, \mathbf{R}) &\approx \lambda P^{1/2} k^{-m-d} + \lambda f'(0) k^{-m-d} (\mathbf{R}\cdot\mathbf{k}) \omega(k) P^{-1/2} k^{-2} \\ &= n_0(k) + \delta n(\mathbf{k})\ . \end{aligned} \tag{1.7}$$

The ratio $\delta n / n_0 \propto \omega(k)/k$ increases with k for waves with the decay dispersion law (when the three-wave kinetic equation is relevant). That is the spectrum of the weak turbulence generated by weakly anisotropic pumping is getting more anisotropic as we go into the inertial interval of scales.

To verify that (1.7) is the stationary solution of the linearized kinetic equation, let us substitute $n(\mathbf{k}) = k^a[1 + k^b(\mathbf{q}\mathbf{R})]$ with $\mathbf{q} = \mathbf{k}/k$ into the linearized collision integral, which may be represented as $\hat{L}_{\mathbf{k}} \delta n(\mathbf{k}) = \mathbf{R}\cdot\mathbf{I}'(\mathbf{k})$,

$$\begin{aligned} \mathbf{I}'(\mathbf{k}) = \int & d\mathbf{k}_1 d\mathbf{k}_2 [U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\mathbf{f}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) - U(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k})\mathbf{f}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \\ & - U(\mathbf{k}_2, \mathbf{k}, \mathbf{k}_1)\mathbf{f}(\mathbf{k}_2, \mathbf{k}, \mathbf{k}_1)], \end{aligned} \tag{1.8}$$

$$U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = U_{k12} = \pi |V(k, k_1, k_2)|^2 \delta(\omega_k - \omega_1 - \omega_2) \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2),$$

$$\mathbf{f}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = -\mathbf{q} k^{a+b}(k_1^a + k_2^a) + \mathbf{q}_1 k_1^{a+b}(k_2^a - k^a) + \mathbf{q}_2 k_2^{a+b}(k_1^a - k^a)\ .$$

Isotropy allows us to write $\mathbf{I}' = \mathbf{q} I(\mathbf{k})$ where

$$I = \int d\mathbf{k}_1 d\mathbf{k}_2 [U_{k12}\mathbf{f}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) - U_{12k}\mathbf{f}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) - U_{2k1}\mathbf{f}(\mathbf{k}_2, \mathbf{k}, \mathbf{k}_1)]\ .$$

Similarly to what we did in the Problem 1, let us look for the change of the integration variables that transform the second and the third terms of the integral into the first one (with certain factors). Such transformation was found out by Kats and Kontorovich (see Zakharov et al 1992 and the references therein). Let us relabel the integration variables so that the third term takes the form $\int d\mathbf{k}_1' d\mathbf{k}_2' U(\mathbf{k}_2', \mathbf{k}, \mathbf{k}_1')\mathbf{q}\cdot\mathbf{f}(\mathbf{k}_2', \mathbf{k}, \mathbf{k}_1')$. The integrand is non-vanishing only for those $\mathbf{k}_i'$ which satisfy the conservation of energy and momentum. Figure 1.9, a and c show triangles corresponding to the momentum conservation in the first and third term respectively.

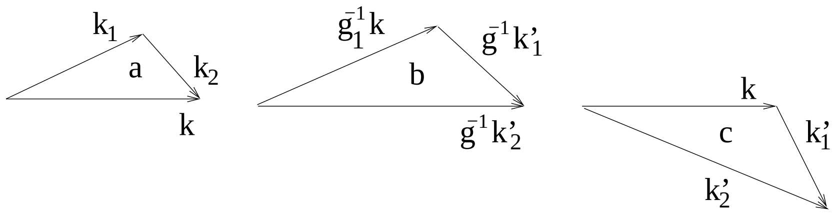

Fig. 1.9. Transformation which converts two triangles (c and a) into one another. The triangles express the laws of conservation of energy and momentum.

