Fluid Mechanics

a short course for physicists

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Preface

Why study fluid mechanics? The primary reason is not even technical, it is cultural: a physicist is defined as one who looks around and understands at least part of the material world. One of the goals of this book is to let you understand how the wind blows and how the water flows so that swimming or flying you may appreciate what is actually going on. The secondary reason is to do with applications: whether you are to engage with astrophysics or biophysics theory or to build an apparatus for condensed matter research, you need the ability to make correct fluid-mechanics estimates; some of the art for doing this will be taught in the book. Yet another reason is conceptual: mechanics is the basis of the whole of physics in terms of intuition and mathematical methods. Concepts introduced in the mechanics of particles were subsequently applied to optics, electromagnetism, quantum mechanics etc; here you will see the ideas and methods developed for the mechanics of fluids, which are used to analyze other systems with many degrees of freedom in statistical physics and quantum field theory. And last but not least: at present, fluid mechanics is one of the most actively developing fields of physics, mathematics and engineering so you may wish to participate in this exciting development.

Even for physicists who are not using fluid mechanics in their work taking a one-semester course on the subject would be well worth their effort. This is one such course. It presumes no prior acquaintance with the subject and requires only basic knowledge of vector calculus and analysis. On the other hand, applied mathematicians and engineers working on fluid mechanics may find in this book several new insights presented from a physicist's perspective. In choosing from the enormous wealth of material produced by the last four centuries of ever-accelerating research, preference was given to the ideas and concepts that teach lessons whose importance transcends the confines of one specific subject as they prove useful time and again across the whole spectrum of modern physics. To much delight, it turned out to be possible to weave the subjects into a single coherent narrative so that the book is a novel rather than a collection of short stories.

We approach every subject as physicists: start from qualitative considerations (dimensional reasoning, symmetries and conservation laws), then use back-of-the-envelope estimates and crown it with concise yet consistent derivations. Fluid mechanics is an essentially experimental science as any branch of physics. Experimental data guide us at each step which is often far from trivial: for example, energy is not conserved even in the frictionless limit and other symmetries can be unexpectedly broken, which makes a profound impact on estimates and derivations.

Lecturers and students using the book for a course will find out that its 12 sections comfortably fit into 12 lectures plus, if needed, problem-solving sessions. Sections 2.3 and 3.1 each contain one extra subsection treated at a problem-solving session (specifically, Sects. 2.3.5 and 3.1.2, but the choice may be different). For 2nd year students, one can use a shorter version, excluding Sects. 3.2-3.4 and two small-font parts in Sects. 2.2.1 and 2.3.4. The lectures are supposed to be self-contained so that no references are included in the text. Epilogue and endnotes provide guidance for further reading; the references, that are cited there more than once, are collected in the reference list at the end. Those using the book for self-study will find out that in about two intense weeks one is able to master the basic elements of fluid physics. Those reading for amusement can disregard the endnotes, skip all the derivations and half of the resulting formulas and still be able to learn a lot about fluids and a bit about the world around us, helped by numerous pictures.

In many years of teaching this course at the Weizmann Institute, I have benefitted from the generations of brilliant students who taught me never stop looking for simpler explanations and deeper links between branches of physics. I also learnt from V. Arnold, E. Balkovsky, E. Bodenschatz, G. Boffetta, A. Celani, M. Chertkov, B. Chirikov, G. Eyink, U. Frisch, K. Gawedzki, V. Geshkenbein, L. Kadanoff, K. Khanin, D. Khmelnitskii, I. Kolokolov, G. Kotkin, R. Kraichnan, E. Kuznetsov, A. Larkin, V. Lebedev, V. L'vov, B. Lugovtsov, S. Lukaschuk, K. Moffatt, A. Newell, A. Polyakov, I. Procaccia, A. Pumir, A. Rubenchik, D. Ryutov, V. Serbo, E. Siggia, A. Shafarenko, M. Shats, B. Shraiman, Ya. Sinai, M. Spektor, K. Sreenivasan, V. Steinberg, S. Turitsyn, K. Turitsyn, G. Vekshtein, M. Vergassola, P. Wiegmann, V. Zakharov, A. Zamolodchikov, Ya. Zeldovich. Special thanks to Itzhak Fouxon and Marija Vucelja who were instructors in problem-solving sessions and wrote draft solutions for some of the exercises. Errors, both of omission and of commission, is my responsibility alone. The book is dedicated to my family.

Contents

Prologue

"The water's language was a wondrous one,
some narrative on a recurrent subject..."
A. Tarkovsky [1]

There are two protagonists in this story: inertia and friction. One meets them first in the mechanics of particles and solids where their interplay is not very complicated: inertia tries to keep the motion while friction tries to stop it. Going from a finite to an infinite number of degrees of freedom is always a game-changer. We will see in this book how an infinitesimal viscous friction makes fluid motion infinitely more complicated than inertia alone would ever manage to produce. Without friction, most incompressible flows would stay potential i.e. essentially trivial. At solid surfaces, friction produces vorticity which is carried away by inertia and changes the flow in the bulk. Instabilities then bring about turbulence, and statistics emerges from dynamics. Vorticity penetrating the bulk makes life interesting in ideal fluids though in a way different from superfluids and superconductors.

On the other hand, compressibility makes even potential flows non-trivial as it allows inertia to develop a finite-time singularity (shock), which friction manages to stop. Only in a wave motion, inertia is able to have an interesting life in the absence of friction, when it is instead partnered with medium anisotropy or inhomogeneity, which cause the dispersion of waves. The soliton is a happy child of that partnership. Yet even there, a modulational instability can bring a finite-time singularity in the form of self-focussing or collapse. At the end we discuss how inertia, friction and dispersion may act together.

On a formal level, inertia of a continuous medium is described by a nonlinear term in the equation of motion. Friction and dispersion are described by linear terms which, however, have the highest spatial derivatives so that the limit of zero friction and zero dispersion is singular. Friction is not only singular but also a symmetry-breaking perturbation, which leads to an anomaly when the effect of symmetry breaking remains finite even in the limit of vanishing viscosity.

The first chapter introduces basic notions and describes stationary flows, inviscid and viscous. Time starts to run in the second chapter where we discuss instabilities, turbulence and sound. The third chapter is devoted to dispersive waves, it progresses from linear to nonlinear waves, solitons, collapses and wave turbulence. Epilogue gives a guide to further reading and briefly describes the present-day activities in fluid mechanics. At the end, detailed solutions of the exercises are given.

1
Basic equations and steady flows

In this Chapter, we define the subject, derive the equations of motion and describe their fundamental symmetries. We start from hydrostatics where all forces are normal. We then try to consider flows this way as well, neglecting friction. That allows us to understand some features of inertia, most important induced mass, but the overall result is a failure to describe a fluid flow past a body. We then are forced to introduce friction and learn how it interacts with inertia producing real flows. We briefly describe an Aristotelean world where friction dominates. In an opposite limit we discover that the world with a little friction is very much different from the world with no friction at all.

1.1 Definitions and basic equations

Continuous media. Absence of oblique stresses in equilibrium. Pressure and density as thermodynamic quantities. Continuous motion. Continuity equation and Euler's equation. Boundary conditions. Entropy equation. Isentropic flows. Steady flows. Bernoulli equation. Limiting velocity for the efflux into vacuum. Vena contracta.

1.1.1 Definitions

We deal with *continuous media* where matter may be treated as homogeneous in structure down to the smallest portions. Term *fluid* embraces both liquids and gases and relates to the fact that even though any fluid may resist deformations, that resistance cannot prevent deformation from happening. The reason is that the resisting force vanishes with the rate of deformation. Whether one treats the matter as a fluid or a solid may depend on the time available for observation. As prophetess Deborah sang, "The mountains flowed before the Lord" (Judges 5:5). The ratio of the relaxation time to the observation time is called the Deborah number [1]. The smaller the number the more fluid the material.

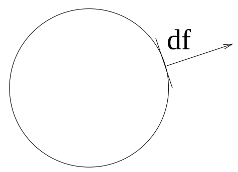
A fluid can be in equilibrium only if all the mutual forces between two adjacent parts are normal to the common surface. That *experimental* observation is the basis of Hydrostatics. If one applies a force parallel (tangential) to the common surface then the fluid layer on one side of the surface start sliding over the layer on the other side. Such sliding motion will lead to a friction between layers. For example, if you cease to stir tea in a glass it could come to rest only because of such tangential forces i.e. friction. Indeed, if the mutual action between the portions on the same radius was wholly normal i.e. radial, then the conservation of the moment of momentum about the rotation axis would cause the fluid to rotate forever.

Since tangential forces are absent at rest or for a uniform flow, it is natural to consider first the flows where such forces are small and can be neglected. Therefore, a natural first step out of hydrostatics into hydrodynamics is to restrict ourselves with a purely normal forces, assuming velocity gradients small (whether such step makes sense at all and how long such approximation may last is to be seen). Moreover, the intensity of a normal force per unit area does not depend on the direction in a fluid, the statement called the Pascal law (see Exercise 1.1). We thus characterize the internal force (or stress) in a fluid by a single scalar function $p(\mathbf{r},t)$ called pressure which is the force per unit area. From the viewpoint of the internal state of the matter, pressure is a macroscopic (thermodynamic) variable. To describe completely the internal state of the fluid, one needs the second thermodynamical quantity, e.g. the density $\rho(\mathbf{r},t)$, in addition to the pressure.

What *analytic properties* of the velocity field $\mathbf{v}(\mathbf{r},t)$ we need to presume? We suppose the velocity to be finite and a continuous function of $\mathbf{r}$. In addition, we suppose the first spatial derivatives to be everywhere finite. That makes the *motion continuous*, i.e. trajectories of the fluid particles do not cross. The equation for the distance $\delta\mathbf{r}$ between two close fluid particles is $d\delta\mathbf{r}/dt = \delta\mathbf{v}$ so, mathematically speaking, finiteness of $\nabla\mathbf{v}$ is Lipschitz condition for this equation to have a unique solution [a simple example of non-unique solutions for non-Lipschitz equation is $dx/dt = |x|^{1-\alpha}$ with *two* solutions, $x(t) = (\alpha t)^{1/\alpha}$ and $x(t) = 0$ starting from zero for $\alpha > 0$]. For a continuous motion, any surface moving with the fluid completely separates matter on the two sides of it. We don't yet know when exactly the continuity assumption is consistent with the equations of the fluid motion. Whether velocity derivatives may turn into infinity after a finite time is a subject of active research for an incompressible viscous fluid (and a subject of the one-million-dollar Clay prize). We shall see below that a compressible inviscid flow generally develops discontinuities called shocks.

1.1.2 Equations of motion for an ideal fluid

The Euler equation. The force acting on any fluid volume is equal to the pressure integral over the surface: $-\oint p\, d\mathbf{f}$. The surface area element $d\mathbf{f}$ is a vector directed as outward normal:



Let us transform the surface integral into the volume one: $-\oint p\, d\mathbf{f} = -\int \nabla p\, dV$. The force acting on a unit volume is thus $-\nabla p$ and it must be equal to the product of the mass ρ and the acceleration $d\mathbf{v}/dt$. The latter is not the rate of change of the fluid velocity at a fixed point in space but the rate of change of the velocity of a given fluid particle as it moves about in space. One uses the chain rule differentiation to express this (substantial or material) derivative in terms of quantities referring to points fixed in space. During the time dt the fluid particle changes its velocity by $d\mathbf{v}$ which is composed of two parts, temporal and spatial:

$$d\mathbf{v} = dt\frac{\partial \mathbf{v}}{\partial t} + (d\mathbf{r}\cdot\nabla)\mathbf{v} = dt\frac{\partial \mathbf{v}}{\partial t} + dx\frac{\partial \mathbf{v}}{\partial x} + dy\frac{\partial \mathbf{v}}{\partial y} + dz\frac{\partial \mathbf{v}}{\partial z}\ . \tag{1.1}$$

It is the change in the fixed point plus the difference at two points $d\mathbf{r}$ apart where $d\mathbf{r} = \mathbf{v}dt$ is the distance moved by the fluid particle during dt. Dividing (1.1) by dt we obtain the substantial derivative as local derivative plus convective derivative:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\ .$$

Any function $F(\mathbf{r}(t), t)$ varies for a moving particle in the same way according to the chain rule differentiation:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\mathbf{v}\cdot\nabla)F\ .$$

Writing now the second law of Newton for a unit mass of a fluid, we come to the equation derived by Euler (Berlin, 1757; Petersburg, 1759):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} \ . \tag{1.2}$$

Before Euler, the acceleration of a fluid had been considered as due to the difference of the pressure exerted by the enclosing walls. Euler introduced the pressure field *inside* the fluid. We see that even when the flow is steady, $\partial \mathbf{v}/\partial t = 0$, the acceleration is nonzero as long as $(\mathbf{v} \cdot \nabla)\mathbf{v} \neq 0$, that is if the velocity field changes in space along itself. For example, for a steadily rotating fluid shown in Figure 1.1, the vector $(\mathbf{v} \cdot \nabla)\mathbf{v}$ has a nonzero radial component v^2/r. The radial acceleration times the density must be given by the radial pressure gradient: $dp/dr = \rho v^2/r$.

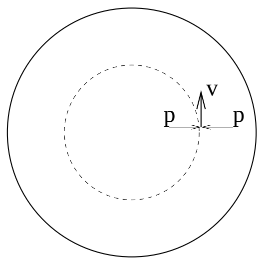


Figure 1.1 Pressure p is normal to circular surfaces and cannot change the moment of momentum of the fluid inside or outside the surface; the radial pressure gradient changes the direction of velocity $\mathbf{v}$ but does not change its modulus.

We can also add an external body force per unit mass (for gravity $\mathbf{f} = \mathbf{g}$):

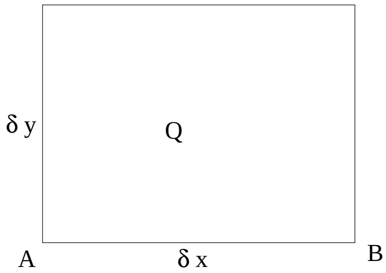
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{f} \ . \tag{1.3}$$

The term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ describes inertia and makes the equation (1.3) nonlinear.

Continuity equation expresses conservation of mass. If Q is the volume of a moving element then $d\rho Q/dt = 0$ that is

$$Q\frac{d\rho}{dt} + \rho\frac{dQ}{dt} = 0 \ . \tag{1.4}$$

The volume change can be expressed via $\mathbf{v}(\mathbf{r}, t)$.



The horizontal velocity of the point B relative to the point A is $\delta x \partial v_x/\partial x$. After the time interval dt, the length of the AB edge is $\delta x(1 + dt\partial v_x/\partial x)$. Overall, after dt, one has the volume change

$$dQ = dt\frac{dQ}{dt} = \delta x \delta y \delta z dt \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = Q\, dt\, div\, \mathbf{v} \ .$$

Substituting that into (1.4) and canceling (arbitrary) Q we obtain the continuity equation

$$\frac{d\rho}{dt} + \rho div\, \mathbf{v} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho + \rho div \mathbf{v} = \frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0 \ . \qquad (1.5)$$

The last equation is almost obvious since for any *fixed volume of space* the decrease of the total mass inside, $-\int(\partial\rho/\partial t)dV$, is equal to the flux $\oint \rho \mathbf{v} \cdot d\mathbf{f} = \int div(\rho \mathbf{v})dV$.

Entropy equation. We have now four equations (1.3,1.5) for five quantities p, ρ, v_x, v_y, v_z, so we need one extra equation. In deriving (1.3,1.5) we have taken no account of energy dissipation neglecting thus internal friction (viscosity) and heat exchange. Fluid without viscosity and thermal conductivity is called *ideal.* The motion of an ideal fluid is adiabatic that is the entropy of any fluid particle remains constant: $ds/dt = 0$, where s is the entropy per unit mass. We can turn this equation into a continuity equation for the entropy density in space

$$\frac{\partial(\rho s)}{\partial t} + div(\rho s \mathbf{v}) = 0 \ . \qquad (1.6)$$

At the boundaries of the fluid, the continuity equation (1.5) is replaced by the *boundary conditions*:

1) On a fixed boundary, $v_n = 0$;

2) On a moving boundary between two immiscible fluids,

$p_1 = p_2$ and $v_{n1} = v_{n2}$.

These are particular cases of the general surface condition. Let $F(\mathbf{r}, t) =$

0 be the equation of the bounding surface. Absence of any fluid flow across the surface requires

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla)F = 0\,,$$

which means, as we now know, the zero rate of F variation for a fluid particle. For a stationary boundary, $\partial F/\partial t = 0$ and $\mathbf{v} \perp \nabla F \Rightarrow v_n = 0$.

Eulerian and Lagrangian descriptions. We thus encountered two alternative ways of description. The equations (1.3,1.6) use the coordinate system fixed in space, like field theories describing electromagnetism or gravity. That way of description is called Eulerian in fluid mechanics. Another approach is called Lagrangian, it is a generalization of the approach taken in particle mechanics. This way one follows fluid particles[2] and treats their current coordinates, $\mathbf{r}(\mathbf{R}, t)$, as functions of time and their initial positions $\mathbf{R} = \mathbf{r}(\mathbf{R}, 0)$. The substantial derivative is thus the Lagrangian derivative since it sticks to a given fluid particle, that is keeps $\mathbf{R}$ constant: $d/dt = (\partial/\partial t)_R$. Conservation laws written for a unit-mass quantity $\mathcal{A}$ have a Lagrangian form:

$$\frac{d\mathcal{A}}{dt} = \frac{\partial \mathcal{A}}{\partial t} + (\mathbf{v}\nabla)\mathcal{A} = 0\ .$$

Every Lagrangian conservation law together with mass conservation generates an Eulerian conservation law for a unit-volume quantity $\rho\mathcal{A}$:

$$\frac{\partial(\rho\mathcal{A})}{\partial t} + div(\rho\mathcal{A}\mathbf{v}) = \mathcal{A}\left[\frac{\partial \rho}{\partial t} + div(\rho\mathbf{v})\right] + \rho\left[\frac{\partial \mathcal{A}}{\partial t} + (\mathbf{v}\nabla)\mathcal{A}\right] = 0\ .$$

On the contrary, if the Eulerian conservation law has the form

$$\frac{\partial(\rho\mathcal{B})}{\partial t} + div(\mathbf{F}) = 0$$

and the flux is not equal to the density times velocity, $\mathbf{F} \neq \rho\mathcal{B}\mathbf{v}$, then the respective Lagrangian conservation law does not exist. That means that fluid particles can exchange $\mathcal{B}$ conserving the total space integral — we shall see below that the conservation laws of energy and momentum have that form.

1.1.3 Hydrostatics

A necessary and sufficient condition for fluid to be in a mechanical equilibrium follows from (1.3):

$$\nabla p = \rho\mathbf{f}\ . \tag{1.7}$$

Not any distribution of $\rho(\mathbf{r})$ could be in equilibrium since $\rho(\mathbf{r})\mathbf{f}(\mathbf{r})$ is not necessarily a gradient. If the force is potential, $\mathbf{f} = -\nabla\phi$, then taking *curl* of (1.7) we get

$$\nabla\rho \times \nabla\phi = 0.$$

That means that the gradients of ρ and ϕ are parallel and their level surfaces coincide in equilibrium. The best-known example is gravity with $\phi = gz$ and $\partial p/\partial z = -\rho g$. For an incompressible fluid, it gives

$$p(z) = p(0) - \rho g z \ .$$

For an ideal gas under a homogeneous temperature, which has $p = \rho T/m$, one gets

$$\frac{dp}{dz} = -\frac{pgm}{T} \quad \Rightarrow \quad p(z) = p(0)\exp(-mgz/T) \ .$$

For air at $0°C$, $T/mg \simeq 8\,km$. The Earth atmosphere is described by neither linear nor exponential law because of an inhomogeneous temperature. Assuming a linear temperature decay, $T(z) = T_0 - \alpha z$, one gets a

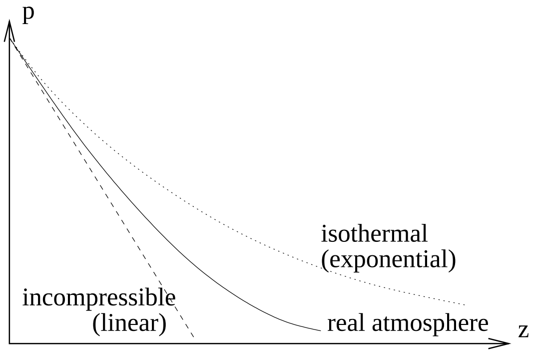


Figure 1.2 Pressure-height dependence for an incompressible fluid (broken line), isothermal gas (dotted line) and the real atmosphere (solid line).

better approximation:

$$\frac{dp}{dz} = -\rho g = -\frac{pmg}{T_0 - \alpha z} \ ,$$
$$p(z) = p(0)(1 - \alpha z/T_0)^{mg/\alpha} \ ,$$

which can be used not far from the surface with $\alpha \simeq 6.5°/km$.

In a (locally) homogeneous gravity field, the density depends only on vertical coordinate in a mechanical equilibrium. According to $dp/dz = -\rho g$, the pressure also depends only on z. Pressure and density determine temperature, which then must also be independent of the horizontal coordinates. Different temperatures at the same height necessarily produce fluid motion, that is why winds blow in the atmosphere and currents flow in the ocean. Another source of atmospheric flows is thermal convection due to a negative vertical temperature gradient. Let us derive the stability criterium for a fluid with a vertical profile $T(z)$. If a fluid element is shifted up adiabatically from z by dz, it keeps its entropy $s(z)$ but acquires the pressure $p' = p(z + dz)$ so its new density is $\rho(s, p')$. For stability, this density must exceed the density of the displaced air at the height $z+dz$, which has the same pressure but different entropy $s' = s(z + dz)$. The condition for stability of the stratification is as follows:

$$\rho(p', s) > \rho(p', s') \;\Rightarrow\; \left(\frac{\partial\rho}{\partial s}\right)_p \frac{ds}{dz} < 0 \ .$$

Entropy usually increases under expansion, $(\partial\rho/\partial s)_p < 0$, and for stability we must require

$$\frac{ds}{dz} = \left(\frac{\partial s}{\partial T}\right)_p \frac{dT}{dz} + \left(\frac{\partial s}{\partial p}\right)_T \frac{dp}{dz} = \frac{c_p}{T}\frac{dT}{dz} - \left(\frac{\partial V}{\partial T}\right)_p \frac{g}{V} > 0 \ . \qquad (1.8)$$

Here we used specific volume $V = 1/\rho$. For an ideal gas the coefficient of the thermal expansion is as follows: $(\partial V/\partial T)_p = V/T$ and we end up with

$$-\frac{dT}{dz} < \frac{g}{c_p} \ . \qquad (1.9)$$

For the Earth atmosphere, $c_p \sim 10^3 J/kg \cdot Kelvin$, and the convection threshold is $10°/km$, not far from the average gradient $6.5°/km$, so that the atmosphere is often unstable with respect to thermal convection[3]. Human body always excites convection in a room-temperature air [4].

The convection stability argument applied to an incompressible fluid rotating with the angular velocity $\Omega(r)$ gives the Rayleigh's stability criterium, $d(r^2\Omega)^2/dr > 0$, which states that the angular momentum of the fluid $L = r^2|\Omega|$ must increase with the distance r from the rotation axis [5]. Indeed, if a fluid element is shifted from r to r' it keeps its angular momentum $L(r)$, so that the local pressure gradient $dp/dr = \rho r'\Omega^2(r')$ must overcome the centrifugal force $\rho r'(L^2 r^4/r'^4)$.

1.1.4 Isentropic motion

The simplest motion corresponds to s =const and allows for a substantial simplification of the Euler equation. Indeed, it would be convenient to represent $\nabla p/\rho$ as a gradient of some function. For this end, we need a function which depends on p, s, so that at s =const its differential is expressed solely via dp. There exists the thermodynamic potential called *enthalpy* defined as $W = E + pV$ per unit mass (E is the internal energy of the fluid). For our purposes, it is enough to remember from thermodynamics the single relation $dE = Tds - pdV$ so that $dW = Tds + Vdp$ [one can also show that $W = \partial(E\rho)/\partial\rho)$]. Since s =const for an isentropic motion and $V = \rho^{-1}$ for a unit mass then $dW = dp/\rho$ and without body forces one has

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla W \ . \qquad (1.10)$$

Such a gradient form will be used extensively for obtaining conservation laws, integral relations etc. For example, representing

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla v^2/2 - \mathbf{v}\times(\nabla\times\mathbf{v})\,,$$

we get

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v}\times(\nabla\times\mathbf{v}) - \nabla(W + v^2/2) \ . \qquad (1.11)$$

The first term in the right-hand side is perpendicular to the velocity. To project (1.11) along the velocity and get rid of this term, we define streamlines as the lines whose tangent is everywhere parallel to the instantaneous velocity. The streamlines are then determined by the relations

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \ .$$

Note that for time-dependent flows streamlines are different from particle trajectories: tangents to streamlines give velocities at a given time while tangents to trajectories give velocities at subsequent times. One records streamlines experimentally by seeding fluids with light-scattering particles; each particle produces a short trace on a short-exposure photograph, the length and orientation of the trace indicates the magnitude and direction of the velocity. Streamlines can intersect only at a point of zero velocity called stagnation point.

Let us now consider a steady flow assuming $\partial\mathbf{v}/\partial t = 0$ and take the component of (1.11) along the velocity at a point:

$$\frac{\partial}{\partial l}(W + v^2/2) = 0 \ . \tag{1.12}$$

We see that $W + v^2/2 = E + p/\rho + v^2/2$ is constant along any given streamline, but may be different for different streamlines (Bernoulli, 1738). Why W rather than E enters the conservation law is discussed after (1.16) below. In a gravity field, $W + gz + v^2/2$ =const. Let us consider several applications of this useful relation.

Incompressible fluid. Under a constant temperature and a constant density and without external forces, the energy E is constant too. One can obtain, for instance, the limiting velocity with which such a liquid escapes from a large reservoir into vacuum:

$$v = \sqrt{2p_0/\rho} \ .$$

For water ($\rho = 10^3\, kg\, m^{-3}$) at atmospheric pressure ($p_0 = 10^5\, N\, m^{-2}$) one gets $v = \sqrt{200} \approx 14\, m/s$.

Adiabatic gas flow. The adiabatic law, $p/p_0 = (\rho/\rho_0)^\gamma$, gives the enthalpy as follows:

$$W = \int \frac{dp}{\rho} = \frac{\gamma p}{(\gamma - 1)\rho} \ .$$

The limiting velocity for the escape into vacuum is

$$v = \sqrt{\frac{2\gamma p_0}{(\gamma - 1)\rho}}$$

that is $\sqrt{\gamma/(\gamma - 1)}$ times larger than for an incompressible fluid (because the internal energy of the gas decreases as it flows, thus increasing the kinetic energy). In particular, a meteorite-damaged spaceship looses the air from the cabin faster than the liquid fuel from the tank. We shall see later that $(\partial P/\partial \rho)_s = \gamma P/\rho$ is the sound velocity squared, c^2, so that $v = c\sqrt{2/(\gamma - 1)}$. For an ideal gas with n degrees of freedom, $W = E + p/\rho = nT/2m + T/m$ so that $\gamma = (2 + n)/n$. For bi-atomic gas at not very high temperature, $n = 5$.

Efflux from a small orifice under the action of gravity. Supposing the external pressure to be the same at the horizonal surface and at the orifice, we apply the Bernoulli relation to the streamline which originates at the upper surface with almost zero velocity and exits with the velocity $v = \sqrt{2gh}$ (Torricelli, 1643). The Torricelli formula is not of much use practically to calculate the rate of discharge as the orifice area times $\sqrt{2gh}$ (the fact known to wine merchants long before physicists). Indeed, streamlines converge from all sides towards the orifice so that the jet continues to converge for a while after coming out. Moreover, that converging motion makes the pressure in the interior of the jet somewhat greater that at the surface so that the velocity in the interior is somewhat less than $\sqrt{2gh}$. The experiment shows that contraction ceases and

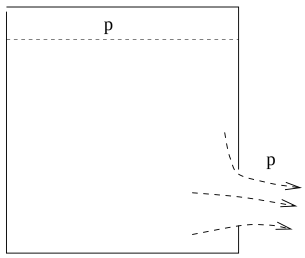


Figure 1.3 Streamlines converge coming out of the orifice.

the jet becomes cylindrical at a short distance beyond the orifice. That point is called "vena contracta" and the ratio of jet area there to the orifice area is called the coefficient of contraction. The estimate for the discharge rate is $\sqrt{2gh}$ times the orifice area times the coefficient of contraction. For a round hole in a thin wall, the coefficient of contraction is experimentally found to be 0.62. The Exercise 1.3 presents a particular case where the coefficient of contraction can be found exactly.

Bernoulli relation is also used in different devices that measure the flow velocity. Probably, the simplest such device is the *Pitot tube* shown in Figure 1.4. It is open at both ends with the horizontal arm facing upstream. Since the liquid does not move inside the tube than the velocity is zero at the point labelled B. On the one hand, the pressure difference at two pints on the same streamline can be expressed via the velocity at A: $P_B - P_A = \rho v^2/2$. On the other hand, it is expressed via the height h by which liquid rises above the surface in the vertical arm of the tube: $P_B - P_A = \rho gh$. That gives $v^2 = 2gh$.