The two triangles have a common vector $\mathbf{k}$ and the map c→a is the combination of rotation $\hat{g}_1^{-1}$, depicted in Figure c in 1.9 which maps the original triangle to the re-scaled desired triangle and then re-scaling $\hat{\lambda}$ with the coefficient $\lambda_1 = k/k_1 = k_2'/k$ that finishes the transformation. Such map means the change of variables $\mathbf{k}_2' = (\lambda_1\hat{g}_1)^2\mathbf{k}_1$ and $\mathbf{k}_1' = \lambda_1\hat{g}_1\mathbf{k}_2$. Taking into account the Jacobian and using $\mathbf{k} = \lambda_1\hat{g}_1\mathbf{k}_1$ we find

$$\int d\mathbf{k}_1' d\mathbf{k}_2' U(\mathbf{k}_2', \mathbf{k}, \mathbf{k}_1')\mathbf{q}\cdot\mathbf{f}(\mathbf{k}_2', \mathbf{k}, \mathbf{k}_1') = \int d\mathbf{k}_1 d\mathbf{k}_2 U(\lambda_1\hat{g}_1\mathbf{k}, \lambda_1\hat{g}_1\mathbf{k}_1, \lambda_1\hat{g}_1\mathbf{k}_2)$$
$$\times\mathbf{q}\cdot\mathbf{f}(\lambda_1\hat{g}_1\mathbf{k}, \lambda_1\hat{g}_1\mathbf{k}_1, \lambda_1\hat{g}_1\mathbf{k}_2)\lambda_1^{3d}\,.$$

Since $U(\lambda_1\hat{g}_1\mathbf{k}, \lambda_1\hat{g}_1\mathbf{k}_1, \lambda_1\hat{g}_1\mathbf{k}_2) = \lambda_1^{2m-d-\alpha}U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ and

$$\mathbf{q}\cdot\mathbf{f}(\lambda_1\hat{g}_1\mathbf{k}, \lambda_1\hat{g}_1\mathbf{k}_1, \lambda_1\hat{g}_1\mathbf{k}_2) = \lambda_1^{2a+b}\mathbf{q}\cdot\hat{g}_1\mathbf{f}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \lambda_1^{2a+b}\mathbf{q}_1\cdot\mathbf{f}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\,,$$

$\int d\mathbf{k}_1' d\mathbf{k}_2' U(\mathbf{k}_2', \mathbf{k}, \mathbf{k}_1')\mathbf{q}\cdot\mathbf{f}(\mathbf{k}_2', \mathbf{k}, \mathbf{k}_1') = \lambda^w \int d\mathbf{k}_1 d\mathbf{k}_2 U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\mathbf{q}_1\cdot\mathbf{f}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ with $w = 2m + 2d - \alpha + 2a + b$. Interchanging 1 and 2 we obtain $I(\mathbf{k})$ in the factorized form:

$$I(\mathbf{k}) = \int d\mathbf{k}_1 d\mathbf{k}_2 U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\mathbf{f}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\Big[\mathbf{q} - \mathbf{q}_1(k/k_1)^w - \mathbf{q}_2(k/k_2)^w\Big]\,.$$

When we choose $w = 2m + 2d - \alpha + 2a + b = -1$, then $I(\mathbf{k}) = 0$ due to the δ-function of wave vectors. It is easily seen that the index obtained corresponds to δn from (1.7). To conclude that the drift solutions are steady modes, we have to verify their locality, i.e., the convergence of $I(\mathbf{k})$ which is done similarly to the isotropic case in the Problem 1. The only difference is that the divergences are reduced by the power of k rather than ω_k, due to which the locality strip is compressed by $2(\alpha - 1)$.

The direction of the momentum flux (to large or small k) is given by the sign of the collision integral derivative with respect to the index of the solution: $\operatorname{sign} R = -\operatorname{sign}(\partial I/\partial b)$.

Exercise 1.3 Our consideration here is more intuitive and less formal than that of Gawędzki, Sect. 3.2. Since $\dot{W} = \sigma W$ then $\dot{I} = \sigma I + I\sigma^T$. We assume the following ordering of the eigenvalues $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_d$. The equation on I becomes

$$\partial_t \rho_i = \tilde{\sigma}_{ii}, \quad \tilde{\sigma} = O\sigma O^T, \quad \partial_t O = \Omega O, \quad \Omega_{ij} = \frac{e^{2\rho_i}\tilde{\sigma}_{ji} + e^{2\rho_j}\tilde{\sigma}_{ij}}{e^{2\rho_i} - e^{2\rho_j}}\,, \tag{1.9}$$

where $\rho_i(0) = 0$ and $O_{ij}(0) = \delta_{ij}$. Here and below we do not sum over repeated indices unless stated otherwise. Note that Ω is antisymmetric to preserve $O^T O = 1$. We assume the spectrum of Lyapunov exponents to be non-degenerate, then at times much larger than the maximum of $(\lambda_i - \lambda_{i+1})^{-1}$ we have $\rho_1 \gg \rho_2 \gg \ldots \gg \rho_d$ and matrix Ω becomes independent of ρ_i

$$\Omega_{ik} = \tilde{\sigma}_{ki}, \quad i < k, \quad \Omega_{ik} = -\tilde{\sigma}_{ik}, \quad i > k. \tag{1.10}$$

The above independence allows us to resolve explicitly the equation on ρ_i as follows

$$\rho_i(t) = \int_0^t \tilde{\sigma}_{ii}(t')dt', \quad \lambda_i = \lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{\sigma}_{ii}(t')dt'. \tag{1.11}$$

The above representation of Lyapunov exponents is a (non-rigorous) proof that the limits defining Lyapunov exponents exist as Oseledec theorem states. Indeed let us make the (ergodic) hypothesis that the above time-average can be calculated as an average over the statistics of velocity field. Then, provided the statistics of $\tilde{\sigma}_{ii}(t)$ becomes stationary at large t, the Lyapunov exponents λ_i become the subject of the law of large numbers and we have

$$\lambda_i = \lim_{t\to\infty} \langle \tilde{\sigma}_{ii}(t) \rangle = \lim_{t\to\infty} \sum_{jk} \langle O_{ij}(t) O_{ik}(t) \sigma_{jk}(t) \rangle\ . \tag{1.12}$$

While generally the analytic calculation of that average is not possible, it is readily accomplished in a statistically isotropic case when the correlation time τ of σ is small comparing to the rms value σ_c. For Δt satisfying $\tau \ll \Delta t \ll \sigma_c^{-1}$ we have $O_{ij}(t) = O_{ij}(t-\Delta t) + \int_{t-\Delta t}^{t} \Omega_{ik}(t')O_{kj}(t')dt' + o(\sigma_c \Delta t)$ where $O_{ij}(t-\Delta t)$, being determined by $\sigma(t)$ at times smaller than $t-\Delta t$, is approximately independent of $\sigma(t)$ by $\Delta t \gg \tau$. First iteration of O_{ij} gives at large time

$$\lambda_i = \langle O_{ij}(t-\Delta t)O_{ik}(t-\Delta t)\sigma_{jk}(t)\rangle + \sum_{jk}\int_{t-\Delta t}^{t} dt' \Big\langle \sigma_{jk}(t)$$
$$\times \Big[\Omega_{il}(t')O_{lj}(t')O_{ik}(t-\Delta t) + \Omega_{il}(t')O_{lk}(t')O_{ij}(t-\Delta t)\Big]\Big\rangle + o(\sigma_c \Delta t)\,,$$

Introducing $\tilde{\sigma}'(t) = O(t-\Delta t)\sigma(t)O^T(t-\Delta t)$, performing second iteration of $O_{ij}(t')$ and using (1.10) and the symmetry of thr indices we find

$$\lambda_i = \langle \tilde{\sigma}'_{ii}\rangle + \int\limits_{t-\Delta t}^{t} dt' \left[\sum_{l>i}\langle(\tilde{\sigma}'_{il}(t)+\tilde{\sigma}'_{li}(t))\tilde{\sigma}'_{il}(t')\rangle - \sum_{l<i}\langle(\tilde{\sigma}'_{il}(t)+\tilde{\sigma}'_{li}(t))\tilde{\sigma}'_{il}(t')\rangle\right].$$