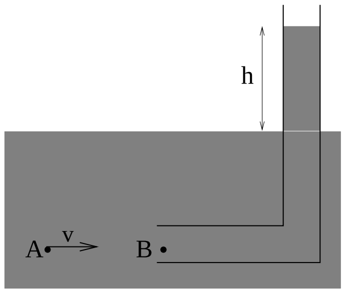


Figure 1.4 Pitot tube that determines the velocity v at the point A by measuring the height h.

1.2 Conservation laws and potential flows

Kinematics: Strain and Rotation. Kelvin's theorem of conservation of circulation. Energy and momentum fluxes. Irrotational flow as a potential one. Incompressible fluid. Conditions of incompressibility. Potential flows in two dimensions.

1.2.1 Kinematics

The relative motion near a point is determined by the velocity difference between neighbouring points:

$$\delta v_i = r_j \partial v_i / \partial x_j \ .$$

It is convenient to analyze the tensor of the velocity derivatives by decomposing it into symmetric and antisymmetric parts: $\partial v_i/\partial x_j = S_{ij} + A_{ij}$. The symmetric tensor $S_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ is called strain, it can be always transformed into a diagonal form by an orthogonal transformation (i.e. by the rotation of the axes). The diagonal components are the rates of stretching in different directions. Indeed, the equation for the distance between two points along a principal direction has a form: $\dot{r}_i = \delta v_i = r_i S_{ii}$ (no summation over i). The solution is as follows:

$$r_i(t) = r_i(0) \exp\left[\int_0^t S_{ii}(t')\, dt'\right] \ .$$

For a permanent strain, the growth/decay is exponential in time. One recognizes that a purely straining motion converts a spherical material element into an ellipsoid with the principal diameters that grow (or decay) in time, the diameters do not rotate. Indeed, consider a circle of the radius R at $t = 0$. The point that starts at $x_0, y_0 = \sqrt{R^2 - x_0^2}$ goes into

$$x(t) = e^{S_{11}t} x_0 \,,$$
$$y(t) = e^{S_{22}t} y_0 = e^{S_{22}t}\sqrt{R^2 - x_0^2} = e^{S_{22}t}\sqrt{R^2 - x^2(t)e^{-2S_{11}t}} \,,$$
$$x^2(t)e^{-2S_{11}t} + y^2(t)e^{-2S_{22}t} = R^2 \,. \tag{1.13}$$

The equation (1.13) describes how the initial fluid circle turns into the ellipse whose eccentricity increases exponentially with the rate $|S_{11} - S_{22}|$.

The sum of the strain diagonal components is $div\,\mathbf{v} = S_{ii}$ which determines the rate of the volume change: $Q^{-1}dQ/dt = -\rho^{-1}d\rho/dt = div\,\mathbf{v} = S_{ii}$.

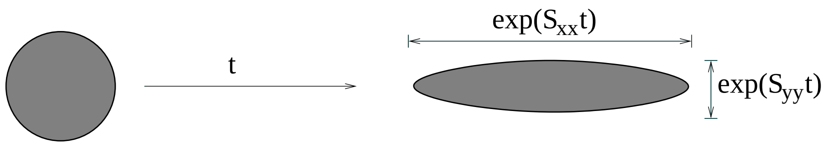


Figure 1.5 Deformation of a fluid element by a permanent strain.

The antisymmetric part $A_{ij} = (\partial v_i/\partial x_j - \partial v_j/\partial x_i)/2$ has only three independent components so it could be represented via some vector ω: $A_{ij} = -\epsilon_{ijk}\omega_k/2$. The coefficient $-1/2$ is introduced to simplify the relation between $\mathbf{v}$ and ω:

$$\omega = \nabla \times \mathbf{v} \,.$$

The vector ω is called *vorticity* as it describes the rotation of the fluid element: $\delta\mathbf{v} = [\omega \times \mathbf{r}]/2$. It has a meaning of twice the effective local angular velocity of the fluid. Plane shearing motion like $v_x(y)$ corresponds to strain and vorticity being equal in magnitude.

1.2.2 Kelvin's theorem

That theorem describes the conservation of velocity circulation for isentropic flows. For a rotating cylinder of a fluid, the momentum of momentum is proportional to the velocity circulation around the cylinder circumference. The momentum of momentum and circulation are both conserved when there are only normal forces, as was already mentioned

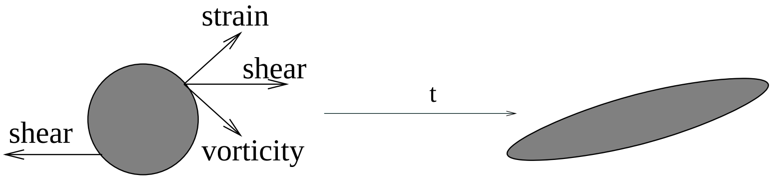


Figure 1.6 Deformation and rotation of a fluid element in a shear flow. Shearing motion is decomposed into a straining motion and rotation.

at the beginning of Sect. 1.1.1. Let us show that this is also true for every "fluid" contour which is made of fluid particles. As fluid moves, both the velocity and the contour shape change:

$$\frac{d}{dt}\oint \mathbf{v}\cdot d\mathbf{l} = \oint \mathbf{v}(d\mathbf{l}/dt) + \oint (d\mathbf{v}/dt)\cdot d\mathbf{l} = 0 \ .$$

The first term here disappears because it is a contour integral of the complete differential: since $d\mathbf{l}/dt = \delta\mathbf{v}$ then $\oint \mathbf{v}(d\mathbf{l}/dt) = \oint \delta(v^2/2) = 0$. In the second term we substitute the Euler equation for isentropic motion, $d\mathbf{v}/dt = -\nabla W$, and use the Stokes formula which tells that the circulation of a vector around the closed contour is equal to the flux of the curl through any surface bounded by the contour: $\oint \nabla W \cdot d\mathbf{l} = \int \nabla \times \nabla W \, d\mathbf{f} = 0$.

Stokes formula also tells us that $\oint \mathbf{v}d\mathbf{l} = \int \omega\cdot d\mathbf{f}$. Therefore, the conservation of the velocity circulation means the conservation of the vorticity flux. To better appreciate this, consider an alternative derivation. Taking curl of (1.11) we get

$$\frac{\partial\omega}{\partial t} = \nabla \times (\mathbf{v}\times\omega) \ . \tag{1.14}$$

This is the same equation that describes the magnetic field in a perfect conductor: substituting the condition for the absence of the electric field in the frame moving with the velocity $\mathbf{v}$, $cE + \mathbf{v}\times H = 0$, into the Maxwell equation $\partial H/\partial t = -c\nabla\times E$, one gets $\partial H/\partial t = \nabla\times(\mathbf{v}\times H)$. The magnetic flux is conserved in a perfect conductor and so is the vorticity flux in an isentropic flow. One can visualize vector field introducing field lines which give the direction of the field at any point while their density is proportional to the magnitude of the field. Kelvin's theorem means that vortex lines move with material elements in an inviscid fluid exactly like magnetic lines are frozen into a perfect conductor. One way to prove that is to show that ω/ρ (and H/ρ) satisfy the same equation as the distance $\mathbf{r}$ between two fluid particles: $d\mathbf{r}/dt = (\mathbf{r}\cdot\nabla)\mathbf{v}$. This is done using $d\rho/dt = -\rho div\,\mathbf{v}$ and applying the general relation

$$\nabla\times(A\times B) = A(\nabla\cdot B) - B(\nabla\cdot A) + (B\cdot\nabla)A - (A\cdot\nabla)B \qquad (1.15)$$

to $\nabla\times(\mathbf{v}\times\omega) = (\omega\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\omega - \omega\, div\,\mathbf{v}$. We then obtain

$$\frac{d}{dt}\frac{\omega}{\rho} = \frac{1}{\rho}\frac{d\omega}{dt} - \frac{\omega}{\rho^2}\frac{d\rho}{dt} = \frac{1}{\rho}\left[\frac{\partial\omega}{\partial t} + (\mathbf{v}\cdot\nabla)\omega\right] + \frac{div\,\mathbf{v}}{\rho}$$
$$= \frac{1}{\rho}\left[(\omega\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\omega - \omega\, div\,\mathbf{v} + (\mathbf{v}\cdot\nabla)\omega\right] + \frac{div\,\mathbf{v}}{\rho} = \left(\frac{\omega}{\rho}\cdot\nabla\right)\mathbf{v}\ .$$

Since $\mathbf{r}$ and ω/ρ move together, then any two close fluid particles chosen on the vorticity line always stay on it. Consequently any fluid particle stays on the same vorticity line so that any fluid contour never crosses vorticity lines and the flux is indeed conserved.

1.2.3 Energy and momentum fluxes

Let us now derive the equation that expresses the conservation law of energy. The energy density (per unit volume) in the flow is $\rho(E+v^2/2)]$. For isentropic flows, one can use $\partial\rho E/\partial\rho = W$ and calculate the time derivative

$$\frac{\partial}{\partial t}\left(\rho E + \frac{\rho v^2}{2}\right) = \left(W + v^2/2\right)\frac{\partial\rho}{\partial t} + \rho v\cdot\frac{\partial v}{\partial t} = -div\left[\rho v(W+v^2/2)\right]\ .$$

Since the right-hand side is a total derivative then the integral of the energy density over the whole space is conserved. The same Eulerian conservation law in the form of a continuity equation can be obtained in a general (non-isentropic) case as well. It is straightforward to calculate the time derivative of the kinetic energy:

$$\frac{\partial}{\partial t}\frac{\rho v^2}{2} = -\frac{v^2}{2}div\,\rho v - v\cdot\nabla p - \rho v\cdot(v\nabla)v$$
$$= -\frac{v^2}{2}div\,\rho v - v(\rho\nabla W - \rho T\nabla s) - \rho v\cdot\nabla v^2/2\ .$$

For calculating $\partial(\rho E)/\partial t$ we use $dE = Tds - pdV = Tds + p\rho^{-2}d\rho$ so that $d(\rho E) = Ed\rho + \rho dE = Wd\rho + \rho T ds$ and

$$\frac{\partial(\rho E)}{\partial t} = W\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t} = -W div\,\rho v - \rho T v\cdot\nabla s\ .$$

Adding everything together one gets

$$\frac{\partial}{\partial t}\left(\rho E + \frac{\rho v^2}{2}\right) = -div\left[\rho v(W+v^2/2)\right]\ . \qquad (1.16)$$

As usual, the rhs is the divergence of the flux, indeed:

$$\frac{\partial}{\partial t}\int\left(\rho E+\frac{\rho v^2}{2}\right)dV=-\oint\rho(W+v^2/2)]\mathbf{v}\cdot d\mathbf{f}\ .$$

Note the remarkable fact that the energy flux is

$$\rho\mathbf{v}(W+v^2/2)=\rho\mathbf{v}(E+v^2/2)+p\mathbf{v}$$

which is not equal to the energy density times $\mathbf{v}$ but contains an extra pressure term which describes the work done by pressure forces on the fluid. In other terms, any unit mass of the fluid carries an amount of energy $W+v^2/2$ rather than $E+v^2/2$. That means, in particular, that for energy there is no (Lagrangian) conservation law for unit mass $d(\cdot)/dt=0$ that is valid for passively transported quantities like entropy. This is natural because different fluid elements exchange energy by doing work.

Momentum is also exchanged between different parts of fluid so that the conservation law must have the form of a continuity equation written for the momentum density. The momentum of the unit volume is the vector $\rho\mathbf{v}$ whose every component is conserved so it should satisfy the equation of the form

$$\frac{\partial\rho v_i}{\partial t}+\frac{\partial\Pi_{ik}}{\partial x_k}=0\ .$$

Let us find the momentum flux Π_{ik} — the flux of the i-th component of the momentum across the surface with the normal along k. Substitute the mass continuity equation $\partial\rho/\partial t=-\partial(\rho v_k)/\partial x_k$ and the Euler equation $\partial v_i/\partial t=-v_k\partial v_i/\partial x_k-\rho^{-1}\partial p/\partial x_i$ into

$$\frac{\partial\rho v_i}{\partial t}=\rho\frac{\partial v_i}{\partial t}+v_i\frac{\partial\rho}{\partial t}=-\frac{\partial p}{\partial x_i}-\frac{\partial}{\partial x_k}\rho v_i v_k\ ,$$

that is

$$\Pi_{ik}=p\delta_{ik}+\rho v_i v_k\ . \tag{1.17}$$

Plainly speaking, along $\mathbf{v}$ there is only the flux of parallel momentum $p+\rho v^2$ while perpendicular to $\mathbf{v}$ the momentum component is zero at the given point and the flux is p. For example, if we direct the x-axis along velocity at a given point then $\Pi_{xx}=p+v^2$, $\Pi_{yy}=\Pi_{zz}=p$ and all the off-diagonal components are zero.

We have finished the formulations of the equations and their general properties and will discuss now the simplest case which allows for an analytic study. This involves several assumptions.

1.2.4 Irrotational and incompressible flows

Irrotational flows are defined as having zero vorticity: $\omega = \nabla \times \mathbf{v} \equiv 0$. In such flows, $\oint \mathbf{v} \cdot d\mathbf{l} = 0$ round any closed contour, which means, in particular, that there are no closed streamlines for a single-connected domain. Note that the flow has to be isentropic to stay irrotational (i.e. inhomogeneous heating can generate vortices). A zero-curl vector field is potential, $\mathbf{v} = \nabla\phi$, so that the Euler equation (1.11) takes the form

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + W\right) = 0 \ .$$

After integration, one gets

$$\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + W = C(t)$$

and the space independent function $C(t)$ can be included into the potential, $\phi(r,t) \to \phi(r,t) + \int^t C(t')dt'$, without changing velocity. Eventually,

$$\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + W = 0 \ . \tag{1.18}$$

For a steady flow, we thus obtained a more strong Bernoulli theorem with $v^2/2 + W$ being the same constant along all the streamlines in distinction from a general case where it may be a different constant along different streamlines.

Absence of vorticity provides for a dramatic simplification which we exploit in this Section and the next one. Unfortunately, irrotational flows are much less frequent than Kelvin's theorem suggests. The main reason is that (even for isentropic flows) the viscous boundary layers near solid boundaries generate vorticity as we shall see in Sect. 1.5. Yet we shall also see there that large regions of the flow can be unaffected by the vorticity generation and effectively described as irrotational. Another class of potential flows is provided by small-amplitude oscillations (like waves or motions due to oscillations of an immersed body). If the amplitude of oscillations a is small comparatively to the velocity scale of change l then $\partial v/\partial t \simeq v^2/a$ while $(v\nabla)v \simeq v^2/l$ so that the nonlinear term can be neglected and $\partial v/\partial t = -\nabla W$. Taking curl of this equation we see that ω is conserved but its average is zero in oscillating motion so that $\omega = 0$.

Incompressible fluid can be considered as such if the density can be considered constant. That means that in the continuity equation,

$\partial\rho/\partial t + (v\nabla)\rho + \rho div\, v = 0$, the first two terms are much smaller than the third one. Let the velocity v change over the scale l and the time τ. The density variation can be estimated as

$$\delta\rho \simeq (\partial\rho/\partial p)_s \delta p \simeq (\partial\rho/\partial p)_s \rho v^2 \simeq \rho v^2/c^2 \,, \tag{1.19}$$

where the pressure change was estimated from the Bernoulli relation. Requiring

$$(v\nabla)\rho \simeq v\delta\rho/l \ll \rho div\, v \simeq \rho v/l \,,$$

we get the condition $\delta\rho \ll \rho$ which, according to (1.19), is true as long as the velocity is much less than the speed of sound. The second condition, $\partial\rho/\partial t \ll \rho div\, v$, is the requirement that the density changes slow enough:

$$\partial\rho/\partial t \simeq \delta\rho/\tau \simeq \delta p/\tau c^2 \simeq \rho v^2/\tau c^2 \ll \rho v/l \simeq \rho div\, v \ .$$

That suggests $\tau \gg (l/c)(v/c)$ — that condition is actually more strict since the comparison of the first two terms in the Euler equation suggests $l \simeq v\tau$ which gives $\tau \gg l/c$. We see that the extra condition of incompressibility is that the typical time of change τ must be much larger than the typical scale of change l divided by the sound velocity c. Indeed, sound equilibrates densities in different points so that all flow changes must be slow to let sound pass.

For an incompressible fluid, the continuity equation is thus reduced to

$$\mathrm{div}\, \mathbf{v} = 0 \ . \tag{1.20}$$

For isentropic motion of an incompressible fluid, the internal energy does not change ($dE = Tds + p\rho^{-2}d\rho$) so that one can put everywhere $W = p/\rho$. Since density is no more an independent variable, the equations can be chosen that contain only velocity: one takes (1.14) and (1.20).

In two dimensions, incompressible flow can be characterized by a single scalar function. Since $\partial v_x/\partial x = -\partial v_y/\partial y$ then we can introduce the *stream function* ψ defined by $v_x = \partial\psi/\partial y$ and $v_y = -\partial\psi/\partial x$. Recall that the streamlines are defined by $v_x dy - v_y dx = 0$ which now correspond to $d\psi = 0$ that is indeed the equation $\psi(x, y)$ =const determines streamlines. Another important use of the stream function is that the flux through any line is equal to the difference of ψ at the endpoints (and is thus independent of the line form - an evident consequence of incompressibility):

$$\int_1^2 v_n dl = \int_1^2 (v_x dy - v_y dx) = \int d\psi = \psi_2 - \psi_1 \ . \tag{1.21}$$

Here v_n is the velocity projection on the normal that is the flux is equal to the modulus of the vector product $\int |\mathbf{v} \times d\mathbf{l}|$, see Figure 1.7. Solid boundary at rest has to coincide with one of the streamlines.

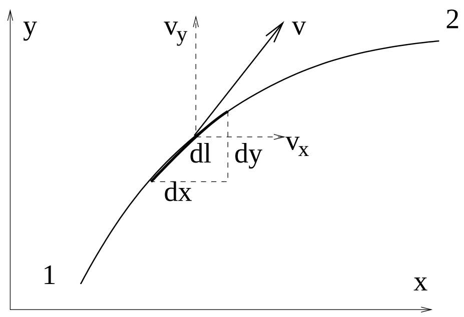
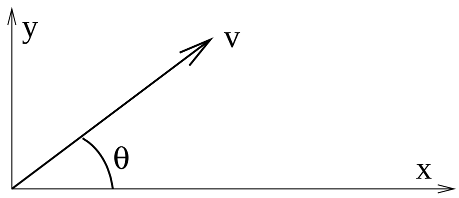


Figure 1.7 The flux through the line element dl is the flux to the right $v_x dy$ minus the flux up $v_y dx$ in agreement with (1.21).

Potential flow of an incompressible fluid is described by a linear equation. By virtue of (1.20) the potential satisfies the Laplace equation[6]

$$\Delta\phi = 0 \ ,$$

with the condition $\partial\phi/\partial n = 0$ on a solid boundary at rest.



Particularly beautiful is the description of two-dimensional (2d) potential incompressible flows. Both potential and stream function exist in this case. The equations

$$v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \ , \qquad v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \ , \tag{1.22}$$

could be recognized as the Cauchy-Riemann conditions for the complex potential $w = \phi + \imath\psi$ to be an analytic function of the complex argument $z = x + \imath y$. That means that the rate of change of w does not depend on the direction in the x, y-plane, so that one can define the complex derivative dw/dz, which exists everywhere. For example, both choices $dz = dx$ and $dz = \imath dy$ give the same answer by virtue of (1.22):

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + \imath \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\imath \partial y} + \frac{\partial \psi}{\partial y} = v_x - \imath v_y = v e^{-\imath\theta} , \quad \mathbf{v} = v_x + i v_y = \frac{d\bar{w}}{d\bar{z}} .$$

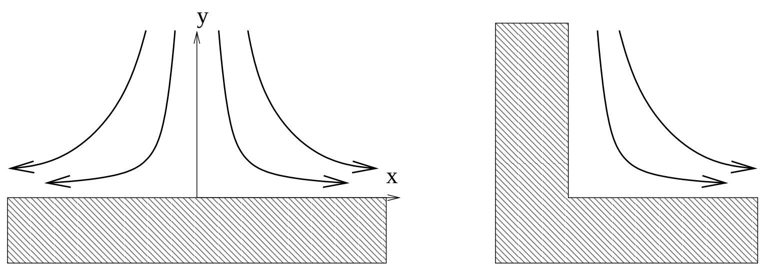
Complex form allows one to describe many flows in a compact form and find flows in a complex geometry by mapping a domain into a standard one. Such transformation must be conformal i.e. done by an analytic function so that the equations (1.22) preserve their form in the new coordinates [7].

We thus get our first (infinite) family of flows: any complex function analytic in a domain and having a constant imaginary part on the boundary describes a potential flow of an incompressible fluid in this domain. Uniform flow is just $w = (v_x - \imath v_y)z$. Few other examples:

1) Potential flow near a stagnation point $\mathbf{v} = 0$ (inside the domain or on a smooth boundary) is expressed via the rate-of-strain tensor S_{ij}: $\phi = S_{ij} x_i x_j / 2$ with $div\, \mathbf{v} = S_{ii} = 0$. In the principal axes of the tensor, one has $v_x = kx$, $v_y = -ky$ which corresponds to

$$\phi = k(x^2 - y^2)/2 , \quad \psi = kxy , \quad w = kz^2/2$$

The streamlines are rectangular hyperbolae. This is applied, in particular, on the boundary which has to coincide with one of the principal axes (x or y) or both. The Figure presents the flows near the boundary along x and along x and y (half of the previous one):



2) Consider the potential in the form $w = Az^n$ that is $\phi = Ar^n \cos n\theta$ and $\psi = Ar^n \sin n\theta$. Zero-flux boundaries should coincide with the streamlines so two straight lines $\theta = 0$ and $\theta = \pi/n$ could be seen as boundaries. Choosing different n, one can have different interesting particular cases.

Velocity modulus

$$v = \left|\frac{dw}{dz}\right| = n|A|r^{n-1}$$

at $r \to 0$ either turns to 0 ($n > 1$) or to ∞ ($n < 1$).

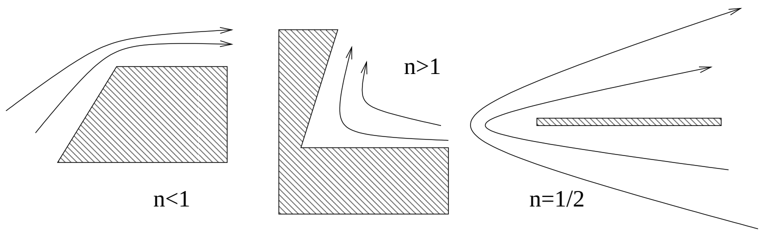


Figure 1.8 Flows described by the complex potential $w = Az^n$.

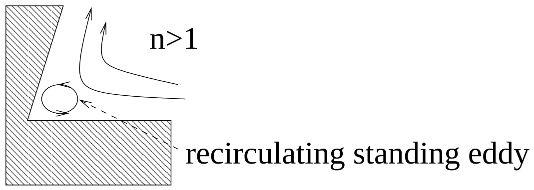
One can think of those solutions as obtained by a conformal transformation $\zeta = z^n$ which maps z-domain into the full ζ-plane. The potential $w = Az^n = A\zeta$ describes a uniform flow in the ζ-plane. Respective z and ζ points have the same value of the potential so that the transformation maps streamlines into streamlines. The velocity in the transformed domain is as follows: $dw/d\zeta = (dw/dz)(dz/d\zeta)$, that is the velocity modulus is inversely proportional to the stretching factor of the transformation. That has two important consequences: First, the energy of the potential flow is invariant with respect to conformal transformations i.e. the energy inside every closed curve in z-plane is the same as the energy inside the image of the curve in ζ-plane. Second, flow dynamics is not conformal invariant even when it proceeds along the conformal invariant streamlines (which coincide with particle trajectories for a steady flow). Indeed, when the flow shifts the fluid particle from z to $z + vdt = z + dt(d\bar{w}/d\bar{z})$, the new image,

$$\zeta(z + vdt) = \zeta(z) + dtv\frac{d\zeta}{dz} = \zeta(z) + dt\frac{d\bar{w}}{d\bar{z}}\frac{d\zeta}{dz},$$

does not coincide with the new position of the old image,

$$\zeta(z) + dt\frac{d\bar{w}}{d\bar{\zeta}} = \zeta(z) + dt\frac{d\bar{w}}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}} \ .$$

Despite the beauty of conformal flows, their applications are limited. Real flow usually separates at discontinuities, it does not turn over the corner for $n < 1$ and does not reach the inside of the corner for $n > 1$:



The phenomenon of separation is due to a combined action of friction and inertia and is discussed in detail in Section 1.5.2. Separation produces vorticity, which makes it impossible to introduce the potential ϕ and use the complex potential w (rotational flows are not conformal invariant).

1.3 Flow past a body

Here we go from two-dimensional to three-dimensional flows, starting from the most symmetric case of a moving sphere and then consider a moving body of an arbitrary shape. Our aim is to understand and describe what we know from everyday experience: fluids apply forces both when we try to set a body into motion and when we try to maintain a motion with a constant speed. In addition to resistance forces, for non-symmetric cases we expect to find a force perpendicular to the motion (called lift), which is what keeps birds and planes from falling from the skies. We consider the motion of a body in an ideal fluid and a body set in motion by a moving fluid.

Flow is assumed to be four "i": infinite, irrotational, incompressible and ideal. The algorithm to describe such a flow is to solve the Laplace equation

$$\Delta\phi = 0 \ . \tag{1.23}$$

The boundary condition on the body surface is the requirement that the normal components of the body and fluid velocities coincide, that is at any given moment one has $\partial\phi/\partial n = u_n$, where $\mathbf{u}$ is body velocity. After finding the potential, one calculates $\mathbf{v} = \nabla\phi$ and then finds pressure from the Bernoulli equation:

$$p = -\rho(\partial\phi/\partial t + v^2/2) \ . \tag{1.24}$$

It is the distinctive property of an irrotational incompressible flow that the velocity distribution is defined completely by a *linear* equation. Due to linearity, velocity potentials can be superimposed (but not pressure distributions).

1.3.1 Incompressible potential flow past a body

Before going into calculations, one can formulate several general statements. First, note that the Laplace equation is elliptic which means that the solutions are smooth inside the domains, singularities could exist on boundaries only, in contrast to hyperbolic (say, wave) equations [8]. Second, integrating (1.23) over any volume one gets

$$\int \Delta\phi \, dV = \int \mathrm{div}\nabla\phi \, dV = \oint \nabla\phi \cdot d\mathbf{f} = 0 \, ,$$

that is the flux is zero through any closed surface (as is expected for an incompressible fluid). That means, in particular, that $\mathbf{v} = \nabla\phi$ changes sign on any closed surface so that extrema of ϕ could be on the boundary only. The same can be shown for velocity components (e.g. for $\partial\phi/\partial x$) since they also satisfy the Laplace equation. That means that for any point P inside one can find P' having higher $|v_x|$. If we choose the x-direction to coincide at P with $\nabla\phi$ we conclude that for any point inside one can find another point in the immediate neighborhood where $|v|$ is greater. In other terms, v^2 cannot have a maximum inside (but can have a minimum). Similarly for pressure, taking Laplacian of the Bernoulli relation (1.24),

$$\Delta p = -\rho \Delta v^2/2 = -\rho(\nabla v)^2 \, ,$$

and integrating it over volume, one obtains

$$\oint \nabla p \cdot d\mathbf{f} = -\rho \int (\nabla v)^2 dV < 0 \, ,$$

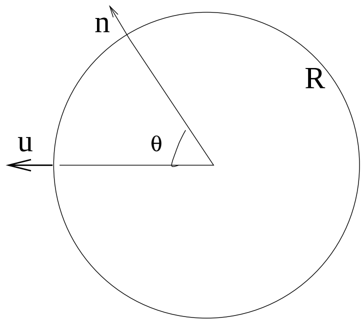
that is a pressure minimum could be only on a boundary (although a maximum can occur at an interior point). For steady flows, $v^2/2 + p/\rho =$const so that the points of $\max v^2$ coincide with those of $\min p$ and all are on a boundary [9]. The knowledge of points of minimal pressure is important for cavitation which is a creation of gas bubbles when the pressure falls below the vapour pressure; when such bubbles then experience higher pressure, they may collapse producing shock waves that do severe damage to moving boundaries like turbine blades and ships' propellers. Likewise, we shall see in Section 2.3.2 that when local fluid velocity exceeds the velocity of sound, shock is created; this is again must happen on the boundary of a potential flow.

1.3.2 Moving sphere

Solutions of the equation $\Delta\phi = 0$ that vanish at infinity are $1/r$ and its derivatives, $\partial^n(1/r)/\partial x^n$. Due to the complete symmetry of the sphere, its motion is characterized by a single vector of its velocity $\mathbf{u}$. Linearity requires $\phi \propto \mathbf{u}$ so the flow potential could be only made as a scalar product of the vectors $\mathbf{u}$ and the gradient, which is the dipole field:

$$\phi = a\left(\mathbf{u}\cdot\nabla\frac{1}{r}\right) = -a\frac{(\mathbf{u}\cdot\mathbf{n})}{r^2}$$

where $\mathbf{n} = \mathbf{r}/r$. On the body, $r = R$ and $\mathbf{v}\cdot\mathbf{n} = \mathbf{u}\cdot\mathbf{n} = u\cos\theta$. Using $\phi = -ua\cos\theta/r^2$ and $v_R = 2auR^{-3}\cos\theta$, this condition gives $a = R^3/2$.



$$\phi = \frac{R^3(\mathrm{un})}{2r^2} \qquad \text{(Stokes, 1843)}$$

$$v = \frac{R^3}{2r^3}\,[3n(\mathrm{un}) - u]$$

Now one can calculate the pressure

$$p = p_0 - \rho v^2/2 - \rho\partial\phi/\partial t\,,$$

having in mind that our solution moves with the sphere that is $\phi(\mathbf{r} - \mathbf{u}t, \mathbf{u})$ and

$$\frac{\partial\phi}{\partial t} = \dot{\mathbf{u}}\cdot\frac{\partial\phi}{\partial\mathbf{u}} - \mathbf{u}\cdot\nabla\phi\,,$$

which gives

$$p = p_0 + \rho u^2\frac{9\cos^2\theta - 5}{8} + \rho R\mathbf{n}\cdot\dot{\mathbf{u}}\ .$$

The force is $\oint p\,d\mathbf{f}$. For example,

$$F_x = \oint p\cos\theta\,d\mathbf{f} = \rho R^3\dot{u}\pi\int\cos^2\theta\,d\cos\theta = 2\pi\rho R^3\dot{u}/3\ . \qquad (1.25)$$

If the radius depends on time too then $F_x \propto \partial\phi/\partial t \propto \partial(R^3u)/\partial t$. For a uniformly moving sphere with a constant radius, $\dot{R} = \dot{\mathbf{u}} = 0$, the force is zero: $\oint p\,d\mathbf{f} = 0$. This flies in the face of our common experience: fluids do resist attempts to move through them. Maybe we obtained zero force in a steady case due to a symmetrical shape?

1.3.3 Moving body of an arbitrary shape

At large distances from the body, a solution of $\Delta\phi = 0$ is again sought in the form of the first non-vanishing multipole. The first (charge) term $\phi = a/r$ cannot be present because it corresponds to the velocity $\mathbf{v} = -a\mathbf{r}/r^3$ with the radial component $v_R = a/R^2$ providing for a non-vanishing flux $4\pi\rho a$ through a closed sphere of radius R; existence of a flux contradicts incompressibility. So the first non-vanishing term is again a dipole:

$$\phi = \mathbf{A}\cdot\nabla(1/r) = -(\mathbf{A}\cdot\mathbf{n})r^{-2}\,,$$
$$\mathbf{v} = [3(\mathbf{A}\cdot\mathbf{n})\mathbf{n} - \mathbf{A}]r^{-3}\ .$$

For the sphere above, $\mathbf{A} = \mathbf{u}R^3/2$, but for nonsymmetric bodies the vectors $\mathbf{A}$ and $\mathbf{u}$ are not collinear, though linearly related $A_i = \alpha_{ik}u_k$, where the tensor α_{ik} (having the dimensionality of volume) depends on the body shape.

What can one say about the force acting on the body if only flow at large distances is known? That's the main beauty of the potential theory that one often can say something about "here" by considering field "there". Let us start by calculating the energy $E = \rho\int v^2\,dV/2$ of the moving fluid outside the body and inside the large sphere of the radius R. We present $v^2 = u^2 + (\mathbf{v}-\mathbf{u})(\mathbf{v}+\mathbf{u})$ and write $\mathbf{v}+\mathbf{u} = \nabla(\phi + \mathbf{u}\cdot\mathbf{r})$. Using $\operatorname{div} v = \operatorname{div} u = 0$ one can write

$$\begin{aligned}\int_{r<R} v^2\,dV &= u^2(V-V_0) + \int_{r<R} \operatorname{div}[(\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})]\,dV \\ &= u^2(V-V_0) + \oint_{S+S_0} (\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\,d\mathbf{f} \\ &= u^2(V-V_0) + \oint_{S} (\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\,d\mathbf{f}\ .\end{aligned}$$

Substituting

$$\phi = -(\mathbf{A}\cdot\mathbf{n})R^{-2}\,, \qquad \mathbf{v} = [3\mathbf{n}(\mathbf{A}\cdot\mathbf{n}) - \mathbf{A}]R^{-3}\ .$$

and integrating over angles,

$$\begin{aligned}\int(\mathbf{A}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n})\,d\Omega = A_iu_k\int n_in_k\,d\Omega = A_iu_k\delta_{ik}\int\cos^2\theta\sin\theta\,d\theta d\varphi \\ = (4\pi/3)(\mathbf{A}\cdot\mathbf{u})\,,\end{aligned}$$

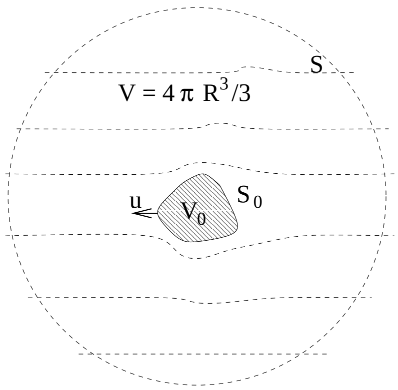
we obtain the energy in the form

$$E = \rho[4\pi(\mathbf{A}\cdot\mathbf{u}) - V_0u^2]/2 = m_{ik}u_iu_k/2 \ . \qquad (1.26)$$

Here we introduced the *induced-mass tensor*:

$$m_{ik} = 4\pi\rho\alpha_{ik} - \rho V_0\delta_{ik} \ .$$

For sphere, $m_{ik} = \rho V_0\delta_{ik}/2$ that is half the displaced fluid.



We now have to pass from the energy to the force acting on the body which is done by considering the change in the energy of the body (the same as minus the change of the fluid energy dE) being equal to the work done by force F on the path udt: $dE = -\mathbf{F}\cdot\mathbf{u}dt$. The change of the momentum of the body is $d\mathbf{P} = -\mathbf{F}dt$ so that $dE = \mathbf{u}\cdot d\mathbf{P}$. That relation is true for changes caused by the velocity change by force (not by the change in the body shape) so that the change of the body momentum is $dP_i = m_{ik}du_k$ and the force is

$$F_i = -m_{ik}\dot{u}_k \ , \qquad (1.27)$$

i.e. the presence of potential flow means only an additional mass but not resistance.

How to generalize (1.27) for the case when both m_{ik} and $\mathbf{u}$ change? Our consideration of the pressure for a sphere suggests that the proper generalization is

$$F_i = -\frac{d}{dt}m_{ik}u_k \ . \qquad (1.28)$$

It looks as if $m_{ik}u_k$ is the momentum of the fluid yet it is not (it is quasi-momentum), as explained in the next section [10].

Equation of motion for the body under the action of an external force $\mathbf{f}$,

$$\frac{d}{dt} M u_i = f_i + F_i = f_i - \frac{d}{dt} m_{ik} u_k \ ,$$

could be written in a form that makes the term induced mass clear:

$$\frac{d}{dt}(M\delta_{ik} + m_{ik})u_k = f_i \ . \tag{1.29}$$

This is one of the simplest examples of renormalization in physics: the body moving through a fluid acquires additional mass. For example, a spherical air bubble in a liquid has the mass which is half of the mass of the displaced liquid; since the buoyancy force is the displaced mass times g then the bubble acceleration is close to $2g$ when one can neglect other forces and the mass of the air inside.

Body in a flow. Consider now an opposite situation when the fluid moves in an oscillating way while a small body is immersed into the fluid. For example, a long sound wave propagates in a fluid. We do not consider here the external forces that move the fluid, we wish to relate the body velocity $\mathbf{u}$ to the fluid velocity $\mathbf{v}$, which is supposed to be homogeneous on the scale of the body size. If the body moved with the same velocity, $\mathbf{u} = \mathbf{v}$, then it would be under the action of force that would act on the fluid in its place, $\rho V_0 \dot{\mathbf{v}}$. Relative motion gives the reaction force $dm_{ik}(v_k - u_k)/dt$. The sum of the forces gives the body acceleration

$$\frac{d}{dt} M u_i = \rho V_0 \dot{v}_i + \frac{d}{dt} m_{ik}(v_k - u_k) \ .$$

Integrating over time with the integration constant zero (since $u = 0$ when $v = 0$) we get the relation between the velocities of the body and of the fluid:

$$(M\delta_{ik} + m_{ik})u_k = (m_{ik} + \rho V_0 \delta_{ik})v_k \ .$$

For a sphere, $\mathbf{u} = \mathbf{v}3\rho/(\rho + 2\rho_0)$, where ρ_0 is the density of the body. For a spherical air bubble in a liquid, $\rho_0 \ll \rho$ and $u \approx 3v$.

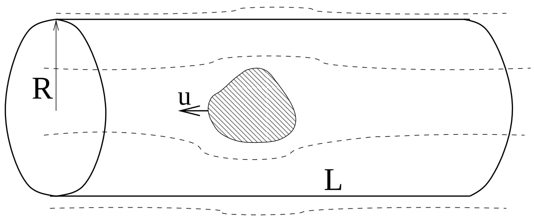
1.3.4 Quasi-momentum and induced mass

In the previous Section, we obtained the force acting on an accelerating body via the energy of the fluid and the momentum of the body because the momentum of the fluid, $\mathbf{M} = \rho \int \mathbf{v}\, dV$, is not well-defined for a potential flow around the body. For example, the integral of $v_x = D(3\cos^2\theta - 1)r^{-3}$ depends on the form of the volume chosen: it is zero for a spherical volume and nonzero for a cylinder of the length L and the radius $\mathcal{R}$ set around the body:

$$\int_{-1}^{1}(3\cos^2\theta - 1)\,d\cos\theta = 0\,,$$

$$M_x = 4\pi\rho D\int_{-L}^{L}dz\int_0^{\mathcal{R}} rdr\frac{2z^2 - r^2}{(z^2+r^2)^{5/2}} = \frac{4\pi\rho DL}{(L^2+\mathcal{R}^2)^{1/2}}\,. \quad (1.30)$$

That dependence means that the momentum stored in the fluid depends on the boundary conditions at infinity. For example, the motion by the sphere in the fluid enclosed by rigid walls must be accompanied by the displacement of an equal amount of fluid in the opposite direction, then the momentum of the fluid must be $-\rho V_0 u = -4\pi\rho R^3 u/3$ rather than $\rho V_0 u/2$. The negative momentum $-3\rho V_0 u/2$ delivered by the walls is absorbed by the whole body of fluid and results in an infinitesimal backflow, while the momentum $\rho V_0 u/2$ delivered by the sphere results in a finite localized flow. From (1.30) we can get a shape-independent answer $4\pi\rho D$ only in the limit $L/\mathcal{R} \to \infty$. To recover the answer $4\pi\rho D/3$ ($=\rho V_0 u/2 = 2\pi R^3\rho u/3$ for a sphere) that we expect from (1.28), one needs to subtract the reflux $8\pi\rho D/3 = 4\pi R^3\rho u/3$ compensating the body motion [11].



It is the quasi-momentum of the fluid particles which is independent of the remote boundary conditions and whose time derivative gives the inertial force (1.28) acting on the body. Conservation laws of the momentum and the quasi-momentum follow from different symmetries. The momentum expresses invariance of the Hamiltonian $\mathcal{H}$ with respect to the shift of coordinate system. If the space is filled by a medium (fluid or solid), then the quasi-momentum expresses invariance of the Hamiltonian with respect to a space shift, *keeping the medium fixed.* That invariance follows from the identity of different elements of the medium. In a crystal, such shifts are allowed only by the lattice spacing. In a continuous medium, shifts are arbitrary. In this case, the system Hamiltonian must be independent of the coordinates:

$$\frac{\partial \mathcal{H}}{\partial x_i} = \frac{\partial \mathcal{H}}{\partial \pi_j}\frac{\partial \pi_j}{\partial x_i} + \frac{\partial \mathcal{H}}{\partial q_j}\frac{\partial q_j}{\partial x_i} = 0 \;, \tag{1.31}$$

where the vectors $\pi(\mathbf{x},t), q(\mathbf{x},t)$ are canonical momentum and coordinate respectively. We need to define the quasi-momentum $\mathbf{K}$ whose conservation is due to invariance of the Hamiltonian: $\partial K_i/\partial t = \partial\mathcal{H}/\partial x_i = 0$. Recall that the time derivative of any function of canonical variables is given by the Poisson bracket of this function with the Hamiltonian:

$$\begin{aligned}\frac{\partial K_i}{\partial t} &= \{K_i, \mathcal{H}\} = \frac{\partial K_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial \pi_j} - \frac{\partial K_i}{\partial \pi_j}\frac{\partial \mathcal{H}}{\partial q_j} \\ &= \frac{\partial \mathcal{H}}{\partial x_i} = \frac{\partial \mathcal{H}}{\partial \pi_j}\frac{\partial \pi_j}{\partial x_i} + \frac{\partial \mathcal{H}}{\partial q_j}\frac{\partial q_j}{\partial x_i} \;,\end{aligned}$$

That gives the partial differential equations on the quasi-momentum,

$$\frac{\partial K_i}{\partial \pi_j} = -\frac{\partial q_j}{\partial x_i}\;, \quad \frac{\partial K_i}{\partial q_j} = \frac{\partial \pi_j}{\partial x_i}\;,$$

whose solution is as follows:

$$K_i = -\int d\mathbf{x}\pi_j \frac{\partial q_j}{\partial x_i}\;. \tag{1.32}$$

For isentropic (generally compressible) flow of an ideal fluid, the hamiltonian description can be done in Lagrangian coordinates, which describe the current position of a fluid element (particle) $\mathbf{r}$ as a function of time and its initial position $\mathbf{R}$. The canonical coordinate is the displacement $\mathbf{q} = \mathbf{r} - \mathbf{R}$, which is the continuum limit of the variable that describes lattice vibrations in the solid state physics. The canonical momentum is $\pi(\mathbf{R},t) = \rho_0(\mathbf{R})\mathbf{v}(\mathbf{R},t)$ where the velocity is $\mathbf{v} = (\partial\mathbf{r}/\partial t)_{\mathbf{R}} \equiv \dot{\mathbf{r}}$. Here ρ_0 is the density in the reference (initial) state, which can always be chosen uniform. The Hamiltonian is as follows:

$$\mathcal{H} = \int \rho_0 \left[W(\mathbf{q}) + v^2/2\right] d\mathbf{R}\,, \tag{1.33}$$

where $W = E + p/\rho$ is the enthalpy. Canonical equations of motion, $\dot{q}_i = \partial\mathcal{H}/\partial\pi_i$ and $\dot{\pi}_i = -\partial\mathcal{H}/\partial q_i$, give respectively $\dot{r}_i = v_i$ and $\dot{v}_i = -\partial W/\partial r_i = -\rho^{-1}\partial p/\partial r_i$. The velocity $\mathbf{v}$ now is an independent variable and not a function of the coordinates $\mathbf{r}$. All the time derivatives are for fixed $\mathbf{R}$ i.e. they are substantial derivatives. The quasi-momentum (1.32) is as follows:

$$K_i = -\rho_0 \int v_j \frac{\partial q_j}{\partial R_i}\, d\mathbf{R} = \rho_0 \int v_j \left(\delta_{ij} - \frac{\partial r_j}{\partial R_i}\right) d\mathbf{R}\;, \tag{1.34}$$

In plain words, only those particles contribute quasi-momentum whose motion is disturbed by the body so that for them $\partial r_j/\partial R_i \neq \delta_{ij}$. The integral (1.34) converges for spatially localized flows since $\partial r_j/\partial R_i \to \delta_{ij}$ when $R \to \infty$. Unlike (1.30), the quasi-momentum (1.34) is independent of the form of a distant surface. Using $\rho_0 d\mathbf{R} = \rho d\mathbf{r}$ one can also present

$$K_i = \rho_0 \int v_j \left(\delta_{ij} - \frac{\partial r_j}{\partial R_i}\right) d\mathbf{R} = \int \rho v_i \, d\mathbf{r} - \rho_0 \int v_j \frac{\partial r_j}{\partial R_i} \, d\mathbf{R} \,, \quad (1.35)$$

i.e. indeed the quasi-momentum is the momentum minus what can be interpreted as a reflux.

The conservation can now be established substituting the equation of motion $\rho\dot{\mathbf{v}} = -\partial p/\partial \mathbf{r}$ into

$$\begin{aligned}
\dot{K}_i &= -\rho_0 \int \left(\dot{v}_j \frac{\partial q_j}{\partial R_i} + v_j \frac{\partial v_j}{\partial R_i}\right) d\mathbf{R} \\
&= -\rho_0 \int \left[\dot{v}_j \left(\frac{\partial r_j}{\partial R_i} - \delta_{ij}\right) + v_j \frac{\partial v_j}{\partial R_i}\right] d\mathbf{R} \\
&= -\int \frac{\partial p}{\partial r_i} \, d\mathbf{r} + \int \frac{\partial}{\partial R_i} \left(W - \frac{v^2}{2}\right) d\mathbf{R} \qquad (1.36) \\
&= -\int \frac{\partial p}{\partial r_i} \, d\mathbf{r} = \oint p \, df_i \ .
\end{aligned}$$

In (1.36), the integral over the reference space $\mathbf{R}$ of the total derivative in the second term is identically zero while the integral over $\mathbf{r}$ in the first term excludes the volume of the body, so that the boundary term remains which is minus the force acting on the body. Therefore, the sum of the quasi-momentum of the fluid and the momentum of the body is conserved in an ideal fluid. That means, in particular, a surprising effect: when a moving body shrinks it accelerates. Indeed, when the induced mass and the quasi-momentum of the fluid decrease then the body momentum must increase.

This quasi-momentum is defined for any flow. For a potential flow, the quasi-momentum can be obtained much easier than doing the volume integration (1.34), one can just integrate the potential over the body surface: $\mathbf{K} = \int \rho\phi \, d\mathbf{f}$. Indeed, consider very short and strong pulse of pressure needed to bring the body from rest into motion, formally $p \propto \delta(t)$. During the pulse, the body doesn't move so its position and surface are well-defined. In the Bernoulli relation (1.18) one can then neglect v^2-term:

$$\frac{\partial \phi}{\partial t} = -\frac{v^2}{2} - \frac{p}{\rho} \approx -\frac{p}{\rho} \,,$$

Integrating the relation $\rho\phi = -\int p(t)\,dt$ over the body surface we get minus the change of the body momentum i.e. the quasi-momentum of the fluid. For example, integrating $\phi = R^3 u\cos\theta/2r^2$ over the sphere we get

$$K_x = \int \rho\phi\cos\theta\, d\mathbf{f} = 2\pi\rho R^3 u \int_{-1}^{1} \cos^2\theta\, d\cos\theta = 2\pi\rho R^3 u/3\,,$$

as expected. The difference between momentum and quasi-momentum can be related to the momentum flux across the infinite surface due to pressure which decreases as r^{-2} for a potential flow.

The quasi-momentum of the fluid is related to the body velocity via the induced mass, $K_i = m_{ik}u_k$, so that one can use (1.34) to evaluate the induced mass. For this, one needs to solve the Lagrangian equation of motion $\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}, t)$, then one can show that the induced mass can be associated with the displacement of the fluid after the body pass. Fluid particles displaced by the body do not return to their previous positions after the body pass but are shifted to the direction of the fluid motion as shown in Figure 1.9. The permanently displaced mass enclosed between the broken lines is in fact the induced mass itself (Darwin, 1953).

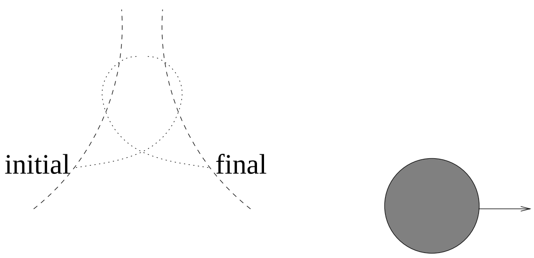


Figure 1.9 Displacement of the fluid by the passing body. The dotted line is the trajectory of the fluid particle. Two broken lines (chosen to be symmetrical) show the initial and final positions of the particles before and after the passage of the body.

Notice the loop made by every fluid particle; for a sphere, the horizontal component of the fluid velocity changes sign when $3\cos^2\theta = 1$ i.e. at the angle equal 30 degrees. Note also the striking difference between the particle trajectories and instantaneous streamlines (see also Exercise 1.6) [12].

Let us summarize: neglecting tangential forces (i.e. internal friction) we were able to describe the inertial reaction of the fluid to the body acceleration (quantified by the induced mass). For a motion with a constant speed, we failed to find any force, including the force perpendicular to **u** called lift. If that was true, flying would be impossible. Physical intuition also suggests that the resistance force opposite to **u** called drag must be given by the amount of momentum transferred to the fluid in front of the body per unit time:

$$F = CR^2 \rho u^2 \ , \tag{1.37}$$

where C is some order-unity dimensionless constant (called drag coefficient) depending on the body shape[13]. This is the correct estimate for the resistance force in the limit of vanishing internal friction (called viscosity). Unfortunately, I don't know any other way to show its validity but to introduce viscosity first and then consider the limit when it vanishes. That limit is quite non-trivial: even an arbitrary small friction makes an infinite region of the flow (called wake) very much different from the potential flow described above. Introducing viscosity and describing wake will take the next two Sections.

1.4 Viscosity

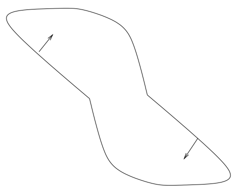
In this section we try to find our way out of paradoxes of ideal flows towards a real world. This will require considering internal friction that is viscosity.

1.4.1 Reversibility paradox

Let us discuss the absence of resistance in a more general way. We have made five assumptions on the flow: incompressible, irrotational, inviscid (ideal), infinite, steady. The last can be always approached with any precision by waiting enough time (after body passes a few its sizes is usually enough). An irrotational flow of an incompressible fluid is completely determined by the instantaneous body position and velocity. When the body moves with a constant velocity, the flow pattern moves along without changing its form, neither quasi-momentum nor kinetic energy of the fluid change so there are no forces acting between the body and the fluid. Let us also show that an account of compressibility does not give the drag resistance for a steady flow. That follows from *reversibility* of the continuity and Euler equations: the reverse of the flow [defined as $\mathbf{w}(\mathbf{r},t) = -\mathbf{v}(\mathbf{r},-t)$] is also a solution with the velocity at infinity $\mathbf{u}$ instead of $-\mathbf{u}$ but with the same pressure and density fields. For the steady flow, defined by the boundary problem

$$\mathrm{div}\,\rho\mathbf{v} = 0\ , \quad v_n = 0\ \text{(on the body surface)}\,, \quad \mathbf{v} \to -\mathbf{u}\ \text{at infinity}$$
$$\frac{v^2}{2} + \int \frac{dp}{\rho(p)} = \text{const}\ ,$$

the reverse flow $\mathbf{w}(\mathbf{r}) = -\mathbf{v}(\mathbf{r})$ has the same pressure field so it must give the same drag force on the body. Since the drag is supposed to change sign when you reverse the direction of motion then the drag is zero in an ideal irrotational flow. For the particular case of a body with a central symmetry, reversibility gives D'Alembert paradox: the pressure on the symmetrical surface elements is the same and the resulting force is a pure couple [14].



If fluid is finite that is has a surface, a finite drag arises due to surface waves. If surface is far away from the body, that drag is negligible.

Exhausting all the other possibilities, we conclude that without friction we cannot describe drag and lift acting on a body moving through the fluid.

1.4.2 Viscous stress tensor

We define the stress tensor σ_{ij} as having ij entry equal to the i component of the force acting on a unit area perpendicular to j direction. The diagonal components present normal stress, they are equal to each other due to the Pascal law, we called this quantity pressure. Internal friction in a fluid must lead to the appearance of the non-diagonal components of the stress tensor: $\sigma_{ik} = -p\delta_{ik} + \sigma'_{ik}$ (here the stress is applied to the fluid element under consideration so that the pressure is negative). That changes the momentum flux, $\Pi_{ik} = p\delta_{ik} - \sigma'_{ik} + \rho v_i v_k$, as well as the Euler equation: $\partial \rho v_i/\partial t = -\partial \Pi_{ik}/\partial x_k$.

To avoid infinite rotational accelerations, the stress tensor must be symmetric: $\sigma_{ij} = \sigma_{ji}$. Indeed, consider the moment of force (with respect

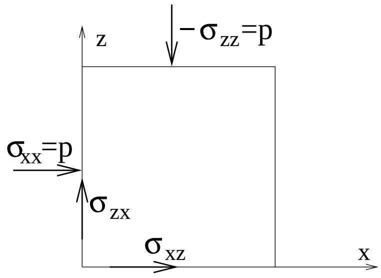
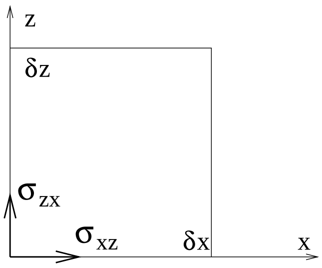


Figure 1.10 Diagonal and non-diagonal components of the stress tensor.

to the axis at the upper right corner) acting on an infinitesimal element with the sizes $\delta x, \delta y, \delta z$:



If the stress tensor was not symmetric, then the moment of force $(\sigma_{xz} - \sigma_{zx})\,\delta x\delta y\delta z$ is nonzero. That moment then must be equal to the time derivative of the moment of momentum which is the moment of inertia $\rho\delta x\delta y\delta z\left[(\delta x)^2 + (\delta z)^2\right]$ times the angular velocity Ω:

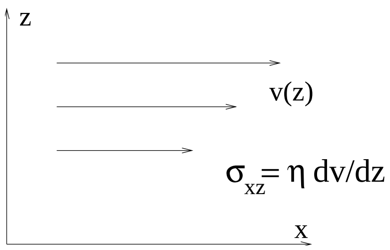
$$(\sigma_{xz} - \sigma_{zx})\,\delta x\delta y\delta z = \rho\,\delta x\delta y\delta z\left[(\delta x)^2 + (\delta z)^2\right]\frac{\partial\Omega}{\partial t}\ .$$

We see that to avoid $\partial\Omega/\partial t \to \infty$ as $(\delta x)^2 + (\delta z)^2 \to 0$ we must assume that $\sigma_{xz} = \sigma_{zx}$.

To connect the frictional part of the stress tensor σ' and the velocity $\mathbf{v}(\mathbf{r})$, note that $\sigma' = 0$ for a uniform flow, so σ' must depend on the velocity spatial derivatives. Supposing these derivatives to be small (comparatively to the velocity changes on a molecular level) one could *assume* that the tensor σ' is linearly proportional to the tensor of velocity derivatives (Newton, 1687). Fluids with that property are called *newtonian*. Non-newtonian fluids are those of elaborate molecular structure (e.g. with long molecular chains like polymers), where the relation may be nonlinear already for moderate strains, and rubber-like liquids, where the stress depends on history. For newtonian fluids, to relate linearly two second-rank tensors, σ'_{ij} and $\partial v_i/\partial x_j$, one generally needs a tensor of the fourth rank. Yet another simplification comes from the fact that vorticity (that is the antisymmetric part of $\partial v_i/\partial x_j$) gives no contribution since it corresponds to a solid-body rotation where no sliding of fluid layers occurs. We thus need to connect two symmetric tensors, the stress σ'_{ij} and the rate of strain $S_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$. In the isotropic medium, the principal axes of σ'_{ij} have to coincide with those of S_{ij} so that just two constants, η and μ, are left out of the scary fourth-rank tensor:

$$\sigma'_{ij} = \eta(\partial v_i/\partial x_j + \partial v_j/\partial x_i) + \mu\delta_{ij}\partial v_l/\partial x_l \ . \tag{1.38}$$

Dimensionally $[\eta] = [\mu] = g/cm{\cdot}sec$. To establish the sign of η, consider a simple shear flow shown in the Figure and recall that the stress is applied to the fluid. The stress component $\sigma_{xz} = \eta dv_x/dz$ is the x-component of the force by which an upper layer of the fluid acts on the lower layer so that it must be positive which requires $\eta > 0$.