For $\Delta t \gg \tau$ the matrix $\sigma_{ij}(t)$ is independent of $O_{ij}(t-\Delta t)$ which is determined by σ at times earlier than $t-\Delta t$. Due to isotropy, the matrix $\tilde{\sigma}'$ has the same statistics as σ and can be replaced by it in the correlation functions. The first average in is independent of i and equals $\langle tr\sigma\rangle/d = \sum \lambda_i/d$ while the second is independent of l so that

$$\lambda_i = d^{-1}\sum_p \lambda_p + (d-2i+1)\int_{t-\Delta t}^{t} dt' \langle(\sigma_{il}(t)+\sigma_{li}(t))\sigma_{il}(t')\rangle\,,$$

where there is no summation over the repeated indices in the second term (note that summation over i produces identity). We may write

$$\int_{t-\Delta t}^{t} dt'\,\langle \sigma_{ij}(t)\sigma_{kl}(t')\rangle = \int_{t-\Delta t}^{t} dt' \langle\langle \sigma_{ij}(t)\sigma_{kl}(t')\rangle\rangle + \Delta t\langle\sigma_{ij}\rangle\langle\sigma_{kl}\rangle$$

where double brackets stand for dispersion. Noting from $\langle\sigma_{jk}\rangle = \sum_p \lambda_p \delta_{jk}/d$ that the last term contains additional factor $\sum \lambda_i \Delta t \ll \sigma_c\tau \ll 1$ with respect to the first term we conclude that it can be neglected. Here we used $\sum \lambda_i \sim \sigma_c^2 \tau$ and $\Delta t \ll \sigma_c^{-1}$. Using stationarity of the statistics of $\sigma_{ij}(t)$ at large times we may write

$$\begin{aligned}\lambda_i &= d^{-1}\sum \lambda_p + (d-2i+1)(C_{ijij}+C_{ijji})/2,\\ &= -(2d)^{-1}\sum_{j,l} C_{jjll} + (d-2i+1)(C_{ikik}+C_{kiik})/2. \qquad (1.13)\\ C_{ijkl} &= \lim_{t\to\infty}\int dt' \langle\langle \sigma_{ij}(t)\sigma_{kl}(t')\rangle\rangle\,,\end{aligned}$$

where there is no summation in the last term in λ_i. Note the symmetry

$C_{ijkl} = C_{ilkj}$. Due to isotropy and the symmetry $C_{ijkl} + C_{ilkj}$ we have $C_{ijkl} = A\delta_{ik}\delta_{jl} + B(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$, where we also assumed parity. It is convenient to write the two remaining constants as

$$C_{ijkl} = 2D_1 \left[(d+1-2\Gamma)\delta_{ik}\delta_{jl} + (\Gamma d - 1)(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})\right] . \quad (1.14)$$

Then it is easy to show using dispersion non-negativity conditions that D_1 and Γ are non-negative. While D_1 measures the overall rate of strain, Γ is the degree of compressibility, it changes between zero (for an incompressible flow) and unity (for a potential flow). Since Γ vanishes for incompressible flow one has $\sum \lambda_i \propto \Gamma$:

$$\sum \lambda_i = -\sum_{jl} C_{jjll}/2 = -\Gamma D_1 d(d-1)(d+2). \quad (1.15)$$

The final answer takes the form

$$\lambda_i = D_1 \left[d(d-2i+1) - 2\Gamma(d+(d-2)i)\right] . \quad (1.16)$$

Formulas (1.14) and (1.16) correspond respectively to (3.9) and (3.17) from Gawędzki. The senior Lyapunov exponent, $\lambda_1 = D_1(d-1)\left[d - 4\Gamma\right]$ decreases linearly when compressibility degree grows. Thus the effect of compressibility is to suppress the exponential divergence of nearby trajectories. For an incompressible random flow where $\Gamma = 0$, the first Lyapunov exponent is positive. Generally $\lambda_1 \geq 0$ for incompressible flow because volume conservation implies $\sum \lambda_i = 0 \leq \lambda_1$. On the hand in the case $d = 1$ where compressibility is always maximal $\Gamma = 1$ (the only incompressible flow in one dimension is a constant one) we always have $\lambda_1 < 0$ (to define this limit one should assume that $D_1(d-1)$ is a finite constant). In dimensions 2 and 3, Γ becomes negative at the critical compressibility $\Gamma_{\rm cr} = d/4$. Finally λ_1 is always positive at $d > 4$ while in four dimensions $\lambda_1 > 0$ unless the flow is potential where $\lambda_1 = 0$.

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