1.4.3 Navier-Stokes equation

Now we substitute σ' into the Euler equation

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial}{\partial x_k}\left[p\delta_{ik} - \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - \mu\delta_{ik}\frac{\partial v_l}{\partial x_l}\right] \ . \tag{1.39}$$

The viscosity is determined by the thermodynamic state of the system that is by p, ρ. When p, ρ depend on coordinates so must $\eta(p, \rho)$ and $\mu(p, \rho)$. However, we consistently assume that the variations of p, ρ are small and put η, μ constant. In this way we get the famous Navier-Stokes equation (Navier, 1822; Stokes, 1845):

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \eta\Delta\mathbf{v} + (\eta + \mu)\nabla\mathrm{div}\,\mathbf{v} \ . \tag{1.40}$$

Apart from the case of rarefied gases we cannot derive this equation consistently from kinetics. That means only that we generally cannot quantitatively relate η and μ to the properties of the material. One can *estimate* the viscosity of the fluid saying that the flux of molecules with the thermal velocity v_T through the plane (perpendicular to the velocity gradient) is nv_T, they come from a layer comparable to the mean free path l, have velocity difference $l\nabla u$, which causes momentum flux $mnv_T l\nabla u \simeq \eta\nabla u$, where m is the molecule mass. Therefore, $\eta \simeq mnv_T l = \rho v_T l$. We also define kinematic viscosity $\nu = \eta/\rho$ which is estimated as $\nu \simeq v_T l$. The thermal velocity is determined by the temperature while the mean free path by the strength of interaction between molecules: the stronger the interaction the shorter is l and the smaller is the viscosity. In other words, it is more difficult to transfer momentum in a system with stronger interaction. For example, air has $\nu = 0.15\, cm^2/sec$ so it is 15 times more viscous than water which has $\nu = 0.01\, cm^2/sec$. The Navier-Stokes equation is valid for liquids as well as for gases as long as the typical scale of the flow is much larger than the mean free path.

The Navier-Stokes equation has higher-order spatial derivatives (second) than the Euler equation so that we need more boundary conditions. Since we accounted (in the first non-vanishing approximation) for the forces between fluid layers, we also have to account for the forces of molecular attraction between a viscous fluid and a solid body surface. Such force makes the layer of adjacent fluid to stick to the surface: $\mathbf{v} = 0$ on the surface (not only $v_n = 0$ as for the Euler equation)[15]. The solutions of the Euler equation do not generally satisfy that no-slip boundary condition. That means that even a very small viscosity must play a role near a solid surface.

Viscosity adds an extra term to the momentum flux, but (1.39,1.40) still have the form of a continuity equation which conserves total momentum. However, viscous friction between fluid layers necessarily leads to some energy dissipation. Consider, for instance, a viscous incompressible fluid with $\operatorname{div}\mathbf{v} = 0$ and calculate the time derivative of the energy at a point:

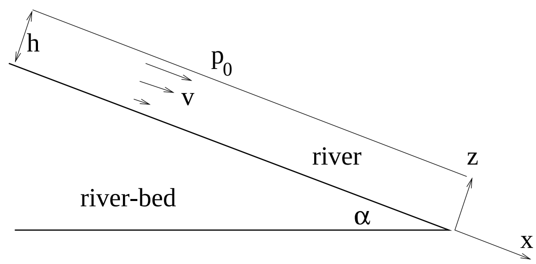
$$\begin{aligned}\frac{\rho}{2}\frac{\partial v^2}{\partial t} &= -\rho\mathbf{v}\cdot(\mathbf{v}\nabla)\mathbf{v} - \mathbf{v}\cdot\nabla p + v_i\frac{\partial\sigma'_{ik}}{\partial x_k}\\ &= -\operatorname{div}\left[\rho\mathbf{v}\left(\frac{v^2}{2}+\frac{p}{\rho}\right) - (\mathbf{v}\cdot\sigma')\right] - \sigma'_{ik}\frac{\partial v_i}{\partial x_k}\ . \qquad (1.41)\end{aligned}$$

The presence of viscosity results in the momentum flux σ' which is accompanied by the energy transfer, $\mathbf{v}\cdot\sigma'$, and the energy dissipation described by the last term. Because of this last term, this equation does not have the form of a continuity equation and the total energy integral is not conserved. Indeed, after the integration over the whole volume,

$$\begin{aligned}\frac{dE}{dt} &= -\int \sigma'_{ik}\frac{\partial v_i}{\partial x_k}\,dV = -\frac{\eta}{2}\int\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)^2 dV\\ &= -\eta\int \omega^2 dV < 0\ . \end{aligned} \tag{1.42}$$

The last equality here follows from $\omega^2 = (\epsilon_{ijk}\partial_j v_k)^2 = (\partial_j v_k)^2 - \partial_k(v_j\partial_j v_k)$, which is true by virtue of $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ and $\partial_i v_i = 0$.

The Navier-Stokes equation is a nonlinear partial differential equation of the second order. Not many steady solutions are known. Particularly easy is to find solutions in the geometry where $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$ and the equation is effectively linear. In particular, symmetry may prescribe that the velocity does not change along itself. One example is the flow along an inclined plane as a model for a river.



Everything depends only on z. The stationary Navier-Stokes equation takes a form

$$-\nabla p + \eta\Delta\mathbf{v} + \rho\mathbf{g} = 0$$

with z and x projections respectively

$$\begin{aligned}&\frac{dp}{dz} + \rho g\cos\alpha = 0\,,\\ &\eta\frac{d^2v}{dz^2} + \rho g\sin\alpha = 0\ .\end{aligned}$$

The boundary condition on the bottom is $v(0) = 0$. On the surface, the boundary condition is that the stress should be normal and balance the pressure: $\sigma_{xz}(h) = \eta dv(h)/dz = 0$ and $\sigma_{zz}(h) = -p(h) = -p_0$. The solution is simple:

$$p(z) = p_0 + \rho g(h - z)\cos\alpha\,, \qquad v(z) = \frac{\rho g \sin\alpha}{2\eta} z(2h - z)\ . \qquad (1.43)$$

Let us see how it corresponds to reality. Take water with the kinematic viscosity $\nu = \eta/\rho = 10^{-2}\,cm^2/sec$. For a rain puddle with the thickness $h = 1\,mm$ on a slope $\alpha \sim 10^{-2}$ we get a reasonable estimate $v \sim 5\,cm/sec$. For slow plain rivers (like Nile or Volga) with $h \simeq 10\,m$ and $\alpha \simeq 0.3\,km/3000\,km \simeq 10^{-4}$ one gets $v(h) \simeq 100\,km/sec$ which is evidently impossible (the resolution of that dramatic discrepancy is that real rivers are turbulent as discussed in Sect. 2.2.2 below). What distinguishes puddle and river, why they are not similar? To answer this question, we need to characterize flows by a dimensionless parameter.

1.4.4 Law of similarity

One can obtain some important conclusions about flows from a dimensional analysis. Consider a steady flow past a body described by the equation

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla(p/\rho) + \nu\Delta\mathbf{v}$$

and by the boundary conditions $\mathbf{v}(\infty) = \mathbf{u}$ and $\mathbf{v} = 0$ on the surface of the body of the size L. For a given body shape, both $\mathbf{v}$ and p/ρ are functions of coordinates $\mathbf{r}$ and three variables, $\mathbf{u}, \nu, L$. Out of the latter, one can form only one dimensionless quantity, called the Reynolds number

$$Re = uL/\nu\ . \qquad (1.44)$$

This is the most important parameter in this book since it determines the ratio of the nonlinear (inertial) term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ to the viscous friction term $\nu\Delta\mathbf{v}$. Since the kinematic viscosity is the thermal velocity times the mean free path then the Reynolds number is

$$Re = uL/v_T l\ .$$

We see that within the hydrodynamic limit ($L \gg l$), Re can be both large and small depending on the ratio $u/v_T \simeq u/c$.

Dimensionless velocity must be a function of dimensionless variables: $\mathbf{v} = u\mathbf{f}(\mathbf{r}/L, Re)$ - it is a unit-free relation. Flows that correspond to the same Re can be obtained from one another by simply changing the units of v and r, such flows are called similar (Reynolds, 1883). In the same way, $p/\rho = u^2\varphi(\mathbf{r}/L, Re)$. For a quantity independent of coordinates, only some function of Re is unknown - the drag or lift force, for instance, must be $F = \rho u^2 L^2 f(Re)$. This law of similarity is exploited in modelling: to measure, say, a drag on the ship one designs, one can build a smaller model yet pull it faster through the fluid (or use a less viscous fluid).

Reynolds number, as a ratio of inertia to friction, makes sense for all types of flows as long as u is some characteristic velocity and L is a scale of the velocity change. For the inclined plane flow (1.43), the nonlinear term (and the Reynolds number) is identically zero since $\mathbf{v} \perp \nabla\mathbf{v}$. How much one needs to perturb this alignment to make $Re \simeq 1$? Denoting $\pi/2 - \beta$ the angle between $\mathbf{v}$ and $\nabla\mathbf{v}$ we get $Re(\beta) = v(h)h\beta/\nu \simeq g\alpha\beta h^3/\nu^2$. For a puddle, $Re(\beta) \simeq 50\beta$ while for a river $Re(\beta) \simeq 10^{12}\beta$. It is then clear that the (so-called laminar) solution (1.43) may make sense for a puddle, but for a river it must be distorted by even tiny violations of this symmetry (say, due to a non-flat bottom).

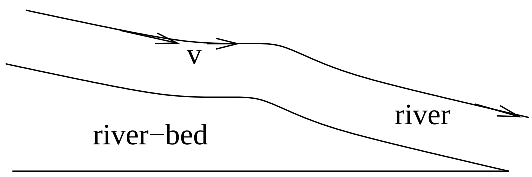


Figure 1.11 Non-flat bottom makes the velocity changing along itself, which leads to a nonzero inertial term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in the Navier-Stokes equation.

Gravity brings another dimensionless parameter, the Froude number $Fr = u^2/Lg$; the flows are similar for the same Re and Fr. Such parameters (whose change brings qualitative changes in the regime even for fixed geometry and boundary conditions) are called control parameters [16].

The law of similarity is a particular case of the so-called π-theorem: **Assume** that among all m variables $\{b_1, \ldots, b_m\}$ we have only $k \le m$ dimensionally independent quantities - that means that the dimensionalities $[b_{k+1}], \ldots [b_m]$ could be expressed via $[b_1], \ldots [b_k]$ like $[b_{k+j}] = \prod_{l=1}^{l=k} [b_l]^{\beta_{jl}}$. **Then** all dimensionless quantities can be expressed in terms of $m - k$ dimensionless variables $\pi_1 = b_{k+1}/\prod_{l=1}^{l=k} b_l^{\beta_{1l}}, \ldots \pi_{m-k} = b_m/\prod_{l=1}^{l=k} b_l^{\beta_{m-k,l}}$.

1.5 Stokes flow and wake

We now return to the flow past a body armed by the knowledge of internal friction. Unfortunately, the Navier-Stokes equation is a nonlinear partial differential equation which we cannot solve in a closed analytical form even for a flow around a sphere. We therefore shall proceed the way physicists often do: solve a limiting case of very small Reynolds numbers and then try to move towards high-Re flow. Remind that we failed spectacularly in Section 1.3 trying to describe high-Re flow as an ideal fluid. This time we shall realize, with the help of qualitative arguments and experimental data, that when viscosity is getting very small its effect stays finite. On the way we shall learn new notions of a boundary layer and a separation phenomenon. The reward will be the resolution of paradoxes and the formulas for the drag and the lift.

1.5.1 Slow motion

Consider such a slow motion of a body through the fluid that the Reynolds number, $Re = uR/\nu$, is small. That means that we can neglect inertia. Indeed, if we stop pushing the body, friction stops it after a time of order R^2/ν, so that inertia moves it by the distance of order $uR^2/\nu = R \cdot Re$, which is much less than the body size R. Formally, neglecting inertia means omitting the nonlinear term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in the Navier-Stokes equation. That makes our problem linear so that the fluid velocity is proportional to the body velocity: $v \propto u$. The viscous stress (1.38) and the pressure are also linear in u and so must be the drag force:

$$F = \int \sigma d\mathbf{f} \simeq \int d\mathbf{f} \eta u/R \simeq 4\pi R^2 \eta u/R = 4\pi\eta u R\,.$$

That crude estimate coincides with the true answer given below by (1.49) up to the dimensionless factor 3/2. Linear proportionality between the force and the velocity makes the low-Reynolds flows an Aristotelean world.

Now, if you wish to know what force would move a body with $Re \simeq 1$ (or $1/6\pi$ for a sphere), you find amazingly that such force, $F \sim \eta^2/\rho$, does not depend on the body size (that is the same for a bacteria and a ship). For water, $\eta^2/\rho \simeq 10^{-4}$ dyn.

Swimming means changing shape in a periodic way to move. Motion on micro and nano scales in fluids usually correspond to very low Reynolds numbers when

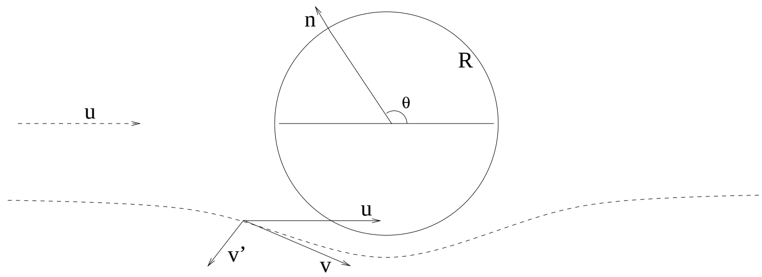
$$\partial v/\partial t \simeq (v\nabla)v \simeq u^2/L \ll \nu\Delta v \simeq \nu u/L^2 \,.$$

Such swimming is very different from pushing water backwards as we do at finite Re. First, there is no inertia so that momentum diffuses instantly through the fluid. Therefore, it does not matter how fast or slow we change the shape. What matters is the shape change itself i.e. low-*Re* swimming is purely geometrical. Second, linearity means that simply retracing the changes back (by inverting the forces i.e. the pressure gradients) we just retrace the motion. One thus needs to change a shape periodically but in a time-irreversible way that is to have a cycle in a configuration space. Microorganisms do that by sending progressive waves along their surfaces. Every point of a surface may move time-reversibly (even by straight lines), time direction is encoded in the phase shift between different points. For example, spermatozoid swims by sending helical waves down its tail[17]. See Exercise 1.10 for another example.

Creeping flow. Consider the steady Navier-Stokes equation without a nonlinear term:

$$\eta\Delta\mathbf{v} = \nabla p \;. \tag{1.45}$$

Let us find the flow around a sphere. In the reference frame of the sphere, the flow at infinity is assumed to have the velocity $\mathbf{u}$. Denote $\mathbf{v} = \mathbf{u}+\mathbf{v}'$.



We wish to repeat the trick we made in considering the potential flow by reducing a vector problem to a scalar one. We do it now by exploiting linearity of the problem. The continuity equation, $div\,\mathbf{v} = 0$, means that the velocity field can be presented in the form $\mathbf{v}' = curl\,\mathbf{A}$ (note that the flow is not assumed potential). The axial vector $\mathbf{A}(\mathbf{r},\mathbf{u})$ has to be linear in $\mathbf{u}$. The only way to make an axial vector from $\mathbf{r}$ and $\mathbf{u}$ is $\mathbf{r}\times\mathbf{u}$ so that it has to be $\mathbf{A} = f'(r)\mathbf{n}\times\mathbf{u}$. We just reduced our problem from finding a vector field $\mathbf{v}(\mathbf{r})$ to finding a scalar function of a single variable, $f(r)$. The vector $f'(r)\mathbf{n}$ can be represented as $\nabla f(r)$ so $\mathbf{v}' = curl\,\mathbf{A} = curl[\nabla f \times \mathbf{u}]$. Since u = const, one can take ∇ out: $\mathbf{v}' = curl\,curl\,(f\mathbf{u})$. If we now apply *curl* to the equation $\eta\Delta\mathbf{v} = \nabla p$ we get the equation to solve

$$\Delta\, curl\,\mathbf{v} = 0\ .$$

Express now $\mathbf{v}$ via f:

$$curl\,\mathbf{v} = \nabla \times \nabla \times \nabla f \times \mathbf{u} = (grad\,div - \Delta)\,\nabla f \times \mathbf{u} = -\Delta\,\nabla f \times \mathbf{u}\,.$$

So our final equation to solve is

$$\Delta^2\nabla f \times \mathbf{u} = 0\ .$$

Since $\nabla f \parallel \mathbf{n}$ so $\Delta^2\nabla f$ cannot always be parallel to $\mathbf{u}$ and we get

$$\Delta^2\nabla f = 0\ . \tag{1.46}$$

Integrating it once and remembering that the velocity derivatives vanish at infinity we obtain $\Delta^2 f = 0$. In spherical coordinates $\Delta = r^{-2}\partial_r r^2\partial_r$ so that $\Delta f = 2a/r$ - here again one constant of integration has to be zero because velocity $\mathbf{v}'$ itself vanishes at infinity. Eventually,

$$f = ar + b/r\ .$$

Taking curl of $\mathbf{A} = f'(r)\mathbf{n} \times \mathbf{u} = (a - br^{-2})\mathbf{n} \times \mathbf{u}$ we get

$$\mathbf{v} = \mathbf{u} - a\frac{\mathbf{u} + \mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} + b\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}}{r^3}\ . \tag{1.47}$$

The last term is the potential part. Boundary condition $\mathbf{v}(R) = 0$ gives $\mathbf{u}$-component $1 - a/R - b/R^3 = 0$ and $\mathbf{n}$-component $-a/R + 3b/R^3 = 0$ so that $a = 3R/4$ and $b = R^3/4$. In spherical components

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)\,,$$
$$v_\theta = -u\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)\ . \tag{1.48}$$

The pressure can be found from $\eta\Delta\mathbf{v} = \nabla p$, but it is easier to note that $\Delta p = 0$. We need the solution of this equation with a dipole source since equal positive and negative pressure changes are generated on the surface of the sphere:

$$p = p_0 + \frac{c(\mathbf{u}\cdot\mathbf{n})}{r^2}\,,$$

where $c = -3\eta R/2$ from $p - p_0 = \eta \mathbf{u}\nabla\Delta f$. Fluid flows down the pressure gradient. The vorticity is a dipole field too:

$$\Delta\, curl\, \mathbf{v} = \Delta\omega = 0 \qquad \Rightarrow \qquad \omega = c' \frac{[\mathbf{u}\times\mathbf{n}]}{r^2}$$

with $c' = -3R/2$ from $\nabla p = \eta\Delta\mathbf{v} = -\eta\, curl\, \omega$.

Stokes formula for the drag. The force acting on a unit surface is the momentum flux through it. On a solid surface $\mathbf{v} = 0$ and $F_i = -\sigma_{ik}n_k = pn_i - \sigma'_{ik}n_k$. In our case, the only nonzero component is along $\mathbf{u}$:

$$\begin{aligned} F_x &= \int (-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)\, df \\ &= (3\eta u/2R)\int df = 6\pi R\eta u\ . \end{aligned} \tag{1.49}$$

Here, we substituted $\sigma'_{rr} = 2\eta\partial v_r/\partial r = 0$ at $r = R$ and

$$\begin{aligned} p(R) &= -\frac{3\eta u}{2R}\cos\theta\,, \\ \sigma'_{r\theta}(R) &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) = -\frac{3\eta u}{2R}\sin\theta\ . \end{aligned}$$

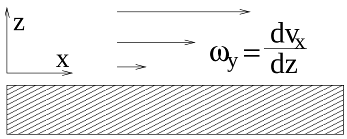
The viscous force is tangential while the pressure force is normal to the surface. The vertical components of the forces cancel each other at every point since the sphere pushes fluid strictly forward so the force is purely horizontal. The viscous and pressure contributions sum into the horizontal force $3\eta\mathbf{u}/2R$, which is independent of θ, i.e. the same for all points on the sphere. The viscous force and the pressure contribute equally into the total force (1.49). That formula is called Stokes law, it works well until $Re \simeq 0.5$.

1.5.2 Boundary layer and separation phenomenon

It is clear that the law of decay $v \propto 1/r$ from (1.47) cannot be realized at arbitrary large distances. Indeed, our assumption of small Reynolds number requires

$$v\nabla v \simeq u^2R/r^2 \ll \nu\Delta v \simeq \nu u R^2/r^3\,,$$

so that (1.47) is valid for $r \ll \nu/u$. One can call ν/u the width of the viscous boundary layer. The Stokes flow is realized inside the boundary layer under the assumption that the size of the body is much less than the width of the layer. So what is the flow outside the viscous boundary layer, that is for $r > \nu/u$? Is it potential? The answer is "yes" only for very small Re. For finite Re, there is an infinite region (called *wake*) behind the body where it is impossible to neglect viscosity whatever the distance from the body. The reason for that is that viscosity produces vorticity in the boundary layer:



At small Re, the process that dominates the flow is vorticity diffusion away from the body. The Stokes approximation, $\omega \propto [\mathbf{u} \times \mathbf{n}]/r^2$, corresponds to symmetrical diffusion of vorticity in all directions. In particular, the flow has a left-right (fore -and-aft) symmetry. For finite Re, it is intuitively clear that the flow upstream and downstream from the body must be different since body leaves vorticity behind it. There should exist some downstream region reached by fluid particles which move along streamlines passing close to the body. The flow in this region (wake) is essentially rotational. On the other hand, streamlines that do not pass through the boundary layer correspond to almost potential motion.

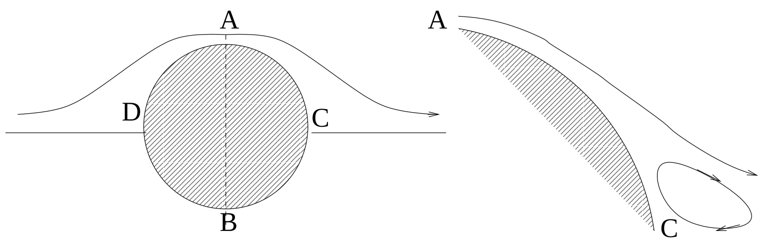
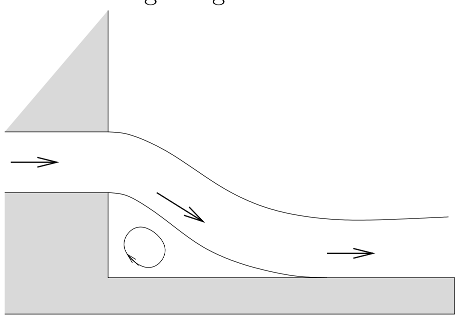


Figure 1.12 Symmetric streamlines for an ideal flow (left) and appearance of separation and a recirculating vortex in a viscous fluid (right).

Let us describe qualitatively how the wake arises. The phenomenon called *separation* is responsible for wake creation (Prandtl, 1905). Consider, for instance, a flow around a cylinder shown in Figure 1.12. The ideal fluid flow is symmetrical with respect to the plane AB. The point D is a stagnation point. On the upstream half DA, the fluid particles accelerate and the pressure decreases according to the Bernoulli theorem. On the downstream part AC, the reverse happens, that is every particle moves against the pressure gradient. Small viscosity changes pressure only slightly across the boundary layer. Indeed, if the viscosity is small, the boundary layer is thin and can be considered as locally flat. In the boundary layer $\mathbf{v} \approx v_x$ and the pressure gradient, $\nabla p = -\rho(v\nabla)v - \eta\Delta v$, has only x component that is $\partial p/\partial z \approx 0$. In other words, the pressure inside the boundary layer is almost equal to that in the main stream, that is the pressure of the ideal fluid flow. But the velocities of the fluid particles that reach the points A and B are lower in a viscous fluid than in an ideal fluid because of viscous friction in the boundary layer. Then those particles have insufficient energy to overcome the pressure gradient downstream. The particle motion in the boundary layer is stopped by the pressure gradient before the point C is reached. The pressure gradient then becomes the force that accelerates the particles from the point C upwards producing separation [18] and a recirculating vortex. A similar mechanism is responsible for recirculating eddies in the corners [19] shown at the end of Sect. 1.2.4.

Reversing the flow pattern of separation one obtains attachment: jets tend to attach to walls and merge with each other. Consider first a jet in an infinite fluid and denote the velocity along the jet u. The momentum flux through any section is the same: $\int u^2\, df = \text{const}$. On the other hand, the energy flux, $\int u^3\, df$, decreases along the jet due to viscous friction. That means that the mass flux of the fluid, $\int u\, df$, must grow — a phenomenon known as *entrainment* [20]. When the jet has a wall (or another jet) on one side, it draws less fluid into itself from this side and so inclines until it is getting attached as shown in the Figure:



wall−attaching jet

In particular, jet merging explains a cumulative effect of arm-piercing shells which contain a conical void covered by a metal and surrounded by explosives. Explosion turns metal into a fluid which moves towards the axis where it creates a cumulative jet with a high momentum density (Lavrent'ev 1947, Taylor 1948), see Figure 1.13 and Exercise 1.14. Similarly, if one creates a void in a liquid by, say, a raindrop or other falling object then the vertical momentum of the liquid that rushes to fill the void creates a jet seen in Figure 1.14.

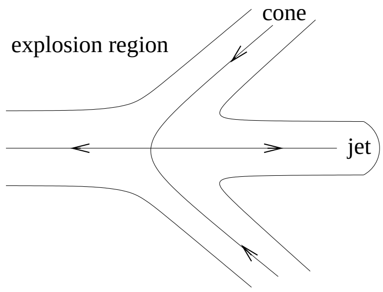


Figure 1.13 Scheme of the flow of a cumulative jet in the reference frame moving with the cone.

Figure 1.14 Jet shooting out after the droplet fall. Upper image - beginning of the jet formation, lower image - jet formed.

1.5.3 Flow transformations

Let us now use the case of the flow past a cylinder to describe briefly how the flow pattern changes as the Reynolds number goes from small to large. The flow is most symmetric for $Re \ll 1$ when it is steady and has an exact up-down symmetry and approximate (order Re) left-right symmetry. Separation and occurrence of eddies is a change of the flow topology, it occurs around $Re \simeq 5$. The first loss of exact symmetries happens around $Re \simeq 40$ when the flow is getting periodic in time. This happens because the recirculating eddies don't have enough time to spread, they are getting detached from the body and carried away by the flow as the new eddies are generated. Periodic flow with shedding eddies has up-down and continuous time shift symmetries broken and replaced by a combined symmetry of up-down reflection and time shift for half a period. Shedding of eddies explains many surprising symmetry-breaking phenomena like, for instance, an air bubble rising through water (or champaign) in a zigzag or a spiral rather than a straight path [21]. For the flow past a body, it results in a double train of vortices called Kármán vortex street [22] behind the body as shown in Figure 1.15.

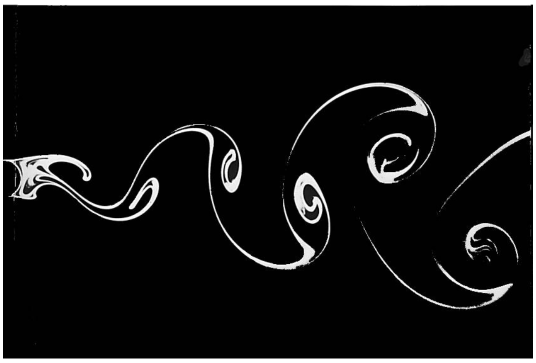


Figure 1.15 Kármán vortex street behind a cylinder at $Re = 105$.

As the Reynolds number increases further, the vortices are getting unstable and produce an irregular turbulent motion downstream as seen in Figure 1.16 [23]. That turbulence is three-dimensional i.e. the translational invariance along the cylinder is broken as well. The higher Re the closer to the body turbulence starts. At $Re \simeq 10^5$, the turbulence reaches the body which brings so-called drag crisis: since a turbulent boundary layer is separated later than a laminar one, then the wake area gets smaller and the drag is lower [24].

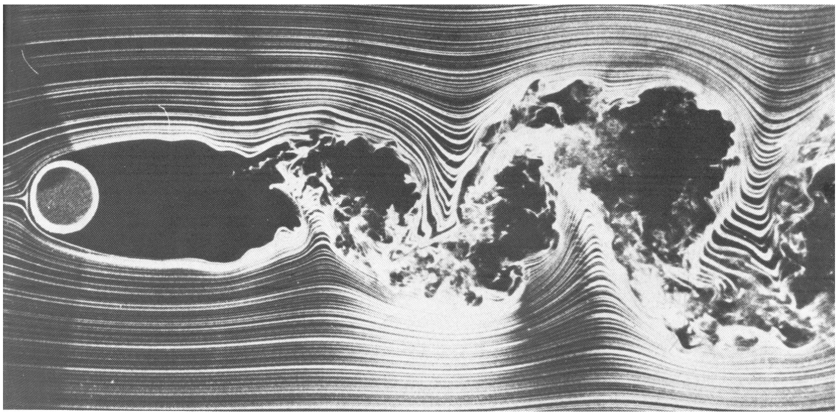


Figure 1.16 Flow past a cylinder at $Re = 10^4$.

1.5.4 Drag and lift with a wake

We can now describe the way Nature resolves reversibility and D'Alembert paradoxes. Like in Sect.1.3, we again consider the steady flow far from the body and relate it to the force acting on the body. The new experimental wisdom we now have is the existence of the wake. The flow is irrotational outside the boundary layer and the wake. First, we consider a laminar wake i.e. assume $v \ll u$ and $\partial v/\partial t = 0$; we shall show that the wake is always laminar far enough from the body. For a steady flow, it is convenient to relate the force to the momentum flux through a closed surface. For a dipole potential flow $v \propto r^{-3}$ from Section 1.3, that flux was zero for a distant surface. Now wake gives a finite contribution. The total momentum flux transported by the fluid through any closed surface is equal to the rate of momentum change which is equal to the force acting on the body:

$$F_i = \oint \Pi_{ik} df_k = \oint \Big[(p_0 + p')\delta_{ik} + \rho(u_i + v_i)(u_k + v_k)\, df_k \ . \qquad (1.50)$$

Mass conservation means that $\rho \oint v_k\, df_k = 0$. Far from the body $v \ll u$ and

$$F_i = \left(\int \int_{X_0} - \int \int_{X}\right) (p'\delta_{ix} + \rho u v_i)\, dydz \ . \qquad (1.51)$$

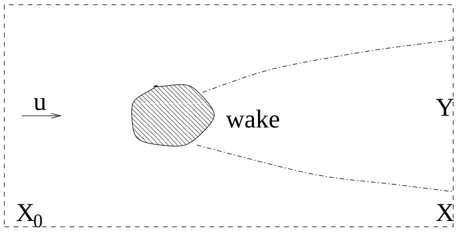


Figure 1.17 Scheme of the wake.

Drag with a wake. Consider the x-component of the force (1.51):

$$F_x = \left(\int\int_{X_0} - \int\int_{X} \right) (p' + \rho u v_x)\, dydz \ .$$

Outside the wake we have potential flow where the Bernoulli relation, $p + \rho|\mathbf{u} + \mathbf{v}|^2/2 = p_0 + \rho u^2/2$, gives $p' \approx -\rho u v_x$ so that the integral outside the wake vanishes. Inside the wake, the pressure is about the same (since it does not change across the almost straight streamlines like we argued in Section 1.5.2) but v_x is shown below to be much larger than outside so that

$$F_x = -\rho u \int\int_{wake} v_x\, dydz \ . \tag{1.52}$$

Force is positive (directed to the right) since v_x is negative. Note that the integral in (1.52) is equal to the deficit of fluid flux Q through the wake area (i.e. the difference between the flux with and without the body). That deficit is x-independent which has dramatic consequences for the potential flow outside the wake, because it has to compensate for the deficit. That means that the integral $\int \mathbf{v}\, d\mathbf{f}$ outside the wake is also r-independent which requires $v \propto r^{-2}$. That corresponds to the potential flow with the source equal to the flow deficit: $\phi = Q/r$. We have thrown away this source flow in Sect. 1.3 but now we see that it exceeds the dipole flow $\phi = \mathbf{A}\cdot\nabla(1/r)$ (which we had without the wake) and dominates sufficiently far from the body.

The wake breaks the fore-and-aft symmetry and thus resolves the paradoxes providing for a nonzero drag in the limit of vanishing viscosity. It is important that the wake has an infinite length, otherwise the body and the finite wake could be treated as a single entity and we are back to paradoxes. The behavior of the drag coefficient $C(Re) = F/\rho u^2 R^2$ is shown in Fig. 1.18. Notice the drag crisis which gives the lowest C. To understand why $C \to$const as $Re \to \infty$ and prove (1.37), one ought to

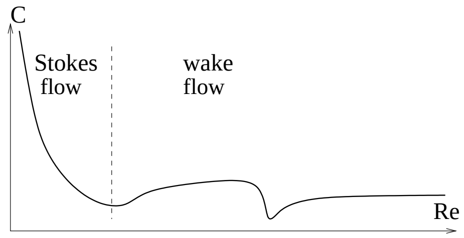


Figure 1.18 Sketch of the drag dependence on the Reynolds number.

pass a long way developing the theory of turbulence briefly described in the next Chapter.

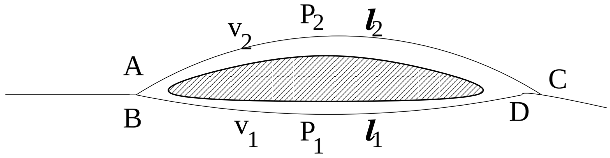
The lift is the force component of (1.51) perpendicular to $\mathbf{u}$:

$$F_y = \rho u \left(\int_{X_0} - \int_X \right) v_y dydz \ . \qquad (1.53)$$

It is also determined by the wake — without the wake the flow is potential with $v_y = \partial\phi/\partial y$ and $v_z = \partial\phi/\partial z$ so that $\int v_y dydz = \int v_z dydz = 0$ since the potential is zero at infinities. We have seen in (1.28) that purely potential flow produces no lift. Without the friction-caused separation, birds and planes would not be able to fly. Let us discuss the lift of the wings which can be considered as slender bodies long in z-direction. The lift force per unit length of the wing can be related to the velocity circulation around the wing. Indeed, adding and subtracting (vanishing) integrals of v_x over two $y = \pm$ const lines we turn (1.53) into

$$F_y = \rho u \oint \mathbf{v} \cdot d\mathbf{l} \ . \qquad (1.54)$$

Circulation over the contour is equal to the vorticity flux through the contour, which is again due to wake. One can often hear a simple explanation of the lift of the wing as being the result of $v_2 > v_1 \ \Rightarrow \ P_2 < P_1$. This is basically true and does not contradict the above argument.



The point is that the circulation over the closed contour ACDB is nonzero: $v_2 l_2 > v_1 l_1$. That would be wrong, however, to argue that $v_2 > v_1$ because $l_2 > l_1$ — neighboring fluid elements A,B do not meet again at the trailing edge; C is shifted relative to D. Nonzero circulation around the body in translational motion requires wake. For a slender wing, the wake is very thin like a cut and a nonzero circulation means a jump of the potential ϕ across the wake [25]. Note that for having lift one needs to break up-down symmetry. Momentum conservation suggests that one can also relate the lift to the downward deflection of the flow by the body.

One can have a nonzero circulation without a wake simply by rotation. When there is a nonzero circulation, then there is a deflecting (Magnus) force acting on a rotating moving sphere. That force is well known to all ball players from soccer to tennis. The air travels faster relative to the center of the ball where the ball surface is moving in the same direction as the air. This reduces the pressure, while on the other side of the ball the pressure increases. The result is the lift force, perpendicular to the motion (As J J Thomson put it, "the ball follows its nose"). One can roughly estimate the magnitude of the Magnus force by the

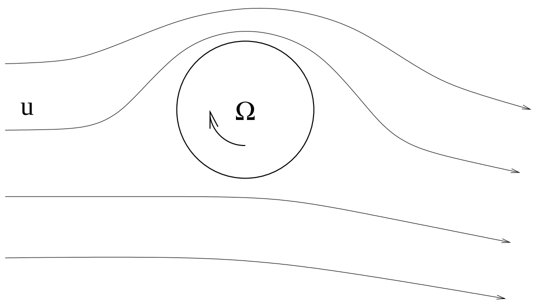


Figure 1.19 Streamlines around a rotating body.

pressure difference between the two sides[26], which is proportional to the translation velocity u times the rotation frequency Ω:

$$\Delta p \simeq \rho[(u+\Omega R)^2 - (u-\Omega R)^2]/2 = 2\rho u \Omega R \ . \qquad (1.55)$$

Magnus force is exploited by winged seeds who travel away from the parent tree superimposing rotation on their descent [27], it also acts on quantum vortices moving in superfluids or superconductors. See Exercise 1.11.

Moral: wake existence teaches us that small viscosity changes the flow not only in the boundary layer but also in the whole space, both inside and outside the wake. Physically, this is because vorticity is produced in the boundary layer and is transported outside[28]. Formally, viscosity is a singular perturbation that introduces the highest spatial derivative and changes the boundary conditions. On the other hand, even for a very large viscosity, inertia dominates sufficiently far from the body[29].

Exercises

1.1 Proceeding from the fact that the force exerted across any plane surface is wholly normal, prove that its intensity (per unit area) is the same for all aspects of the plane (Pascal Law).

1.2 Consider self-gravitating fluid with the gravitational potential ϕ related to the density by

$$\Delta\phi = 4\pi G\rho,$$

G being the constant of gravitation. Assume spherical symmetry and static equilibrium. Describe the radial distribution of pressure for an incompressible liquid.

1.3 Find the discharge rate from a small orifice with a cylindrical tube, projecting inward. Assume h, S and the gravity acceleration g given. Whether such a hole corresponds to the limiting (smallest or largest) value of the "coefficient of contraction" S'/S? Here S is the orifice area and S' is the area of the jet where contraction ceases (vena contracta).

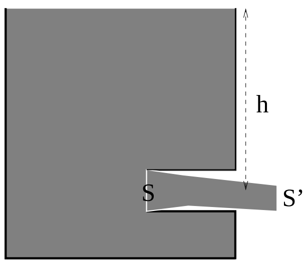


Figure 1.20 Borda mouthpiece

1.4 Prove that if you put a little solid particle — not an infinitesimal point — at any place in the liquid it will rotate with the angular velocity Ω equal to the half of the local vorticity $\omega = curl\,\vec{v}$: $\Omega = \omega/2$.

1.5 There is a permanent source of water on the bottom of a large reservoir. Find the maximal elevation of the water surface for two cases:
i) a straight narrow slit with the constant influx q (g/cm$\cdot$ sec) per unit length;
ii) a point-like source with the influx Q (g/sec) The fluid density is ρ, the depth of the fluid far away from the source is h. Gravity acceleration is g. Assume that the flow is potential.

1.6 Sketch streamlines for the potential inviscid flow and for the viscous Stokes flow in two reference systems, in which: i) fluid at infinity is at rest; ii) sphere is at rest. Hint: Since the flow past a sphere is actually a set of plane flows, one can introduce the stream function analogous to that in two dimensions. If one defines a vector whose only component is perpendicular to the plane and equal to the stream function then the velocity is the curl of that vector and the streamlines are level lines of the stream function.

1.7 A small heavy ball with the density ρ_0 connected to a spring has the oscillation frequency ω_a. The same ball attached to a rope makes a pendulum with the oscillation frequency ω_b. How those frequencies change if such oscillators are placed into an ideal fluid with the density ρ? What change brings an account of a small viscosity of the fluid ($\nu \ll \omega_{a,b}a^2$ where a is the ball radius and ν is the kinematic viscosity).

1.8 Underwater explosion released the energy E and produced a gas bubble oscillating with the period T, which is known to be completely determined by E, the static pressure p in the water and the water density ρ. Find the form of the dependence $T(E,p,\rho)$ (without numerical factors). If the initial radius a is known instead of E, can we determine the form of the dependence $T(a,p,\rho)$?

1.9 At $t = 0$ a straight vortex line exists in a viscous fluid. In cylindrical coordinates, it is described as follows: $v_r = v_z = 0$, $v_\theta = \Gamma/2\pi r$, where Γ is some constant. Find the vorticity $\omega(r,t)$ as a function of time and the time behavior of the total vorticity $\int \omega(r) r\, dr$.

1.10 To appreciate how one swims in a syrup, consider the so-called Purcell swimmer shown in Figure 1.22. It can change its shape by changing separately the angles between the middle link and the arms. Assume that the angle θ is small. Numbers correspond to consecutive shapes. In a position 5 it has the same shape as in

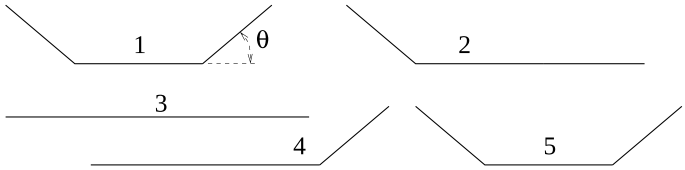


Figure 1.21 Subsequent shapes of the swimmer.

1 but moved in space. Which direction? What distinguishes this direction? How the displacement depends on θ?

1.11 In making a free kick, good soccer players are able to utilize the Magnus force to send the ball around the wall of defenders. Neglecting vertical motion, estimate the horizontal deflection of the ball (with the radius $R = 11$ cm and the weight $m = 450$ g according to FIFA rules) sent with the speed $v_0 = 30$ m/s and the side-spin $\Omega = 10$ revolutions per second towards the goal which is $L = 30$ m away. Take the air density $\rho = 10^{-3}$ g/cm^3.

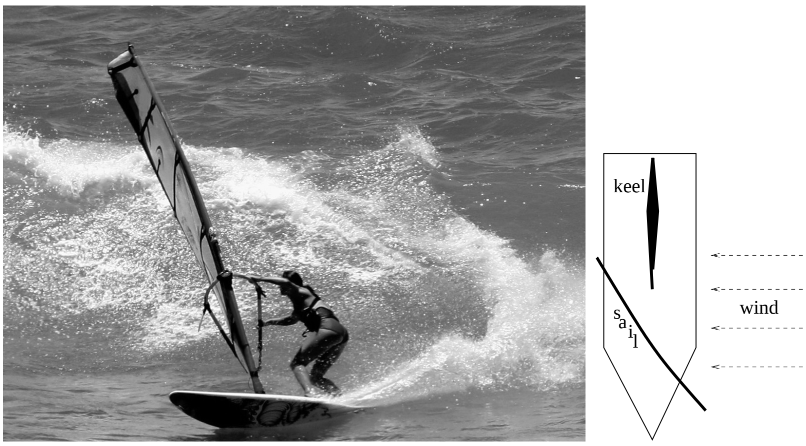


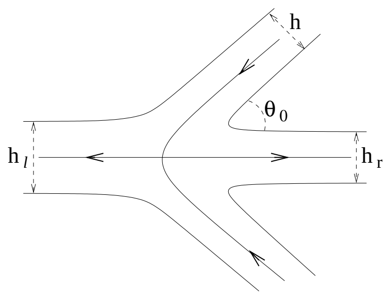
Figure 1.22 Left panel: the sailor holds the sail against the wind which is thus coming from behind her back. Right panel: scheme of the position of the board and its sail with respect to the wind.

1.12 Like flying, sailing also utilizes the lift (perpendicular) force acting on the sails and the keel. The fact that wind provides a force perpendicular to the sail allows one even to move against the wind. But most optimal for starting and reaching maximal speed, as all windsurfers know, is to orient the board perpendicular to the wind and set the sail at about 45 degrees, see Figure 1.22. Why? Draw the forces acting on the board. Does the board move exactly in the direction at which the keel is pointed? Can one move faster than wind?

1.13 Find the fall velocity of a liquid water droplet with the radius 0.01 mm in the air. Air and water viscosities and densities are respectively $\eta_a = 1.8 \cdot 10^{-4}$ g/s · cm, $\eta_w = 0.01$ g/s · cm and $\rho = 1.2 \cdot 10^{-3}$ g/cm^3, $\rho_w = 1$ g/cm^3.

1.14 Describe the motion of an initially small spherical water droplet falling in a saturated cloud and absorbing the vapor in a swept volume so that its volume grows proportionally to its velocity and its cross-section. Consider quasi-steady approximation when the droplet acceleration is much less than the gravity acceleration g.

1.15 Consider plane free jets in an ideal fluid in the geometry shown in the Figure. Find how the widths of the outgoing jets depend on the angle $2\theta_0$ between the impinging jets.



2
Unsteady flows

Fluid flows can be kept steady only for very low Reynolds numbers and for velocities much less than sound velocity. Otherwise, either flow undergoes instabilities and is getting turbulent or sound and shock waves are excited. Both sets of phenomena are described in this Chapter.

A formal reason for instabilities is nonlinearity of the equations of fluid mechanics. For incompressible flows, the only nonlinearity is due to fluid inertia. We shall see below how a perturbation of a steady flow can grow due to inertia, thus causing an instability. For large Reynolds numbers, development of instabilities leads to a strongly fluctuating state of turbulence.

An account of compressibility, on the other hand, leads to another type of unsteady phenomena: sound waves. When density perturbation is small, velocity perturbation is much less than the speed of sound and the waves can be treated within the framework of linear acoustics. We first consider linear acoustics and discuss what phenomena appear as long as one accounts for a finiteness of the speed of sound. We then consider nonlinear acoustic phenomena, creation of shocks and acoustic turbulence.

2.1 Instabilities

At large *Re* most of the steady solutions of the Navier-Stokes equation are unstable and generate an unsteady flow called turbulence.

2.1.1 Kelvin-Helmholtz instability

Apart from a uniform flow in the whole space, the simplest steady flow of an ideal fluid is a uniform flow in a semi-infinite domain with the velocity parallel to the boundary. Physically, it corresponds to one fluid layer sliding along another. Mathematically, it is a tangential velocity discontinuity, which is a formal steady solution of the Euler equation. It is a crude approximation to the description of wakes and shear flows. This simple solution is unstable with respect to arguably the simplest instability described by Helmholtz (1868) and Kelvin (1871). The dynamics of the Kelvin-Helmholtz instability is easy to see from Figure 2.1 where + and − denote respectively increase and decrease in velocity and pressure brought by surface modulation. Velocity over the convex

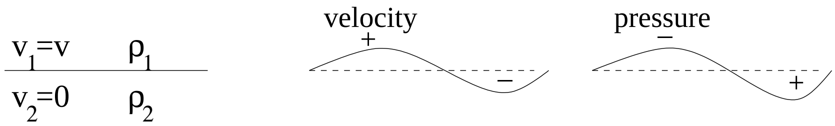


Figure 2.1 Tangential velocity discontinuity (left) and the physics of Kelvin-Helmholtz instability (right).

part is higher and the pressure is lower than over the concave part. Such pressure distribution further increases the modulation of the surface.

The perturbations $\mathbf{v}'$ and p' satisfy the following system of equations

$$\operatorname{div}\mathbf{v}' = 0\,, \qquad \frac{\partial \mathbf{v}'}{\partial t} + v\frac{\partial \mathbf{v}'}{\partial x} = -\frac{\nabla p'}{\rho}\ .$$

Applying divergence operator to the second equation we get $\Delta p' = 0$. That means that the elementary perturbations have the following form

$$p'_1 = \exp[i(kx - \Omega t) - kz]\,,$$
$$v'_{1z} = -ikp'_1/\rho_1(kv - \Omega)\ .$$

Indeed, the solutions of the Laplace equation which are periodic in one direction must be exponential in another direction.

To relate the upper side (indexed 1) to the lower side (indexed 2) we introduce $\zeta(x,t)$, the elevation of the surface, its time derivative is z-component of the velocity:

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v\frac{\partial \zeta}{\partial x} = v'_z\,, \tag{2.1}$$

that is $v'_z = i\zeta(kv - \Omega)$ and $p'_1 = -\zeta\rho_1(kv - \Omega)^2/k$. On the other side, we express in a similar way $p'_2 = \zeta\rho_2\Omega^2/k$. The pressure is continuous across the surface:

$$\rho_1(kv-\Omega)^2 = -\rho_2\Omega^2 \quad \Rightarrow \quad \Omega = kv\frac{\rho_1 \pm i\sqrt{\rho_1\rho_2}}{\rho_1+\rho_2}\ . \tag{2.2}$$

Positive ImΩ means an exponential growth of perturbations i.e. instability [1]. The largest growth rate corresponds to the largest admissible wavenumber. In reality the layer, where velocity increases from zero to v, has some finite thickness δ and our approach is valid only for $k\delta \ll 1$. It is not difficult to show that in the opposite limit, $k\delta \gg 1$ when the flow can be locally considered as a linear profile, it is stable (see Rayleigh criterium below). Therefore, the maximal growth rate corresponds to $k\delta \simeq 1$, i.e. the wavelength of the most unstable perturbation is comparable to the layer thickness.

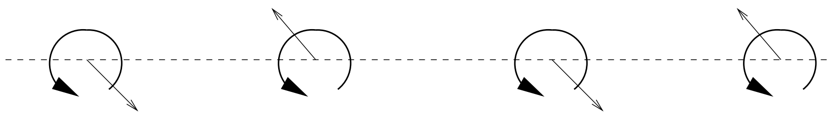


Figure 2.2 Array of vortex lines is instable with respect to the displacements shown by straight arrows.

A complementary insight into the physics of the Kelvin-Helmholtz instability can be obtained from considering vorticity. In the unperturbed flow, vorticity $\partial v_x/\partial z$ is concentrated in the transitional layer which is thus called vortex layer (or *vortex sheet* when $\delta \to 0$). One can consider a discrete version of the vortex layer as a chain of identical vortices shown in Figure 2.2. Due to symmetry, such infinite array of vortex lines is stationary since the velocities imparted to any given vortex by all others cancel. Small displacements shown by straight arrows in Figure 2.2 lead to an instability with the vortex chain breaking into pairs of vortices circling round one another. That circling motion makes an initially sinusoidal perturbation to grow into spiral rolls during the nonlinear stage of the evolution as shown in Figure 2.3 taken from the experiment. Kelvin-Helmholtz instability in the atmosphere is often made visible by corrugated cloud patterns as seen in Figure 2.4, similar patterns are seen on sand dunes. It is also believed to be partially responsible for clear air turbulence (that is atmospheric turbulence unrelated to moist convection). Numerous manifestations of this instability are found in astrophysics, from the interface between the solar wind and the Earth magnetosphere to the boundaries of galactic jets.

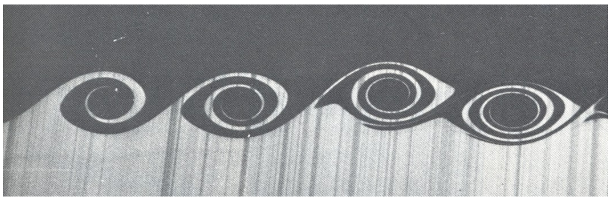


Figure 2.3 Spiral vortices generated by the Kelvin-Helmholtz instability.

Vortex view of the Kelvin-Helmholtz instability suggests that a unidirectional flow depending on a single transverse coordinate, like $v_x(z)$, can only be unstable if it has vorticity maximum on some surface. Such vorticity maximum is an inflection point of the velocity since $d\omega/dx = d^2v_x/dz^2$. That explains why flows without inflection points are linearly stable (Rayleigh, 1880). Examples of such flows are plane linear profile, flows in a pipe or between two planes driven by the pressure gradients, flow between two planes moving with different velocities etc [2].

Our consideration of the Kelvin-Helmholtz instability was completely inviscid which presumes that the effective Reynolds number was large: $Re = v\delta/\nu \gg 1$. In the opposite limit when the friction is very strong, the velocity profile is not stationary but rather evolves according to the equation $\partial v_x(z,t)/\partial t = \nu\partial^2 v_x(z,t)/\partial z^2$ which describes the thickness growing as $\delta \propto \sqrt{\nu t}$. Such diffusing vortex layer is stable because the friction damps all the perturbations. It is thus clear that there must exist a threshold Reynolds number above which instability is possible. We now consider this threshold from a general energetic perspective.

2.1.2 Energetic estimate of the stability threshold

Energy balance between the unperturbed steady flow $\mathbf{v}_0(\mathbf{r})$ and the superimposed perturbation $\mathbf{v}_1(\mathbf{r},t)$ helps one to understand the role of viscosity in imposing an instability threshold. Consider the flow $\mathbf{v}_0(\mathbf{r})$ which is a steady solution of the the Navier-Stokes equation $(\mathbf{v}_0\cdot\nabla)\mathbf{v}_0 = -\nabla p_0/\rho + \nu\Delta\mathbf{v}_0$. The perturbed flow $\mathbf{v}_0(\mathbf{r}) + \mathbf{v}_1(\mathbf{r},t)$ satisfies the equation:

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_1 + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1 = -\frac{\nabla p_1}{\rho} + \nu\Delta\mathbf{v}_1 \ . \qquad (2.3)$$

Figure 2.4 Lower cloud shows the pattern of breaking waves generated by the Kelvin-Helmholtz instability.

Making a scalar product of (2.3) with $\mathbf{v}_1$ and using incompressibility one gets:

$$\frac{1}{2}\frac{\partial v_1^2}{\partial t} = -v_{1i}v_{1k}\frac{\partial v_{0i}}{\partial x_k} - \frac{1}{Re}\frac{\partial v_{1i}}{\partial x_k}\frac{\partial v_{1i}}{\partial x_k}$$
$$-\frac{\partial}{\partial x_k}\left[\frac{v_1^2}{2}(v_{0k}+v_{1k}) + p_1 v_{1k} - \frac{v_{1i}}{Re}\frac{\partial v_{1i}}{\partial x_k}\right] .$$

The last term disappears after the integration over the volume:

$$\frac{d}{dt}\int \frac{v_1^2}{2} d\mathbf{r} = T - \frac{D}{Re} , \tag{2.4}$$

$$T = -\int v_{1i}v_{1k}\frac{\partial v_{0i}}{\partial x_k}\, d\mathbf{r} , \quad D = \int \left(\frac{\partial v_{1i}}{\partial x_k}\right)^2 d\mathbf{r} .$$

The term T is due to inertial forces and the term D is due to viscous friction. We see that for stability (i.e. for decay of the energy of the perturbation) one needs friction dominating over inertia:

$$Re < Re_E = \min_{v_1} \frac{D}{T} . \tag{2.5}$$

The minimum is taken over different perturbation flows. Since both T and D are quadratic in the perturbation velocity then their ratio depends on the orientation and spatial dependence of $\mathbf{v}_1(\mathbf{r})$ but not on its magnitude. For nonzero energy input T one must have $\partial v_0/\partial r \neq 0$ (uniform flow is stable) and the perturbation velocity oriented in such a way as to have both the component v_{1i} along the mean flow and the component v_{1k} along the gradient of the mean flow. One may have positive T if the perturbation velocity is oriented relative to the mean flow gradient as, for instance, in the geometry shown in Fig. 2.5. While the

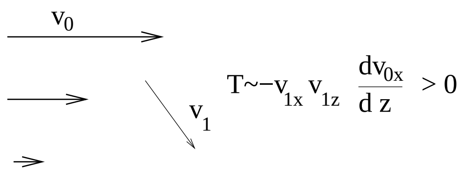


Figure 2.5 Orientation of the perturbation velocity $\mathbf{v}_1$ with respect to the steady shear $\mathbf{v}_0$ that provides for an energy flux from the shear to the perturbation.

flow is always stable for $Re < Re_E$, it is not necessary unstable when one can find a perturbation that breaks (2.5); for instability to develop, the perturbation must also evolve in such a way as to keep $T > D$. As a consequence, the critical Reynolds numbers are usually somewhat higher than those given by the energetic estimate.

2.1.3 Landau law

When the control parameter passes a critical value the system undergoes an instability and goes into a new state. Generally, one cannot say much about this new state except for the case when it is not very much different from the old one. That may happen when the control parameter is not far from critical. Consider $Re > Re_{cr}$ but $Re - Re_{cr} \ll Re_{cr}$. Just above the instability threshold, there is usually only one unstable mode. Let us linearize the equation (2.3) with respect to the perturbation $\mathbf{v}_1(\mathbf{r}, t)$ i.e. omit the term $(\mathbf{v_1} \cdot \nabla)\mathbf{v_1}$. The resulting linear differential equation with time-independent coefficients has the solution in the form $\mathbf{v}_1 = \mathbf{f}_1(\mathbf{r}) \exp(\gamma_1 t - \imath\omega_1 t)$. The exponential growth has to be restricted by the terms nonlinear in $\mathbf{v_1}$. The solution of a weakly nonlinear equation can be sought in the form $\mathbf{v}_1 = \mathbf{f}_1(\mathbf{r})A(t)$. The equation for the amplitude $A(t)$ has to have generally the following form: $d|A|^2/dt = 2\gamma_1|A|^2 +$ third-order terms $+ \ldots$. The fourth-order terms are obtained by expanding further $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2$ and accounting for $\mathbf{v}_2 \propto \mathbf{v}_1^2$ in the equation on $\mathbf{v}_1$. The growth rate turns into zero at $Re = Re_{cr}$ and generally $\gamma_1 \propto Re - Re_{cr}$ while the frequency is usually finite at $Re \to Re_{cr}$. We can thus average the amplitude equation over the time larger than $2\pi/\omega_1$ but smaller than $1/\gamma_1$. Since the time of averaging contains many periods, then among the terms of the third and fourth order only $|A|^4$ gives nonzero contribution:

$$\overline{\frac{d|A|^2}{dt}} = 2\gamma_1|A|^2 - \alpha|A|^4 \ . \tag{2.6}$$

Since the time of averaging is much less than the time of the modulus change, then one can remove the overbar in the left-hand side of (2.6) and solve it as a usual ordinary differential equation. This equation has the solution

$$|A|^{-2} = \alpha/2\gamma_1 + \mathrm{const}\cdot\exp(-2\gamma_1 t) \to \alpha/2\gamma_1 \ .$$

The saturated value changes with the control parameter according to the so-called Landau law:

$$|A|^2_{max} = \frac{2\gamma}{\alpha} \propto Re - Re_{cr} \ .$$

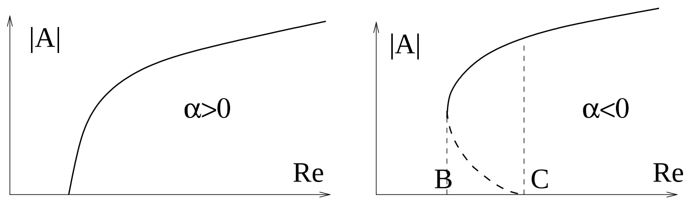
If $\alpha < 0$ then one needs $-\beta|A|^6$ term in (2.6) to stabilize the instability

$$\overline{\frac{d|A|^2}{dt}} = 2\gamma_1|A|^2 - \alpha|A|^4 - \beta|A|^6 \ . \tag{2.7}$$

The saturated value is now

$$|A|^2_{max} = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{\alpha^2}{4\beta^2} + \frac{2\gamma_1}{\beta}} \ .$$

Stability with respect to the variation of $|A|^2$ within the framework of (2.7) is determined by the factor $2\gamma_1 - 2\alpha|A|^2_{max} - 3\beta|A|^4_{max}$. Between B and C, the steady flow is metastable. Broken curve is unstable.



The above description is based on the assumption that at $Re - Re_{cr} \ll Re_{cr}$ the only important dependence is $\gamma_1(Re)$ very much like in the Landau's theory of phase transitions (which also treats loss of stability). The amplitude A, which is non-zero on one side of the transition, is an analog of the order parameter. Cases of positive and negative α correspond to the phase transitions of the second and first order respectively.

2.2 Turbulence

As Reynolds number increases beyond the threshold of the first instability, it eventually reaches a value where the new periodic flow is getting unstable in its own turn with respect to another type of perturbation, usually with smaller scale and consequently higher frequency. Every new instability brings about an extra degree of freedom, characterized by the amplitude and the phase of the new periodic motion. The phases are determined by (usually uncontrolled) initial perturbations. At very large Re, a sequence of instabilities produces *turbulence* as a superposition of motions of different scales. The resulting flow is irregular both spatially and temporally so we need to describe it statistically.

Figure 2.6 Instabilities in three almost identical convective jets lead to completely different flow patterns. Notice also appearance of progressively smaller scales as the instabilities develop.

Flows that undergo instabilities are usually getting temporally chaotic already at moderate *Re* because motion in the phase space of more than three interacting degrees of freedom may tend to sets (called attractors) more complicated than points (steady states) or cycles (periodic motions). Namely, there exist attractors, called strange or chaotic, that consist of saddle-point trajectories. Such trajectories have stable directions by which the system approaches attractor and unstable directions lying within the attractor. Because all trajectories are unstable on the attractor, any two initially close trajectories separate exponentially with the mean rate called the Lyapunov exponent. To intuitively appreciate how the mean stretching rate can be positive in a random flow, note that around a saddle-point more vectors undergo stretching than contraction (Exercise 2.1). Exponential separation of trajectories means instability and unpredictability of the flow patterns. The resulting fluid flow that corresponds to a strange attractor is regular in space and random in time, it is called dynamical chaos [3]. One can estimate the Lyapunov exponent for the Earth atmosphere by dividing the typical wind velocity $20\,m/sec$ by the global scale $10000\,km$. The inverse Lyapunov exponent gives the time one can reasonably hope to predict weather, which is $10^7\,m/(20\,m/sec) = 5\cdot 10^5\,sec$, i.e. about a week.

When the laminar flow is linearly stable at large Re (like uni-directional flows without inflection points), its basin of attraction shrinks when Re grows so that small fluctuations are able to excite turbulence which then sustains itself. In this case, between the laminar flow and turbulence there is no state with simple spatial or temporal structures.

2.2.1 Cascade

Here we discuss turbulence at very large Re. It is a flow random in space and in time. Such flows require statistical description that is an ability to predict mean (expectation) values of different quantities. Despite five centuries of an effort (since Leonardo Da Vinci) a complete description is still lacking but some important elements are established. The most revealing insight into the nature of turbulence presents a cascade picture, which we present in this section. It is a useful phenomenology both from a fundamental viewpoint of understanding a state with many degrees of freedom deviated from equilibrium and from a practical viewpoint of explaining the empirical fact that the drag force is finite in the inviscid limit. The finiteness of the drag coefficient, $C(Re) = F/\rho u^2 L^2 \to \mathrm{const}$ at $Re \to \infty$ (see Figure 1.18), means that the rate of the kinetic energy input per unit mass, $\epsilon = Fu/\rho L^3 = Cu^3/2L$, stays finite when $\nu \to 0$. Where all this energy goes if consider not an infinite wake but a bounded flows, say, generated by a permanently acting fan in a room? Experiments (and everyday experience) tells us that a fan generates some air flow whose magnitude stabilizes after a while which means that the input is balanced by the viscous dissipation. That means that the energy dissipation rate $\epsilon = \nu \int \omega^2\, dV/V$ stays finite when $\nu \to 0$ (if the fluid temperature is kept constant).

Historically, understanding of turbulence started from an empirical law established by Richardson (observing seeds and balloons released in the wind): the mean squared distance between two particles in turbulence increases in a super-diffusive way: $\langle R^2(t)\rangle \propto t^3$. Here the average is over different pairs of particles. The parameter that can relate $\langle R^2(t)\rangle$

and t^3 must have dimensionality cm^2s^{-3} which is that of the dissipation rate ϵ: $\langle R^2(t)\rangle \simeq \epsilon t^3$. Richardson law can be *interpreted* as the increase of the typical velocity difference $\delta v(R)$ with the distance R: since there are vortices of different scales in a turbulent flow, the velocity difference at a given distance is due to vortices with comparable scales and smaller; as the distance increases, more (and larger) vortices contribute the relative velocity, which makes separation faster than diffusive (when the velocity is independent of the distance). Richardson law suggests the law of the relative velocity increase with the distance in turbulence. Indeed, $R(t) \simeq \epsilon^{1/2}t^{3/2}$ is a solution of the equation $dR/dt \simeq (\epsilon R)^{1/3}$; since $dR/dt = \delta v(R)$ then

$$\delta v(R) \simeq (\epsilon R)^{1/3} \quad \Rightarrow \quad \frac{(\delta v)^3}{R} \simeq \epsilon \ . \tag{2.8}$$

The last relation brings the idea of the energy cascade over scales, which goes from the scale L with $\delta v(L) \simeq u$ down to the viscous scale l defined by $\delta v(l)l \simeq \nu$. The energy flux through the given scale R can be estimated as the energy $(\delta v)^2$ divided by the time $R/\delta v$. For the so-called inertial interval of scales, $L \gg R \gg l$, there is neither force nor dissipation so that the energy flux $\epsilon(R) = \langle \delta v^3(R)\rangle/R$ may be expected to be R-independent, as suggested by (2.8). When $\nu \to 0$, the viscous scale l decreases, that is cascade is getting longer, but the amount of the flux and the dissipation rate stay the same. In other words, finiteness of ϵ in the limit of vanishing viscosity can be interpreted as locality of the energy transfer in R-space (or equivalently, in Fourier space). By using an analogy, one may say that turbulence is supposed to work as a pipe with a flux through its cross-section independent of the length of the pipe [4]. Note that the velocity difference (2.8) is expected to increase with the distance slower than linearly, i.e. the velocity in turbulence is non-Lipschitz on average, see Sect. 1.1, so that fluid trajectories are not well-defined in the inviscid limit [5].

The cascade picture is a nice phenomenology but can one support it with any derivation? That support has been obtained by Kolmogorov in 1941 who derived the exact relation that quantifies the flux constancy. Let us derive the equation for the correlation function of the velocity at different points for an idealized turbulence whose statistics is presumed isotropic and homogeneous in space. We assume no external forces so that the turbulence must decay with time. Let us find the time derivative of the correlation function of the components of the velocity difference

Figure 2.7 Cascade.

between the points 1 and 2,

$$\langle (v_{1i} - v_{2i})(v_{1k} - v_{2k}) \rangle = \frac{2\langle v^2 \rangle}{3} \delta_{ik} - 2\langle v_{2i} v_{1k} \rangle \ .$$

The time derivative of the kinetic energy is minus the dissipation rate: $\epsilon = -d\langle v^2 \rangle / 2dt$. To get the time derivative of the two-point velocity correlation function, take the Navier-Stokes equation at some point $\mathbf{r}_1$, multiply it by the velocity $\mathbf{v}_2$ at another point $\mathbf{r}_2$ and average it over time intervals[6] larger than $|\mathbf{r}_1 - \mathbf{r}_2|/|\mathbf{v}_1 - \mathbf{v}_2|$ and smaller than L/u:

$$\frac{\partial}{\partial t}\langle v_{1i}v_{2k}\rangle = -\frac{\partial}{\partial x_{1l}}\langle v_{1l}v_{1i}v_{2k}\rangle - \frac{\partial}{\partial x_{2l}}\langle v_{1i}v_{2k}v_{2l}\rangle$$
$$-\frac{1}{\rho}\frac{\partial}{\partial x_{1i}}\langle p_1 v_{2k}\rangle - \frac{1}{\rho}\frac{\partial}{\partial x_{2k}}\langle p_2 v_{1i}\rangle + \nu(\Delta_1 + \Delta_2)\langle v_{1i}v_{2k}\rangle \ .$$

Statistical isotropy means that the vector $\langle p_1 \mathbf{v}_2\rangle$ has nowhere to look but to $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the only divergence-less such vector, $\mathbf{r}/r^3$, does not satisfy the finiteness at $r = 0$ so that $\langle p_1 \mathbf{v}_2\rangle = 0$. Due to the space homogeneity, all the correlation functions depend only on $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

$$\frac{\partial}{\partial t}\langle v_{1i}v_{2k}\rangle = -\frac{\partial}{\partial x_l}\Big(\langle v_{1l}v_{1i}v_{2k}\rangle + \langle v_{2i}v_{1k}v_{1l}\rangle\Big) + 2\nu\Delta\langle v_{1i}v_{2k}\rangle \ . \quad (2.9)$$

We have used here $\langle v_{1i}v_{2k}v_{2l}\rangle = -\langle v_{2i}v_{1k}v_{1l}\rangle$ since under $1 \leftrightarrow 2$ both $\mathbf{r}$ and a third-rank tensor change sign (the tensor turns into zero when $1 \to 2$). By straightforward yet lengthy derivation[7] one can rewrite (2.9) for the moments of the longitudinal velocity difference called structure functions,

$$S_n(r,t) = \langle [\mathbf{r} \cdot (\mathbf{v}_1 - \mathbf{v}_2)]^n / r^n \rangle \ .$$

It gives the so-called Kármán-Howarth relation

$$\frac{\partial S_2}{\partial t} = -\frac{1}{3r^4}\frac{\partial}{\partial r}\Big(r^4 S_3\Big) - \frac{4\epsilon}{3} + \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial S_2}{\partial r}\right) \ . \quad (2.10)$$

The average quantity S_2 changes only together with a large-scale motion so

$$\frac{\partial S_2}{\partial t} \simeq \frac{S_2 u}{L} \ll \frac{S_3}{r}$$

at $r \ll L$. On the other hand, we consider $r \gg l$, or more formally we consider finite r and take the limit $\nu \to 0$ so that the last term disappears. We *assume* now that ϵ has a finite limit at $\nu \to 0$ and obtain Kolmogorov's 4/5-law:

$$S_3(r) = -4\epsilon r/5 \ . \quad (2.11)$$

That remarkable relation tells that turbulence is irreversible since S_3 does not change sign when $t \to -t$ and $\mathbf{v} \to -\mathbf{v}$. If one screens a movie of turbulence backwards, we can tell that something is indeed wrong! That is what is called "anomaly" in modern field-theoretical language: a symmetry of the inviscid equation (here, time-reversal invariance) is broken by the viscous term even though the latter might have been expected to become negligible in the limit $\nu \to 0$.

Here the good news end. There is no analytic theory to give us other structure functions. One may *assume* following Kolmogorov (1941) that ϵ is the only quantity determining the statistics in the inertial interval, then on dimensional grounds $S_n \simeq (\epsilon r)^{n/3}$. Experiment give the power laws, $S_n(r) \propto r^{\zeta_n}$ but with the exponents ζ_n deviating from $n/3$ for $n \neq 3$. Moments of the velocity difference can be obtained from the probability density function (PDF) which describes the probability to measure the velocity difference $\delta v = u$ at the distance r: $S_n(r) = \int u^n \mathcal{P}(u,r)\,du$. Deviations of ζ_n from $n/3$ means that the PDF $\mathcal{P}(\delta v, r)$ is not scale invariant i.e. cannot be presented as $(\delta v)^{-1}$ times the dimensionless function of the single variable $\delta v/(\epsilon r)^{1/3}$. Apparently, there is more to turbulence than just cascade, and ϵ is not all one must know to predict the statistics of the velocity. We do not really understand the breakdown of scale invariance for three-dimensional turbulence yet we understand it for a simpler one-dimensional case of Burgers turbulence described in Sect. 2.3.4 below [8]. Both symmetries, one broken by pumping (scale invariance) and another by friction (time reversibility) are not restored even when $r/L \to 0$ and $l/r \to 0$.

To appreciate difficulties in turbulence theory, one can cast turbulence problem into that of quantum field theory. Consider the Navier-Stokes equation driven by a random force $\mathbf{f}$ with the Gaussian probability distribution $P(\mathbf{f})$ defined by the variance $\langle f_i(0,0) f_j(\mathbf{r},t)\rangle = D_{ij}(\mathbf{r},t)$. Then the probability of any flow $\mathbf{v}(\mathbf{r},t)$ is given by the Feynman path integral over velocities satisfying the Navier-Stokes equation with different force histories:

$$\int D\mathbf{v}D\mathbf{f}\,\delta\big(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla P/\rho - \nu\Delta\mathbf{v} - \mathbf{f}\big)P(\mathbf{f})$$
$$= \int D\mathbf{v}D\mathbf{p}\exp\big[-D_{ij}p_ip_j + ip_i\big(\partial_t v_i + v_k\nabla_k v_i + \nabla_i P/\rho - \nu\Delta v_i\big)\big] \quad (2.12)$$

Here we presented the delta-function as an integral over an extra field $\mathbf{p}$ and explicitly made Gaussian integration over the force. One can thus see that turbulence is equivalent to the field theory of two interacting fields ($\mathbf{v}$ and $\mathbf{p}$) with large Re corresponding to a strong coupling limit (for incompressible turbulence the pressure is recovered from $\mathrm{div}(\mathbf{v}\cdot\nabla)\mathbf{v} = \Delta P$). For fans of field theory, add that the convective derivative $d/dt = \partial/\partial t + (\mathbf{v}\cdot\nabla)$ can be identified as a covariant derivative in the framework of a gauge theory, here the velocity of the reference frame fixes the gauge.

2.2.2 Turbulent river and wake

With the new knowledge of turbulence as a multi-scale flow, let us now return to the large-Reynolds flows down an inclined plane and past the body.

River. Now that we know that turbulence makes the drag at large Re much larger than the viscous drag, we can understand why the behavior of real rivers is so distinct from a laminar solution from Sect. 1.4.3. At small Re, the gravity force (per unit mass) $g\alpha$ was balanced by the viscous drag $\nu v/h^2$. At large Re, the drag is v^2/h which balances $g\alpha$ so that

$$v \simeq \sqrt{\alpha g h}\ . \tag{2.13}$$

Indeed, as long as viscosity does not enter, this is the only combination with the velocity dimensionality that one can get from h and the effective gravity αg. For slow plain rivers (the inclination angle $\alpha \simeq 10^{-4}$ and the depth $h \simeq 10\,\mathrm{m}$), the new estimate (2.13) gives reasonable $v \simeq 10\,\mathrm{cm/s}$. Another way to describe the drag is to say that molecular viscosity ν is replaced by turbulent viscosity $\nu_T \simeq vh \simeq \nu Re$ and the drag is still given by viscous formula $\nu v/h^2$ but with $\nu \to \nu_T$. Intuitively, one imagines turbulent eddies transferring momentum between fluid layers.

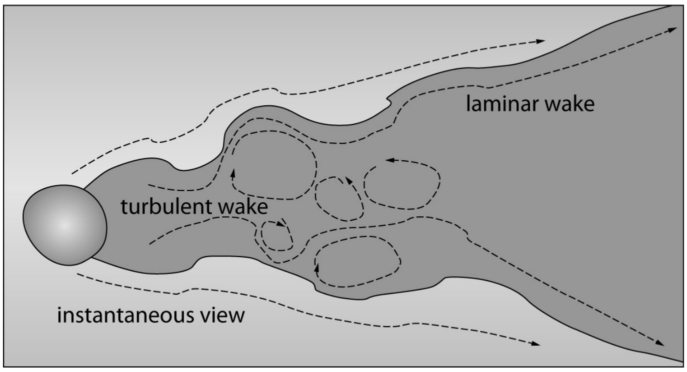


Figure 2.8 Sketch of the wake behind a body.

Wake. Let us now describe the entire wake behind a body at $Re = uL/\nu \gg 1$. Since Re is large then Kelvin's theorem holds outside the boundary layer — every streamline keeps its vorticity. Streamlines are thus divided into those of zero and nonzero vorticity. A separated region of rotational flow (wake) can exist only if streamlines don't go out of it (yet they may come in so the wake grows as one goes away from the body). Instability of the Kelvin-Helmholtz type make the boundary of the wake wavy. Oscillations then must be also present in the velocity field in the immediate outside vicinity of the wake. Still, only large-scale harmonics of turbulence are present in the outside region, because the flow is potential ($\Delta\phi = 0$) so when it changes periodically along the wake it decays exponentially with the distance from the wake boundary. The smaller the scale the faster it decays away from the wake. Therefore, all the small-scale motions and all the dissipation are inside the turbulent wake. The boundary of the turbulent wake fluctuates in time. On the snapshot sketch in Figure 2.8 the wake is dark, broken lines with arrows are streamlines, see Figure 1.16 for a real wake photo.

Let us describe the time-averaged position of the wake boundary $Y(x)$. The average angle between the streamlines and x-direction is $v(x)/u$ where $v(x)$ is the rms turbulent velocity, which can be obtained from the condition that the momentum flux through the wake must be x-independent since it is equal to the drag force $F \simeq \rho u v Y^2$ like in (1.52). Then

$$\frac{dY}{dx} = \frac{v(x)}{u} \simeq \frac{F}{\rho u^2 Y^2} ,$$

so that

$$Y(x) \simeq \left(\frac{Fx}{\rho u^2}\right)^{1/3} , \qquad v(x) \simeq \left(\frac{Fu}{\rho x^2}\right)^{1/3} .$$

One can substitute here $F \simeq \rho u^2 L^2$ and get

$$Y(x) \simeq L^{2/3} x^{1/3} , \qquad v(x) \simeq u(L/x)^{2/3} .$$

Note that Y is independent on u for a turbulent wake. Current Reynolds number, $Re(x) = v(x)Y(x)/\nu \simeq (L/x)^{1/3} uL/\nu = (L/x)^{1/3} Re$, decreases with x and a turbulent wake turns into a laminar one at $x > LRe^3 = L(uL/\nu)^3$ — the transition distance apparently depends on u.

Inside the laminar wake, under the assumption $v \ll u$ we can neglect $\rho^{-1}\partial p/\partial x \simeq v^2/x$ in the steady Navier-Stokes equation which then turns into the (parabolic) diffusion equation with x playing the role of time:

$$u\frac{\partial v_x}{\partial x} = \nu \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right) v_x . \tag{2.14}$$

At $x \gg \nu/u$, the solution of this equation acquires the universal form

$$v_x(x,y,z) = -\frac{F_x}{4\pi\eta x} \exp\left[-\frac{u(z^2+y^2)}{4\nu x}\right] ,$$

where we have used (1.52) in deriving the coefficient. A prudent thing to ask now is why we accounted for the viscosity in (2.14) but not in the stress tensor (1.50). The answer is that $\sigma_{xx} \propto \partial v_x/\partial x \propto 1/x^2$ decays fast while $\int dy \sigma_{yx} = \int dy \partial v_x/\partial y$ vanishes identically.

We see that the laminar wake width is $Y \simeq \sqrt{\nu x/u}$ that is the wake is parabolic. The Reynolds number further decreases in the wake by the law $v_x Y/\nu \propto x^{-1/2}$. Recall that in the Stokes flow $v \propto 1/r$ only for $r < \nu/u$, while in the wake $v_x \propto 1/x$ ad infinitum. Comparing laminar and turbulent estimates, we see that for $x \ll LRe^3$, the turbulent estimate gives a larger width: $Y \simeq L^{2/3}x^{1/3} \gg (\nu x/u)^{1/2}$. On the other hand, in a turbulent wake the width grows and the velocity perturbation decreases with the distance slower than in a laminar wake.

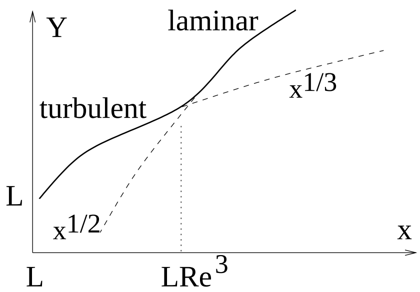


Figure 2.9 Wake width Y versus distance from the body x.

2.3 Acoustics

2.3.1 Sound

Small perturbations of density in an ideal fluid propagate as sound waves that are described by the continuity and Euler equations linearized with respect to the perturbations $p' \ll p_0$, $\rho' \ll \rho_0$:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \mathrm{div}\, \mathbf{v} = 0\,, \qquad \frac{\partial \mathbf{v}}{\partial t} + \frac{\nabla p'}{\rho_0} = 0\ . \tag{2.15}$$

To close the system we need to relate the variations of the pressure and density i.e. specify the equation of state. If we denote the derivative of the pressure with respect to the density as c^2 then $p' = c^2\rho'$. Small oscillations are potential so we introduce $\mathbf{v} = \nabla\phi$ and get from (2.15)

$$\phi_{tt} - c^2\Delta\phi = 0\ . \tag{2.16}$$

We see that indeed c is the velocity of sound. What is left to establish is what kind of the derivative $\partial p/\partial\rho$ one uses, isothermal or adiabatic. For a gas, isothermal derivative gives $c^2 = P/\rho$ while the adiabatic law

$P \propto \rho^\gamma$ gives:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} \,. \tag{2.17}$$

One uses an adiabatic equation of state when one can neglect the heat exchange between compressed (warmer) and expanded (colder) regions. That means that the thermal diffusivity (estimated as thermal velocity times the mean free path) must be less than the sound velocity times the wavelength. Since the sound velocity is of the order of the thermal velocity, it requires the wavelength to be longer than the mean free path, which is always so. Newton already knew that $c^2 = \partial p/\partial \rho$. Experimental data from Boyle showed $p \propto \rho$ (i.e. they were isothermal) which suggested for air $c^2 = p/\rho \simeq 290$ m/s, well off the observed value 340 m/s at 20 C. Only hundred years later Laplace got the true (adiabatic) value with $\gamma = 7/5$.

All velocity components, pressure and density perturbations also satisfy the *wave equation* (2.16). A particular solution of this equation is a monochromatic plane wave, $\phi(\mathbf{r}, t) = \cos(i\mathbf{kr} - i\omega t)$. The relation between the frequency ω and the wavevector $\mathbf{k}$ is called dispersion relation; for acoustic waves it is linear: $\omega = ck$. In one dimension, the general solution of the wave equation is particularly simple:

$$\phi(x, t) = f_1(x - ct) + f_2(x + ct) \,,$$

where f_1, f_2 are given by two initial conditions, for instance, $\phi(x, 0)$ and $\phi_t(x, 0)$. Note that only $v_x = \partial\phi/\partial x$ is nonzero so that sound waves in fluids are longitudinal. Any localized 1d initial perturbation (of density, pressure or velocity along x) thus breaks into two plane wave packets moving in opposite directions without changing their shape. In every such packet, $\partial/\partial t = \pm c\partial/\partial x$ so that the second equation (2.15) gives $v = p'/\rho c = c\rho'/\rho$. The wave amplitude is small when $\rho' \ll \rho$ which requires $v \ll c$. The (fast) pressure variation in a sound wave, $p' \simeq \rho v c$, is much larger than the (slow) variation $\rho v^2/2$ one estimates from the Bernoulli theorem.

Luckily, one can also find the general solution in the spherically symmetric case since the equation

$$\phi_{tt} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) \tag{2.18}$$

turns into $h_{tt} = c^2 \partial^2 h/\partial r^2$ by the substitution $\phi = h/r$. Therefore, the general solution of (2.18) is

$$\phi(r,t) = r^{-1}[f_1(r-ct) + f_2(r+ct)] \ .$$

The energy density of sound waves can be obtained by expanding $\rho E + \rho v^2/2$ up to the second-order terms in perturbations. We neglect the zero-order term $\rho_0 E_0$ because it is constant and the first-order term $\rho' \partial(\rho E)/\partial \rho = w_0 \rho'$ because it is related to the mass change in a given unit volume and disappears after the integration over the whole volume. We are left with the quadratic terms:

$$E_w = \frac{\rho_0 v^2}{2} + \frac{\rho'^2}{2}\frac{\partial^2(\rho E)}{\partial \rho^2} = \frac{\rho_0 v^2}{2} + \frac{\rho'^2}{2}\left(\frac{\partial w_0}{\partial \rho}\right)_s = \frac{\rho_0 v^2}{2} + \frac{\rho'^2 c^2}{2\rho_0} \ .$$

The energy flux with the same accuracy is

$$\mathbf{q} = \rho \mathbf{v}(w + v^2/2) \approx \rho \mathbf{v} w = w' \rho_0 \mathbf{v} + w_0 \rho' \mathbf{v} \ .$$

Again we disregard $w_0 \rho' \mathbf{v}$ which corresponds to $w_0 \rho'$ in the energy and disappears after the integration over the whole volume. The enthalpy variation is $w' = p'(\partial w/\partial p)_s = p'/\rho \approx p'/\rho_0$ and we obtain

$$\mathbf{q} = p' \mathbf{v} \ .$$

The energy and the flux are related by $\partial E_w/\partial t + div\, p'\mathbf{v} = 0$. In a plane wave, $E_w = \rho_0 v^2$ and $q = cE_w$. The energy flux is also called acoustic intensity. To amplify weak sounds and damp strong ones, our ear senses loudness as the logarithm of the intensity for a given frequency. This is why the acoustic intensity is traditionally measured not in watts per square meter but in the units of the intensity logarithm called decibels: $q(\mathrm{dB}) = 120 + 10 \log_{10} q(\mathrm{W/m^2})$.

The momentum density is

$$\mathbf{j} = \rho \mathbf{v} = \rho_0 \mathbf{v} + \rho' \mathbf{v} = \rho_0 \mathbf{v} + \mathbf{q}/c^2 \ .$$

Acoustic perturbation that exists in a finite volume not restricted by walls has a nonzero total momentum $\int \mathbf{q}\, dV/c^2$, which corresponds to the mass transfer. Comment briefly on the momentum of a phonon in solids, which is defined as a sinusoidal perturbation of atom displacements. Monochromatic wave in these (Lagrangian) coordinates has zero momentum [9]. A perturbation, which is sinusoidal in Eulerian coordinates, has a nonzero momentum at second order (where Eulerian and Lagrangian differ). Indeed, let us consider the Eulerian velocity field as a monochromatic wave with a given frequency and a wavenumber:

$v(x,t) = v_0 \sin(kx - \omega t)$. The Lagrangian coordinate $X(t)$ of a fluid particle satisfies the following equation:

$$\dot{X} = v(X,t) = u \sin(kX - \omega t)\ . \tag{2.19}$$

This is a nonlinear equation, which can be solved by iterations, $X(t) = X_0 + X_1(t) + X_2(t)$ assuming $v \ll \omega/k$. The assumption that the fluid velocity is much smaller than the wave phase velocity is equivalent to the assumption that the fluid particle displacement during the wave period is much smaller than the wavelength. Such iterative solution gives oscillations at first order and a mean drift at second order:

$$X_1(t) = \frac{u}{\omega}\cos\bigl(kX_0 - \omega t\bigr)\,,$$
$$X_2(t) = \frac{ku^2 t}{2\omega} + \frac{ku^2}{2\omega^2}\sin 2\bigl(kX_0 - \omega t\bigr)\ . \tag{2.20}$$

We see that at first order in wave amplitude the perturbation propagates while at second order the fluid itself flows.

2.3.2 Riemann wave

As we have seen, an infinitesimally small one-dimensional acoustic perturbation splits into two simple waves which then propagate without changing their forms. Let us show that such purely adiabatic waves of a permanent shape are impossible for finite amplitudes (Earnshaw paradox): In the reference frame moving with the speed c one would have a steady motion with the continuity equation ρv =const= C and the Euler equation $vdv = dp/\rho$ giving $dp/d\rho = (C/\rho)^2$ i.e. $d^2p/d\rho^2 < 0$ which contradicts the second law of thermodynamics. It is thus clear that a simple plane wave must change under the action of a small factor of nonlinearity.

Consider 1d adiabatic motion of a compressible fluid with $p = p_0(\rho/\rho_0)^\gamma$. Let us look for a simple wave where one can express any two of v, p, ρ via the remaining one. This is a generalization for a nonlinear case of what we did for a linear wave. Say, we assume everything to be determined by v that is $p(v)$ and $\rho(v)$. Euler and continuity equations take the form:

$$\frac{dv}{dt} = -\frac{1}{\rho}c^2(v)\frac{d\rho}{dv}\frac{\partial v}{\partial x}\,, \qquad \frac{d\rho}{dv}\frac{dv}{dt} = -\rho\frac{\partial v}{\partial x}\,.$$

Here $c^2(v) \equiv dp/d\rho$. Excluding $d\rho/dv$ one gets

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \pm c(v)\frac{\partial v}{\partial x}\ . \tag{2.21}$$

Two signs correspond to waves propagating in the opposite directions. In a linear approximation we had $u_t + cu_x = 0$ where $c = \sqrt{\gamma p_0/\rho_0}$. Now, we find

$$c(v) = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma \frac{p_0 + \delta p}{\rho_0 + \delta\rho}}$$
$$= c\left(1 + \frac{\delta p}{2p_0} - \frac{\delta\rho}{2\rho_0}\right) = c + v\frac{\gamma - 1}{2} \ , \tag{2.22}$$

since $\delta\rho/\rho_0 = v/c$. The local sound velocity increases with the amplitude since $\gamma > 1$, that is the positive effect of the pressure increase overcomes the negative effect of the density increase.

Taking a plus sign in (2.21) we get the equation for the simple wave propagating rightwards[10]

$$\frac{\partial v}{\partial t} + \left(c + v\frac{\gamma+1}{2}\right)\frac{\partial v}{\partial x} = 0 \ . \tag{2.23}$$

This equation describes the simple fact that the higher the amplitude of the perturbation the faster it propagates, both because of higher velocity and of higher pressure gradient (J S Russel in 1885 remarked that "the sound of a cannon travels faster than the command to fire it"). That means that the fluid particle with faster velocity propagates faster and will catch up slower moving particles. Indeed, if we have the initial distribution $v(x, 0) = f(x)$ then the solution of (2.23) is given by an implicit relation

$$v(x,t) = f\left[x - \left(c + v\frac{\gamma+1}{2}\right)t\right] \ , \tag{2.24}$$

which can be useful for particular f but is not of much help in a general case. Explicit solution can be written in terms of characteristics (the lines in $x - t$ plane that correspond to constant v):

$$\left(\frac{\partial x}{\partial t}\right)_v = c + v\frac{\gamma+1}{2} \quad \Rightarrow x = x_0 + ct + \frac{\gamma+1}{2}v(x_0)t \ , \tag{2.25}$$

where $x_0 = f^{-1}(v)$. The solution (2.25) is called simple or Riemann wave.

In the variables $\xi = x - ct$ and $u = v(\gamma+1)/2$ the equation takes the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} = \frac{du}{dt} = 0$$

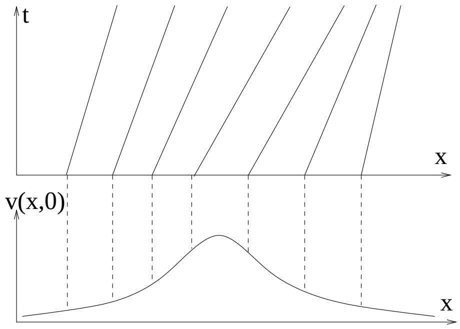


Figure 2.10 Characteristics (upper panel) and the initial velocity distribution (lower panel).

which describes freely moving particles. Indeed, we see that the characteristics are straight lines with the slopes given by the initial distribution $v(x,0)$, that is every fluid particle propagates with a constant velocity. It is seen that the parts where initially $\partial v(x,0)/\partial x$ were positive will decrease their slope while the negative slopes in $\partial v(x,0)/\partial x$ are getting steeper.

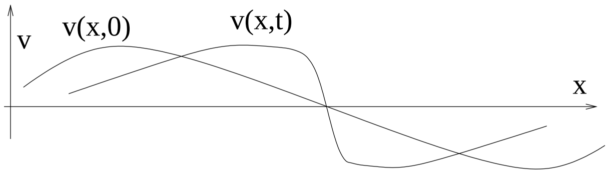


Figure 2.11 Evolution of the velocity distribution towards wave-breaking.

The characteristics are actually Lagrangian coordinates: $x(x_0,t)$. The characteristics cross in $x-t$ plane (and particles hit each other) when $(\partial x/\partial x_0)_t$ turns into zero that is

$$1+\frac{\gamma+1}{2}\frac{dv}{dx_0}t=0\,,$$

which first happens with particles that corresponds to $dv/dx_0=f'(x_0)$ maximal negative that is $f''(x_0)=0$. When characteristics cross, we have different velocities at the same point in space which corresponds to a shock.

General remark: Notice the qualitative difference between the properties of the solutions of the hyperbolic equation $u_{tt}-c^2u_{xx}=0$ and the elliptic equations, say, Laplace equation. As was mentioned in Section 1.3.1, elliptic equations have solutions and its derivatives regular everywhere inside the domain of existence. On the contrary, hyperbolic equations propagate perturbations along the characteristics and characteristics can cross (when c depends on u or x,t) leading to singularities.

2.3.3 Burgers equation

Nonlinearity makes the propagation velocity depending on the amplitude, which leads to crossing of characteristics and thus to wave breaking: any acoustic perturbation tends to create a singularity (shock) in a finite time. Account of higher spatial derivatives is necessary near a shock. In this lecture, we account for the next derivative, (the second one) which corresponds to viscosity:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} = \nu\frac{\partial^2 u}{\partial \xi^2}\ . \tag{2.26}$$

This is the Burgers equation, the first representative of the small family of universal nonlinear equations (two other equally famous members, Korteveg-de-Vries and Nonlinear Schrödinger Equations are considered in the next Chapter where we account, in particular, for the third derivative in acoustic-like perturbations). Burgers equation is a minimal model of fluid mechanics: a single scalar field $u(x,t)$ changes in one dimension under the action of inertia and friction. This equation describes wide classes of systems with hydrodynamic-type nonlinearity, $(u\nabla)u$, and viscous dissipation. It can be written in a potential form $u=\nabla\phi$ then $\phi_t = -(\nabla\phi)^2/2+\nu\Delta\phi$; in such a form it can be considered in 1 and 2 dimensions where it describes in particular the surface growth under uniform deposition and diffusion [11]: the deposition contribution into the time derivative of the surface height $\phi(r)$ is proportional to the flux per unit area, which is inversely proportional to the area: $[1+(\nabla\phi)^2]^{-1/2}\approx 1-(\nabla\phi)^2/2$, as shown in Figure 2.12.

Burgers equation can be linearized by the Hopf substitution $u = -2\nu\varphi_\xi/\varphi$:

$$\frac{\partial}{\partial\xi}\frac{\varphi_t-\nu\varphi_{\xi\xi}}{\varphi}=0\ \Rightarrow\ \varphi_t-\nu\varphi_{\xi\xi}=\varphi C'(t)\,,$$

which by the change $\varphi\to\varphi\exp C$ (not changing u) is brought to the linear diffusion equation:

$$\varphi_t-\nu\varphi_{\xi\xi}=0\ .$$

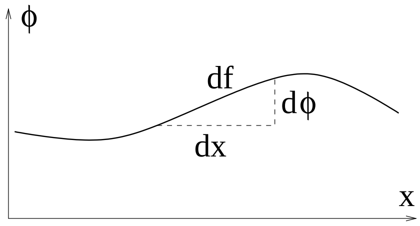


Figure 2.12 If the x-axis is along the direction of the local surface change then the local area element is $df = \sqrt{(dx)^2 + (d\phi)^2} = dx\sqrt{1 + (\nabla\phi)^2}$.

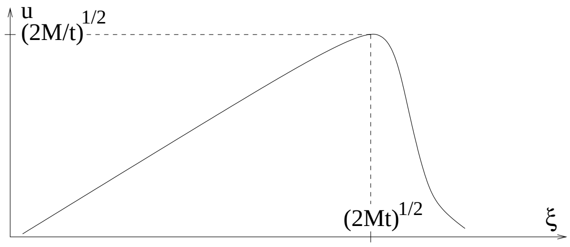
The initial value problem for the diffusion equation is solved as follows:

$$\varphi(\xi, t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \varphi(\xi', 0) \exp\left[-\frac{(\xi - \xi')^2}{4\pi\nu t}\right] d\xi' \tag{2.27}$$
$$= \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(\xi - \xi')^2}{4\pi\nu t} - \frac{1}{2\nu}\int_0^{\xi'} u(\xi'', 0)\, d\xi''\right] d\xi' \ .$$

Despite the fact that the Burgers equation describes a dissipative system, it conserves momentum (as any viscous equation does), $M = \int u(x)\, dx$. If the momentum is finite, then any perturbation evolves into a universal form depending only on M and not on the form of $u(\xi, 0)$. At $t \to \infty$, (2.27) gives $\varphi(\xi, t) \to \pi^{-1/2} F[\xi(4\nu t)^{-1/2}]$ where

$$F(y) = \int_{-\infty}^{\infty} \exp\left[-\eta^2 - \frac{1}{2\nu}\int_0^{(y-\eta)\sqrt{4\nu t}} u(\eta', 0)\, d\eta'\right] d\eta$$
$$\approx e^{-M/4\nu} \int_{-\infty}^{y} e^{-\eta^2} d\eta + e^{M/4\nu} \int_y^{\infty} e^{-\eta^2} d\eta \ . \tag{2.28}$$

Solutions with positive and negative M are related by the transform $u \to -u$ and $\xi \to -\xi$.



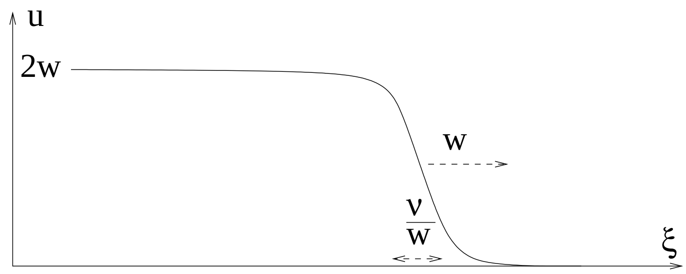
Note that M/ν is the Reynolds number and it does not change while the perturbation spreads. This is a consequence of momentum conservation in one dimension. In a free viscous decay of a d-dimensional flow, usually velocity decays as $t^{-d/2}$ while the scale grows as $t^{1/2}$ so that the Reynolds number evolves as $t^{(1-d)/2}$. For example, we have seen that the Reynolds number decreases in a wake behind the body.

When $M/\nu \gg 1$ the solution looks particularly simple, as it acquires a sawtooth form. In the interval $0 < y < M/2\nu$ (i.e. $0 < \xi < \sqrt{2Mt}$) the first integral in (2.28) is negligible and $F \sim exp(-y^2)$ so that $u(\xi, t) = \xi/t$. For both $\xi < 0$ and $\xi > \sqrt{2Mt}$ we have $F \sim const + \exp(-y^2)$ so that u is exponentially small there.

An example of the solution with an infinite momentum is a steady propagating shock. Let us look for a traveling wave solution $u(\xi - wt)$. Integrating the Burgers equation once and assuming $u \to 0$ at least at one of the infinities, we get $-uw + u^2/2 = \nu u_\xi$. Integrating again:

$$u(\xi, t) = \frac{2w}{1 + C\exp[w(\xi - wt)/\nu]} \tag{2.29}$$

We see that this is a shock having the width ν/w and propagating with the velocity which is half the velocity difference on its sides. A simple explanation is that the shock front is the place where a moving fluid particle hits a standing fluid particle, they stick together and continue with half velocity due to momentum conservation. The form of the shock front is steady since nonlinearity is balanced by viscosity.



Burgers equation is Galilean invariant, that is if $u(\xi, t)$ denotes a solution so does $u(\xi - wt) + w$ for an arbitrary w. In particular, one can transform (2.29) into a standing shock, $u(\xi, t) = w\tanh(w\xi/2\nu)$.

2.3.4 Acoustic turbulence

The shock wave (2.29) dissipates energy with the rate $\nu \int u_x^2\, dx$ independent of viscosity, see (2.30) below. In compressible flows, shock creation is a way to dissipate finite energy in the inviscid limit (in incompressible flows, that was achieved by turbulent cascade). The solution (2.29) shows how it works: velocity derivative goes to infinity as the viscosity goes to zero. In the inviscid limit, the shock is a velocity discontinuity.

Consider now acoustic turbulence produced by a pumping correlated on much larger scales, for example, pumping a pipe from one end by frequencies Ω much less than cw/ν, so that the Reynolds number is large. Upon propagation along the pipe, such turbulence evolves into a set of shocks at random positions with the mean distance between shocks $L \simeq c/\Omega$ far exceeding the shock width ν/w which is a dissipative scale. For every shock (2.29),

$$S_3(x) = \frac{1}{L}\int_{-L/2}^{L/2} [u(x+x') - u(x')]^3\, dx' \approx -8w^3 x/L\,,$$
$$\epsilon = \frac{1}{L}\int_{-L/2}^{L/2} \nu u_x^2\, dx \approx 2w^3/3L\,, \tag{2.30}$$

which gives:

$$S_3 = -12\epsilon x\ . \tag{2.31}$$

This formula is a direct analog of the flux law (2.11). As well as in Sect. 2.2.1, that would be wrong to assume $S_n \simeq (\epsilon x)^{n/3}$, since shocks give much larger contribution for $n > 1$: $S_n \simeq w^n x/L$, here x/L is the probability to find a shock in the interval x.

Generally, $S_n(x) \sim C_n|x|^n + C'_n|x|$ where the first term comes from the smooth parts of the velocity (the right x-interval in Figure 2.13) while the second comes from $O(x)$ probability to have a shock in the interval x.

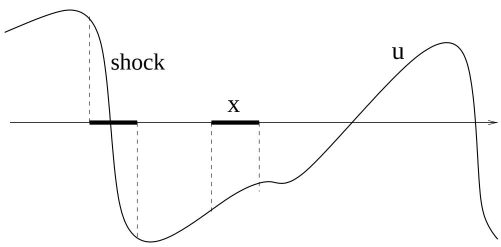


Figure 2.13 Typical velocity profile in Burgers turbulence.

The scaling exponents, $\xi_n = d\ln S_n/d\ln x$, thus behave as follows: $\xi_n = n$ for $0 \le n \le 1$ and $\xi_n = 1$ for $n > 1$. Like for incompressible (vortex) turbulence in Sect. 2.2.1, that means that the probability distribution of the velocity difference $P(\delta u, x)$ is not scale-invariant in the inertial interval, that is the function of the re-scaled velocity difference $\delta u/x^a$ cannot be made scale-independent for any a. Simple bi-modal nature of Burgers turbulence (shocks and smooth parts) means that the PDF is actually determined by two (non-universal) functions, each depending on a single argument: $P(\delta u, x) = \delta u^{-1} f_1(\delta u/x) + x f_2(\delta u/u_{rms})$. Breakdown of scale invariance means that the low-order moments decrease faster than the high-order ones as one goes to smaller scales. That means that the level of fluctuations increases with the resolution: the smaller the scale the more probable are large fluctuations. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when $x/L \to 0$.

Alternatively, one can derive the equation on the structure functions similar to (2.10):

$$\frac{\partial S_2}{\partial t} = -\frac{\partial S_3}{3\partial x} - 4\epsilon + \nu \frac{\partial^2 S_2}{\partial x^2} \ . \tag{2.32}$$

Here $\epsilon = \nu \langle u_x^2 \rangle$. Equation (2.32) describes both a free decay (then ϵ depends on t) and the case of a permanently acting pumping which generates turbulence statistically steady for scales less than the pumping length. In the first case, $\partial S_2/\partial t \simeq S_2 u/L \ll \epsilon \simeq u^3/L$ (where L is a typical distance between shocks) while in the second case $\partial S_2/\partial t = 0$. In both cases, $S_3 = -12\epsilon x + 3\nu \partial S_2/\partial x$. Consider now limit $\nu \to 0$ at fixed x (and t for decaying turbulence). Shock dissipation provides for a finite limit of ϵ at $\nu \to 0$ which gives (2.31). Again, a flux constancy fixes $S_3(x)$ which is universal that is determined solely by ϵ and depends neither on the initial statistics for decay nor on the pumping for steady turbulence. Higher moments can be related to the additional integrals of motion, $E_n = \int u^{2n}\, dx/2$, which are all conserved by the inviscid Burgers equation. Any shock dissipates the finite amount ϵ_n of E_n in the limit $\nu \to 0$ so that one can express S_{2n+1} via these dissipation rates for integer n: $S_{2n+1} \propto \epsilon_n x$ (see the exercise 2.5). That means that the statistics of velocity differences in the inertial interval depends on the infinitely many pumping-related parameters, the fluxes of all dynamical integrals of motion.

For incompressible (vortex) turbulence described in Sect. 2.2.1, we have neither understanding of structures nor classification of the conservation laws responsible for an anomalous scaling.

2.3.5 Mach number

Compressibility leads to finiteness of the propagation speed of perturbations. Here we consider the motions (of the fluid or bodies) with the velocity exceeding the sound velocity. The propagation of perturbations in more than 1 dimension is peculiar for supersonic velocities. Indeed, consider fluid moving uniformly with the velocity $\mathbf{v}$. If there is a small disturbance at some place O, it will propagate with respect to the fluid with the sound velocity c. All possible velocities of propagation in the rest frame are given by $\mathbf{v} + c\mathbf{n}$ for all possible directions of the unit vector $\mathbf{n}$. That means that in a subsonic case ($v < c$) the perturbation

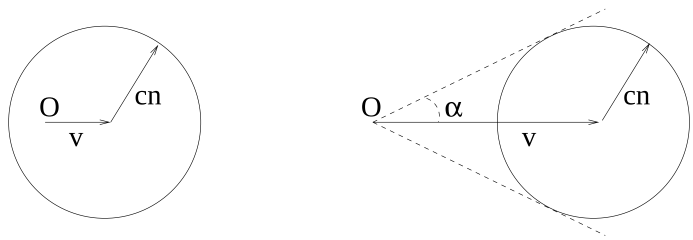


Figure 2.14 Perturbation generated at O in a fluid that moves with a subsonic (left) and supersonic (right) speed v. No perturbation can reach the outside of the Mach cone shown by broken lines.

propagates in all directions around the source O and eventually spreads to the whole fluid. This is seen from Figure 2.14 where the left circle contains O inside. However, in a supersonic case, vectors $\mathbf{v} + c\mathbf{n}$ all lie within a 2α-cone with $\sin\alpha = c/v$ called the Mach angle. Outside the Mach cone shown in Figure 2.14 by broken lines, the fluid stays undisturbed. Dimensionless ratio $v/c = \mathcal{M}$ is called the Mach number, which is a control parameter like Reynolds number, flows are similar for the same Re and $\mathcal{M}$.

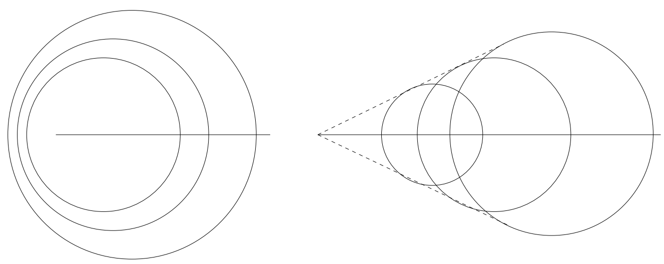


Figure 2.15 Circles are constant-phase surfaces of an acoustic perturbation generated in a fluid that moves to the right with a subsonic (left) and supersonic (right) speed. Alternatively, that may be seen as sound generated by a source moving to the left.

If sound is generated in a moving fluid (by, say, periodic pulsations), the circles in Figure 2.15 correspond to the lines of a constant phase. The

figure shows that the wavelength (the distance between the constant-phase surfaces) is smaller to the left of the source. For the case of a moving source this means that the wavelength is shorter in front of the source and longer behind it. For the case of a moving fluid that means the wavelength is shorter upwind. The frequencies registered by the observer are however different in these two cases:
i) When the emitter and receiver are at rest while the fluid moves, then the frequencies emitted and received are the same; the wavelength upwind is smaller by the factor $1-v/c$ because the propagation speed $c-v$ is smaller in a moving fluid.
ii) When the source moves towards the receiver while the fluid is still, then the propagation speed is c and the smaller wavelength corresponds to the frequency received being larger by the factor $1/(1-v/c)$. This frequency change due to a relative motion of source and receiver is called Döppler effect. This effect is used to determine experimentally the fluid velocity by scattering sound or light on particles carried by the flow [12].

When receiver moves relative to the fluid, it registers the frequency which is different from the frequency ck measured in the fluid frame. Let us find the relation between the frequency and the wavenumber of the sound propagating in a moving fluid and registered in the rest frame. The monochromatic wave is $\exp(i\mathbf{k}\cdot\mathbf{r}'-ckt)$ in a reference frame moving with the fluid. The coordinates in moving and rest frames are related as $\mathbf{r}'=\mathbf{r}-\mathbf{v}t$ so in the rest frame we have $\exp(i\mathbf{k}\cdot\mathbf{r}-ckt-\mathbf{k}\cdot\mathbf{v}t)=\exp(i\mathbf{k}\cdot\mathbf{r}-\omega_k t)$, which means that the frequency measured in the rest frame is as follows:

$$\omega_k = ck + (\mathbf{k}\cdot\mathbf{v})\ . \tag{2.33}$$

This change of the frequency, $(\mathbf{k}\cdot\mathbf{v})$, is called Döppler shift. When sound propagates upwind, one has $(\mathbf{k}\cdot\mathbf{v})<0$, so that a standing person hears a lower tone than those gone with the wind. Another way to put it is that the wave period is larger since more time is needed for a wavelength to pass our ear as the wind sweeps it.

Consider now a wave source that oscillates with the frequency ω_0 and moves with the velocity $\mathbf{u}$. The wave in a still air has the frequency $\omega=ck$. To relate ω and ω_0, pass to the reference frame moving with the source where $\omega_k=\omega_0$ and the fluid moves with $-\mathbf{u}$ so that (2.33) gives $\omega_0=ck-(\mathbf{k}\cdot\mathbf{u})=\omega[1-(u/c)\cos\theta]$, where θ is the angle between $\mathbf{u}$ and $\mathbf{k}$.

Let us now look at (2.33) for $v>c$. We see that the frequency of sound in the rest frame turns into zero on the Mach cone (also called the characteristic surface). Condition $\omega_k = 0$ defines the cone surface $ck = -\mathbf{k}\cdot\mathbf{v}$ or in any plane the relation between the components: $v^2k_x^2 = c^2(k_x^2 + k_y^2)$. The propagation of perturbation in x-y plane is determined by the constant-phase condition $k_x dx + k_y dy = 0$ and $dy/dx = -k_x/k_y = \pm c/\sqrt{v^2 - c^2}$ which again correspond to the broken straight lines in Figure 2.15 with the same Mach angle $\alpha = \arcsin(c/v) = \arctan(c/\sqrt{v^2 - c^2})$. We thus see that there is a stationary perturbation along the Mach surface, acoustic waves inside it and undisturbed fluid outside.

Let us discuss now a flow past a body in a compressible fluid. For a slender body like a wing, flow perturbation can be considered small like we did in Sect. 1.5.4, only now including density: $u + \mathbf{v}$, $\rho_0 + \rho'$, $P_0 + P'$. For small perturbations, $P' = c^2\rho'$. Linearization of the steady Euler and continuity equations gives [13]

$$\rho_0 u \frac{\partial \mathbf{v}}{\partial x} = -\nabla P' = -c^2 \nabla \rho' \,, \qquad u\frac{\partial \rho'}{\partial x} = -\rho_0 \mathrm{div}\, \mathbf{v} \ . \tag{2.34}$$

Taking curl of the Euler equation, we get $\partial\omega/\partial x = 0$ i.e. x-independent vorticity. Since vorticity is zero far upstream it is zero everywhere (in a linear approximation) [14]. We thus have a potential flow, $\mathbf{v} = \nabla\phi$, which satisfies

$$\left(1 - \mathcal{M}^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \ . \tag{2.35}$$

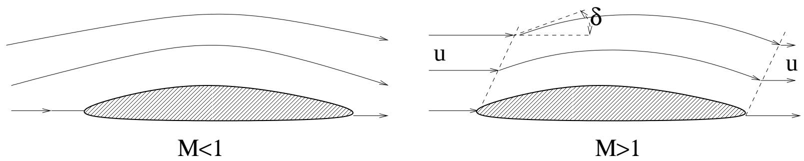


Figure 2.16 Subsonic (left) and supersonic (right) flow around a slender wing.

Here the Mach number, $\mathcal{M} = u/c$, determines whether the equation is elliptic (when $\mathcal{M} < 1$ and the streamlines are smooth) or hyperbolic (when $\mathcal{M} > 1$ and the streamlines are curved only between the Mach planes extending from the ends of the wing and straight outside).

In the elliptic case, the change of variables $x \to x(1 - \mathcal{M}^2)^{-1/2}$ turns (2.35) into Laplace equation, $\mathrm{div}\,\mathbf{v} = \Delta\phi = 0$, which we had for an incompressible case. To put it simply, at subsonic speeds, compressibility of the fluid is equivalent to a longer body. Since the lift is proportional to the velocity circulation i.e. to the wing length then we conclude that compressibility increases the lift by $(1-\mathcal{M}^2)^{-1/2}$.

In the hyperbolic case, the solution is

$$\phi = F(x - By)\ , \quad B = (\mathcal{M}^2 - 1)^{-1/2}\ .$$

The boundary condition on the body having the shape $y = f(x)$ is $v_y = \partial\phi/\partial y = uf'(x)$, which gives $F = -Uf/B$. That means that the streamlines repeat the body shape and turn straight behind the rear Mach surface (in the linear approximation). We see that passing through the Mach surface the velocity and density have a jump proportional to $f'(0)$. That means that Mach surfaces (like planes or cones described here) are actually shocks.One can relate the flow properties before and after the shock by the conservation laws of mass, energy and momentum called in this case Rankine-Hugoniot relations. That means that if w is the velocity component normal to the front then the fluxes ρw, $\rho w(W + w^2/2) = \rho w[\gamma P/(\gamma-1)\rho + w^2/2]$ and $P + \rho w^2$ must be continuous through the shock. That gives three three relations that can be solved for the pressure, velocity and density after the shock (Exercise 2.4). In particular, for a slender body when the streamlines deflect by a small angle $\delta = f'(0)$ after passing through the shock, we get the pressure change due to the velocity decrease:

$$\frac{\Delta P}{P} \propto \frac{u^2 - (u+v_x)^2 - v_y^2}{c^2} = \mathcal{M}^2\left[1 - \left(1 - \frac{\delta}{\sqrt{\mathcal{M}^2-1}}\right)^2 - \delta^2\right]$$
$$\approx \frac{2\delta\mathcal{M}^2}{\sqrt{\mathcal{M}^2-1}}\ . \tag{2.36}$$

The compressibility contribution to the drag is proportional to the pressure drop and thus the drag jumps when $\mathcal{M}$ crosses unity due to appearance of shock and the loss of acoustic energy radiated away between the Mach planes. Drag and lift singularity at $\mathcal{M} \to 1$ is sometimes referred to as "sound barrier". Apparently, our assumption on small perturbations does not work at $\mathcal{M} \to 1$. For comparison, recall that the wake contribution into the drag is proportional to ρu^2, while the shock contribution is proportional to $P\mathcal{M}^2/\sqrt{\mathcal{M}^2-1} \simeq \rho u^2/\sqrt{\mathcal{M}^2-1}$.

We see that in a linear approximation, the steady-state flow perturbation does not decay with distance. Account of nonlinearity leads to the conclusion that the shock amplitude decreases away from the body. We have learnt in Sect. 2.3.2 that the propagation speed depends on the amplitude and so must the angle α, which means that the Mach surfaces are straight only where the amplitude is small, which is usually far away from the body. Weak shocks are described by (2.28) with ξ being coordinate perpendicular to the Mach surface. Generally, Burgers equation can be used only for $\mathcal{M} - 1 \ll 1$. Indeed, according to (2.28,2.29), the front width is

$$\frac{\nu}{u - c} = \frac{\nu}{c(\mathcal{M} - 1)} \simeq \frac{l v_T}{c(\mathcal{M} - 1)} ,$$

which exceeds the mean free path l only for $\mathcal{M} - 1 \ll 1$ since the molecular thermal velocity v_T and the sound velocity c are comparable (see also Sect. 1.4.4). To be consistent in the framework of continuous media, strong shocks must be considered as discontinuities.

Exercises

2.1 i) Consider two-dimensional incompressible saddle-point flow (pure strain): $v_x = \lambda x$, $v_y = -\lambda y$ and the fluid particle with the coordinates x, y that satisfy the equations $\dot{x} = v_x$ and $\dot{y} = v_y$. Whether the vector $\mathbf{r} = (x, y)$ is stretched or contracted after some time T depends on its orientation and on T. Find which fraction of the vectors is stretched.
ii) Consider two-dimensional incompressible flow having both permanent strain λ and vorticity ω: $v_x = \lambda x + \omega y/2$, $v_y = -\lambda y - \omega x/2$. Describe the motion of the particle, $x(t), y(t)$, for different relations between λ and ω.

2.2 Consider a fluid layer between two horizontal parallel plates kept at the distance h at temperatures that differ by Θ. The fluid has kinematic viscosity ν, thermal conductivity χ (both measured in cm^2/sec) and the coefficient of thermal expansion $\beta = -\partial \ln \rho / \partial T$, such that the relative density change due to the temperature difference, $\beta\Theta$, far exceeds the change due to hydrostatic pressure difference, gh/c^2, where c is sound velocity. Find the control parameter(s) for the appearance of the convective (Rayleigh-Bénard) instability.

2.3 Consider a shock wave with the velocity w_1 normal to the front in a polytropic gas having the enthalpy

$$W = c_p T = PV \frac{\gamma}{\gamma - 1} = \frac{P}{\rho} \frac{\gamma}{\gamma - 1} = \frac{c^2}{\gamma - 1} ,$$

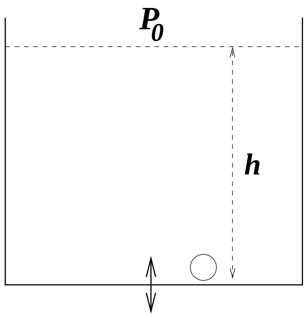
where $\gamma = c_p / c_v$. Write Rankine-Hugoniot relations for this case.

Express the ratio of densities ρ_2/ρ_1 via the pressure ratio P_2/P_1, where the subscripts $1, 2$ denote the values before/after the shock. Express P_2/P_1, ρ_2/ρ_1 and $\mathcal{M}_2 = w_2/c_2$ via the pre-shock Mach number $\mathcal{M}_1 = w_1/c_1$. Consider limits of strong and weak shocks.

2.4 For Burgers turbulence, express the fifth structure function S_5 via the dissipation rate $\epsilon_4 = 6\nu[\langle u^2 u_x^2\rangle + \langle u^2\rangle\langle u_x^2\rangle]$.

2.5 In a standing sound wave, fluid locally moves as follows: $v = a\sin(\omega t)$. Assuming small amplitude, $ka \ll \omega$, describe how a small spherical particle with the density ρ_0 moves in the fluid with the density ρ (find how the particle velocity changes with time). Consider the case when the particle material is dissolved into the fluid so that its volume decreases with the constant rate α: $V(t) = V(0) - \alpha t$. The particle is initially at rest.

2.6 There is an anecdotal evidence that early missiles suffered from an interesting malfunction of the fuel gauge. The gauge was a simple floater (small air-filled rubber balloon) whose position was supposed to signal the level of liquid fuel during the ascending stage. However, when the engine was warming up before the start, the gauge unexpectedly sank to the bottom, signalling zero level of fuel and shutting out the engine. How do engine-reduced vibrations reverse the sign of effective gravity for the floater?



Consider an air bubble in the vessel filled up to the depth h by a liquid with density ρ. The vessel vertically vibrates according to $x(t) = (Ag/\omega^2)\sin(\omega t)$, where g is the static gravity acceleration. Find the threshold amplitude A necessary to keep the bubble near the bottom. The pressure on the free surface is P_0.

2.7 It is a common experience that acoustic intensity drops fast with the distance when the sound propagates upwind. Why it is so difficult to hear somebody shouting against the wind?

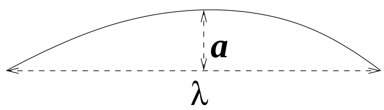
3
Dispersive Waves

In this Chapter, we consider systems that support small-amplitude waves whose speed depends on wavelength. This is in distinction from acoustic waves (or light in the vacuum) that all move with the same speed so that a small-amplitude one-dimensional perturbation propagates without changing its shape. When the speeds of different Fourier harmonics are different then the shape of a perturbation generally changes as it propagates. In particular, initially localized perturbation spreads. That is dispersion of wave speed leads to packet dispersion in space. This is why such waves are called dispersive. Since different harmonics move with different speeds, then they separate with time and can be found subsequently in different places. As a result, for quite arbitrary excitation mechanisms one often finds perturbation locally sinusoidal, the property well known to everybody who observed waves on water surface. Surface waves is the main subject of analysis in this section but the ideas and results apply equally well to numerous other dispersive waves that exist in bulk fluids, plasma and solids (where dispersion usually results from some anisotropy or inhomogeneity of the medium). We shall try to keep our description universal when we turn to consideration of nonlinear dispersive waves having finite amplitudes. We shall consider weak nonlinearity assuming amplitudes to be small and weak dispersion which is possible in two distinct cases: i) when the dispersion relation is close to acoustic and ii) when waves are excited in a narrow spectral interval. These two cases correspond respectively to the Korteveg - de Vries Equation and the Nonlinear Schrödinger Equation, which are as universal for dispersive waves as the Burgers equation for non-dispersive waves. In particular, the results of this Chapter are as applicable to nonlinear optics and quantum physics, as to fluid mechanics.

3.1 Linear waves

To have waves, one either needs inhomogeneity of fluid properties or compressibility. Here we consider an incompressible fluid with an extreme form of inhomogeneity - open surface. In this Section, we shall consider surface waves as an example of dispersive wave systems, account for gravity and surface tension as restoring forces and for viscous friction. We then introduce general notions of phase and group velocities, which are generally different for dispersive waves. We discuss physical phenomena that appear because of that difference.

Linear waves can be presented as superposition of plane waves $\exp(i\mathbf{kr}-i\omega t)$. Therefore, linear waves of every type are completely characterized by their so called dispersion relation between wave frequency ω and wavelength λ. Before doing formal derivations for surface waves, we show a simple way to estimate $\omega(\lambda)$ by using either the Newton's second law or the virial theorem which states that for small oscillations mean kinetic energy is equal to the mean potential energy.



Consider vertical oscillations of a fluid with the elevation amplitude a and the frequency ω. The fluid velocity can be estimated as ωa and the acceleration as $\omega^2 a$. When the fluid depth is much larger than the wavelength then we may assume that the fluid layer of the order λ is involved into motions. Newton's second law for a unit area then requires that the mass $\rho\lambda$ times the acceleration $\omega^2 a$ must be equal to the gravity force $\rho\lambda a g$:

$$\rho\omega^2 a\lambda \simeq \rho a g \Rightarrow \omega^2 \simeq g\lambda^{-1} \ . \tag{3.1}$$

The same result one obtains using the virial theorem. The kinetic energy per unit area of the surface can be estimated as the mass in motion $\rho\lambda$ times the velocity squared $\omega^2 a^2$. The gravitational potential energy per unit area can be estimated as the elevated mass ρa times g times the elevation a.

Curved surface has extra potential energy which is the product of the coefficient of surface tension α and the surface curvature $(a/\lambda)^2$. Taking potential energy as a sum of gravitational and capillary contributions we obtain the dispersion relation for gravitational-capillary waves on a deep water:

$$\omega^2 \simeq g\lambda^{-1} + \alpha\lambda^{-3}\rho^{-1} \ . \tag{3.2}$$

When the depth h is much smaller than λ then fluid mostly moves horizontally with the vertical velocity suppressed by the geometric factor h/λ. That makes the kinetic energy $\rho\omega^2 a^2\lambda^2 h^{-1}$ while the potential energy does not change. The virial theorem then gives the dispersion relation for waves on a shallow water:

$$\omega^2 \simeq gh\lambda^{-2} + \alpha h\lambda^{-4}\rho^{-1} \ . \tag{3.3}$$

3.1.1 Surface gravity waves

Let us now formally describe fluid motion in a surface wave. As we argued in Sect. 1.2.4, small-amplitude oscillations are irrotational flows. Then, one can introduce the velocity potential which satisfies the Laplace equation $\Delta\phi = 0$ while the pressure is

$$p = -\rho(\partial\phi/\partial t + gz + v^2/2) \approx -\rho(\partial\phi/\partial t + gz) \,,$$

neglecting quadratic terms because the amplitude is small. As in considering Kelvin-Helmholtz instability in Section. 2.1, we describe the surface form by the elevation $\zeta(x,t)$. We can include the atmospheric pressure on the surface into $\phi \to \phi + p_0 t/\rho$ which does not change the velocity field. We then have on the surface

$$\frac{\partial\zeta}{\partial t} = v_z = \frac{\partial\phi}{\partial z} \,, \quad g\zeta + \frac{\partial\phi}{\partial t} = 0 \quad \Rightarrow \quad g\frac{\partial\phi}{\partial z} + \frac{\partial^2\phi}{\partial t^2} = 0 \ . \tag{3.4}$$

The first equation here is the linearized kinematic boundary condition (2.1) which states that the vertical velocity on the surface is equal to the time derivative of the surface height. The second one is the linearized dynamic boundary condition on the pressure being constant on the surface, it can be also obtained writing the equation for the acceleration due to gravity: $dv/dt = -g\partial\zeta/\partial x$. To solve (3.4) together with $\Delta\phi = 0$, one needs the boundary condition at the bottom: $\partial\phi/\partial z = 0$ at $z = -h$. The solution of the Laplace equation periodic in one direction must be exponential in another direction:

$$\phi(x,z,t) = a\cos(kx - \omega t)\cosh[k(z+h)] \,, \tag{3.5}$$

$$\omega^2 = gk\tanh kh \ . \tag{3.6}$$

The respective velocities are

$$v_x = -ak\sin(kx - \omega t)\cosh[k(z+h)]\,, \tag{3.7}$$

$$v_z = ak\cos(kx - \omega t)\sinh[k(z+h)]\,. \tag{3.8}$$

The condition of weak nonlinearity is $\partial v/\partial t \gg v\partial v/\partial x$ which requires $\omega \gg ak^2 = kv$ or $g \gg kv^2$.

The trajectories of fluid particles can be obtained by integrating the equation $\dot{\mathbf{r}} = \mathbf{v}$ assuming small oscillations near some $\mathbf{r_0} = (x_0, z_0)$ like it was done in solving (2.19). Fluid displacement during the period can be estimated as velocity ka times $2\pi/\omega$, this is supposed to be much smaller than the wavelength $2\pi/k$. At first order in the small parameter ak^2/ω, we find

$$x = x_0 - \frac{ak}{\omega}\cos(kx_0 - \omega t)\cosh[k(z_0+h)]\,, \tag{3.9}$$
$$z = z_0 - \frac{ak}{\omega}\sin(kx_0 - \omega t)\sinh[k(z_0+h)]\,.$$

The trajectories are ellipses described by

$$\left(\frac{x - x_0}{\cosh[k(z_0+h)]}\right)^2 + \left(\frac{z - z_0}{\sinh[k(z_0+h)]}\right)^2 = \left(\frac{ak}{\omega}\right)^2\,.$$

We see that as one goes down away from the surface, the amplitude of the oscillations decreases and the ellipses get more elongated as one approaches the bottom.

At second order, one obtains the mean (Stokes) drift with the velocity $a^2k^3/2\omega$ in x-direction exactly like in (2.20).

One can distinguish two limits, depending on the ratio of water depth to the wavelength. For gravity waves on a deep water ($kh \gg 1$), fluid particles move in perfect circles whose radius exponentially decays with the depth with the rate equal to the horizontal wavenumber. This is again the property of the Laplace equation mentioned in Section 2.2.2: if solution oscillated in one direction, it decays exponentially in the other direction. That supports our assumption that for a deep fluid the layer comparable to wavelength is involved into motion, fact known to divers. The frequency, $\omega = \sqrt{gk}$, which is like formula for the period of the pendulum $T = 2\pi/\omega = 2\pi\sqrt{L/g}$. Indeed, a standing surface wave is like a pendulum made of water as seen in the bottom Figure 3.1. On a shallow water ($kh \ll 1$), the oscillations are almost one-dimensional: $v_z/v_x \propto kh$. The dispersion relation is sound-like: $\omega = \sqrt{gh}\,k$ (that formula is all one needs to answer the question in Exercise 3.1).

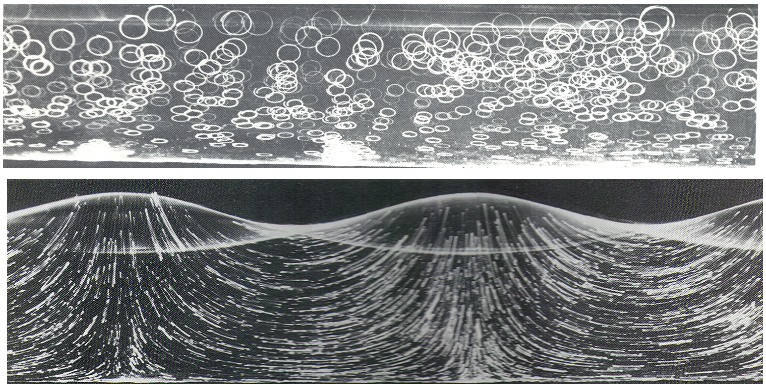


Figure 3.1 White particles suspended in the water are photographed during one period. Top figure shows a wave propagating to the right, some open loops indicate a Stokes drift to the right near the surface and compensating re-flux to the left near the bottom. Bottom figure shows a standing wave, where the particle trajectories are streamlines. In both cases, the wave amplitude is 4% and the depth is 22% of the wavelength.

3.1.2 Viscous dissipation

The moment we account for viscosity, our solution (3.5) does not satisfy the boundary condition on the free surface,

$$\sigma_{ij}n_j = \sigma'_{ij}n_j - pn_i = 0$$

(see Section 1.4.3), because both the tangential stress $\sigma_{xz} = -2\eta\phi_{xz}$ and the oscillating part of the normal stress $\sigma'_{zz} = -2\eta\phi_{zz}$ are nonzero. Note also that (3.7) gives v_x nonzero on the bottom. A true viscous solution has to be rotational but when viscosity is small, vorticity appears only in narrow boundary layers near the surface and the bottom. A standard derivation [1] of the rate of a weak decay is to calculate the viscous stresses from the solution (3.5):

$$\begin{aligned}\frac{dE}{dt} &= -\frac{\eta}{2}\int\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2 dV = -2\eta\int\left(\frac{\partial^2\phi}{\partial x_k\partial x_i}\right)^2 dV\\ &= -2\eta\int\left(\phi_{zz}^2+\phi_{xx}^2+2\phi_{xz}^2\right)dV = -8\eta\int\phi_{xz}^2\,dV\ .\end{aligned}$$

This means neglecting the narrow viscous boundary layers near the surface and the bottom (assuming that there is not much motion there i.e. $kh \gg 1$). Assuming the decay to be weak (that requires $\nu k^2 \ll \omega$ which also guarantees that the boundary layer is smaller than the wavelength) we consider the energy averaged over the period (which is twice the averaged kinetic energy for small oscillations by the virial theorem):

$$\bar{E} = \int_0^{2\pi/\omega} E\,dt\,\omega/2\pi = \rho \int \overline{v^2}\,dV = 2\rho k^2 \int \overline{\phi^2}\,dV \ .$$

The dissipation averaged over the period is related to the average energy:

$$\overline{\frac{dE}{dt}} = \int_0^{2\pi/\omega} \frac{dE}{dt}\,dt\,\omega/2\pi = -8\eta k^4 \int \overline{\phi^2}\,dV = -4\nu k^2 \bar{E} \ . \quad (3.10)$$

Still, something seems strange in this derivation. Notice that our solution (3.5) satisfies the Navier-Stokes equation (but not the boundary conditions) since the viscous term is zero for the potential flow: $\Delta \mathbf{v} = 0$. How then it can give nonzero viscous dissipation? The point is that the viscous stress σ'_{ik} is nonzero but its divergence, which contribute $\rho\partial v_i/\partial t = \ldots + \partial\sigma'_{ik}/\partial x_k$, is zero. In other words, the *net* viscous force on any fluid element is zero, but viscous forces around an element are nonzero as it deforms. These forces bring the energy dissipation $\sigma'_{ik}\partial v_i/\partial x_k$ which has a nonzero mean so that viscosity causes waves to decay. To appreciate how the bulk integration can present the boundary layer distortion, consider a function $U(x)$ on $x \in [0,1]$ which is almost linear but curves in a narrow vicinity near $x = 1$ to have $U'(1) = 0$, as shown in Fig. 3.2. Then the integral $\int_0^1 UU''dx = -\int_0^1 (U')^2 dx$ is nonzero, and the main contribution is from the bulk.

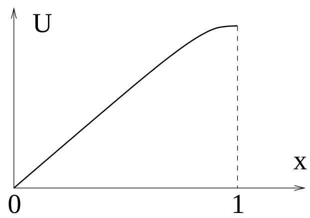


Figure 3.2 Function whose U'' is small everywhere except a small vicinity of $x = 1$ yet $\int_0^1 UU''dx = -\int_0^1 (U')^2 dx$ is due to the bulk.

However, the first derivation of the viscous dissipation (3.10) was done by Stokes in quite an ingenious way, that did not involve any boundary layers. Since the potential solution satisfies the Navier-Stokes equation, he suggested to *imagine* how one may also satisfy the boundary conditions (so that no boundary layers appear). First, we need an extensible bottom to move with $v_x(-h) = -k\sin(kx - \omega t)$. Since $\sigma'_{xz}(-h) = 0$

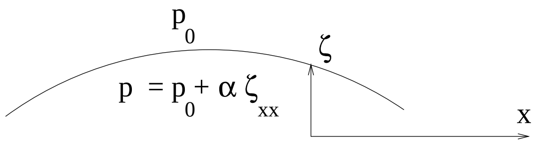
then such bottom movements do no work. In addition, we must apply extra forces to the fluid surface to compensate for $\sigma_{zz}(0)$ and $\sigma_{xz}(0)$. Such forces do work (per unit area per unit time) $v_x\sigma_{xz} + v_z\sigma_{zz} = 2\eta(\phi_x\phi_{xz} + \phi_z\phi_{zz})$. After averaging over the period of the monochromatic wave, it turns into $4\eta k^2\overline{\phi\phi_z}$, which is $8\nu k^2$ times the average kinetic energy per unit surface area, $\rho\overline{\phi\phi_z}/2$ — to obtain this, one writes

$$\rho\int(\nabla\phi)^2\,dV = \rho\int\nabla\cdot(\phi\nabla\phi)\,dV = \int\phi\phi_z\,dS\ .$$

Since we have introduced forces that make our solution steady (it satisfies equations and boundary conditions) then the work of those forces exactly equals the rate of the bulk viscous dissipation which is thus $4\nu k^2$ times the total energy [2].

3.1.3 Capillary waves

Surface tension makes for extra pressure difference proportional to the curvature of the surface:



This changes the second equation of (3.4),

$$g\zeta + \frac{\partial\phi}{\partial t} = \frac{\alpha}{\rho}\frac{\partial^2\zeta}{\partial x^2}\,, \tag{3.11}$$

and adds an extra term to the dispersion relation

$$\omega^2 = \left(gk + \alpha k^3/\rho\right)\tanh kh\ . \tag{3.12}$$

That two restoring factors enter additively into ω^2 is due to the virial theorem as explained at the beginning of Section 3.1.

We see that there is the wavenumber $k_* = \sqrt{\rho g/\alpha}$ which separates gravity-dominated long waves from short waves dominated by surface tension. In general, both gravity and surface tension provide a restoring force for the surface perturbations. For water, $\alpha \simeq 70$ erg/cm^2 and $\lambda_* = 2\pi/k_* \simeq 1.6\,$cm. Now we can answer why water spills out of an overturned glass. The question seems bizarre only to a non-physicist, since physicists usually know that atmospheric pressure at normal conditions is of order of $P_0 \simeq 10^5$ Newton per square meter which is enough to hold up to $P_0/\rho g \simeq 10$ meters of water column [3]. That is if fluid surface remained plane then the atmospheric pressure would keep the water from spilling. The relation (3.12) tells us that with a negative gravity, the plane surface is unstable:

$$\omega^2 \propto -gk + \alpha k^3 < 0 \quad \text{for} \quad k < k_* \ .$$

On the contrary, water can be kept in an upside-down capillary with the diameter smaller than λ_* since the unstable modes cannot fit in (see Exercise 3.9 for the consideration of a more general case).

3.1.4 Phase and group velocity

Let us discuss now general properties of one-dimensional propagation of linear dispersive waves. To describe the propagation, we use the Fourier representation since every harmonics $\exp(ikx - i\omega_k t)$ propagates in a simple way completely determined by the frequency-wavenumber relation ω_k. It is convenient to consider the perturbation in the Gaussian form,

$$\zeta(x,0) = \zeta_0 \exp\bigl(ik_0 x - x^2/l^2\bigr) ,$$

for which the distribution in k-space is Gaussian as well:

$$\zeta(k,0) = \int dx \zeta_0 \exp\left[i(k_0 - k)x - \frac{x^2}{l^2}\right] = \zeta_0 l \sqrt{\pi} \exp\left[-\frac{l^2}{4}(k_0 - k)^2\right] \ .$$

That distribution has the width $1/l$. Consider first quasi-monochromatic wave with $k_0 l \gg 1$ so that we can expand

$$\omega(k) = \omega_0 + (k - k_0)\omega' + (k - k_0)^2 \omega''/2 \tag{3.13}$$

and substitute it into

$$\zeta(x,t) = \int \frac{dk}{2\pi} \zeta(k,0) \exp[ikx - i\omega_k t] \tag{3.14}$$

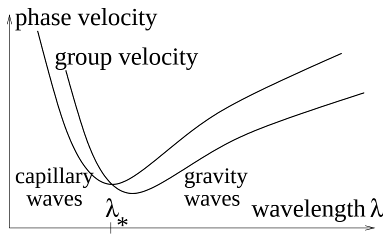
$$\approx \frac{\zeta_0 l}{2} e^{ik_0 x - i\omega_0 t} \exp\left[-\frac{(x - \omega' t)^2}{(l^2 + 2it\omega'')}\right] \left[l^2/4 + it\omega''/2)\right]^{-1/2} \ . \tag{3.15}$$

We see that the perturbation $\zeta(x,t) = \exp(ik_0 x - i\omega_0 t)\Psi(x,t)$ is a monochromatic wave with a complex envelope having the modulus

$$|\Psi(x,t)| \approx \zeta_0 \frac{l}{L(t)} \exp\left[-(x - \omega' t)^2 / L^2(t)\right] ,$$
$$L^2(t) = l^2 + (2t\omega'')^2 l^{-2} \ .$$

The phase propagates with the *phase velocity* ω_0/k_0 while the envelope (and the energy determined by $|\Psi|^2$) propagate with the *group velocity* ω'. For sound waves, $\omega_k = ck$, the group and the phase velocities are equal and are the same for all wavenumbers, the wave packet does not spread since $\omega'' = 0$. On the contrary, when $\omega'' \neq 0$, the waves are called dispersive since different harmonics move with different velocities and *disperse* in space; the wave packet spreads with time and its amplitude decreases since $L(t)$ grows. For $\omega_k \propto k^\alpha$, $\omega' = \alpha\omega_k/k$. In particular, the group velocity for gravity waves on deep water is twice less than the phase velocity so that individual crests can be seen appearing out of nowhere at the back of the packet and disappearing at the front. For capillary waves on deep water, the group velocity is 1.5 times more so that crests appear at the front.

If one considers a localized perturbation in space, which corresponds to a wide distribution in k-space, then in the integral (3.14) for given x, t the main contribution is given by the wavenumber determined by the stationary phase condition $\omega'(k) = x/t$ i.e. waves of wavenumber k are found at positions moving forward with the group velocity $\omega'(k)$ — see Exercise 3.3.



Obstacle to the stream or wave source moving with respect to water can generate a steady wave pattern if the projection of the source relative velocity V on the direction of wave propagation is equal to the phase velocity $c(k)$. For example, if the source creates an elevation of the water surface then it must stay on the wave crest, which moves with the phase velocity. The condition $V\cos\theta = c$ for generation of stationary wave pattern is a direct analog of the Landau criterium for generating excitations in superfluid and the resonance condition for Vavilov-Cherenkov radiation by particles moving faster than light in a medium. For gravity-capillary surface waves on a deep water, the requirement for the Vavilov-Cherenkov resonance means that V must exceed the minimal phase velocity, which is

$$c(k_*) = \frac{\omega(k_*)}{k_*} = \left(\frac{4\alpha g}{\rho}\right)^{1/4} \simeq 23\,cm/s\ .$$

Consider first a source which is long in the direction perpendicular to the motion, say a tree fallen across the stream. Then a one-dimensional pattern of surface waves is generated with two wavenumbers that correspond to $c(k) = V$. These two wavenumbers correspond to the same phase velocity but different group velocities. The waves transfer energy from the source with the group velocity. Since the longer (gravity) wave has its group velocity lower, $\omega' < V$, then it is found behind the source (downstream), while the shorter (capillary) wave can be found upstream[4]. Of course, capillary waves can only be generated by a really thin object (much smaller than λ_*), like a fishing line or a small tree branch.

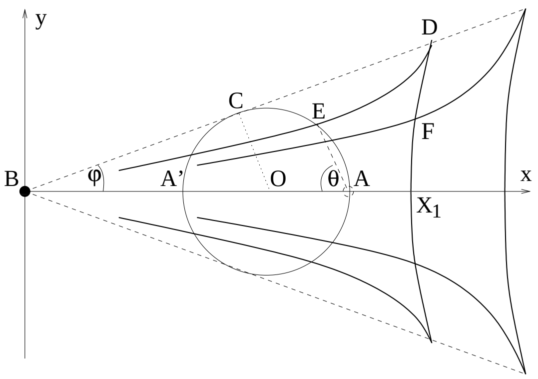


Figure 3.3 Pattern of waves generated by a ship that moves along x-axis to the left. Circle is the locus of points reached by waves generated at A when the ship arrives at B. Broken lines show the Kelvin wedge. Bold lines are wave crests.

Ships generate an interesting pattern of gravity waves that can be understood as follows. The wave generated at the angle θ to the ship motion has its wavelength determined by the condition $V\cos\theta = c(k)$, necessary for ship's bow to stay on the wave crest. That condition means that different wavelengths are generated at different angles in the interval $0 \le \theta \le \pi/2$. Similar to our consideration in Sect. 2.3.5, let us find the locus of points reached by the waves generated at A during the time it takes for the ship to reach B, see Figure 3.3. The waves propagate away from the source with the group velocity $\omega' = c(k)/2$. The fastest is the wave generated in the direction of the ship propagation ($\theta = 0$) which moves with the group velocity equal to half the ship speed and reaches A' such that $AA' = AB/2$. The wave generated at the angle θ reaches E such that $AE = AA' \cos\theta$ which means that AEA' is a right angle. We conclude that the waves generated at different angles reach the circle with the diameter AA'. Since $OB = 3OC$, all the waves generated before the ship reached B are within the *Kelvin wedge* with the angle $\varphi = \arcsin(1/3) \approx 19.5°$, compare with the Mach cone shown in Fig. 2.14. Note the remarkable fact that the angle of the Kelvin wedge is completely universal i.e. independent of the ship speed.

Figure 3.4 Wave pattern of a ship consists of the breaking wave from the bow with its turbulent wake (distinguished by a short trace of white foam) and the Kelvin pattern.

Let us describe now the form of the wave crests, which are neither straight nor parallel since they are produced by the waves emitted at different moments at different angles. The wave propagating at the angle θ makes a crest at the angle $\pi/2 - \theta$. Consider point E on a crest with the coordinates

$$x = AB(2 - \cos^2\theta)/2\,, \quad y = (AB/2)\sin\theta\cos\theta\ .$$

The crest slope must satisfy the equation $dy/dx = \cot\theta$, which gives $dAB/d\theta = -AB\tan\theta$. Solution of this equation, $AB = X_1\cos\theta$, describes how the source point A is related to the angle θ of the wave that creates the given crest. Different integration constants X_1 correspond to different crests. Crest shape is given parametrically by

$$x = X_1 \cos\theta(2 - \cos^2\theta)/2\,, \quad y = (X_1/4)\cos\theta \sin 2\theta \qquad (3.16)$$

and it is shown in Figure 3.3 by the bold solid lines, see also Figure 3.4. As expected, longer (faster) waves propagate at smaller angles. Note that for every point inside the Kelvin wedge one can find two different source points. That means that two constant-phase lines cross at every point like crossing of crests seen at the point F. There are no waves outside the Kelvin wedge whose boundary is thus a caustic [5], where every crest has a cusp like at the point D in Fig. 3.3, determined by the condition that both $x(\theta)$ and $y(\theta)$ have maxima. Differentiating (3.16) and solving $dx/d\theta = dy/d\theta = 0$ we obtain the propagation angle $\cos^2\theta_0 = 2/3$, which is actually the angle CAB since it corresponds to the wave that reached the wedge. We can express the angle COA alternatively as $\pi/2 + \phi$ and $\pi - 2\theta_0$ and relate: $\theta_0 = \pi/4 - \phi/2 \approx 35°$.

3.2 Weakly nonlinear waves

3.2.1 Hamiltonian description

The law of linear wave propagation is completely characterized by the dispersion relation ω_k. It does not matter what physical quantity oscillates in the wave (fluid velocity, density, electromagnetic field, surface elevation etc), waves with the same dispersion relation propagate in the same way. Can we achieve the same level of universality in describing nonlinear waves? As we shall see in this and the next sections, some universality classes can be distinguished but the level of universality naturally decreases as nonlinearity increases.

What else, apart from ω_k, must we know to describe weakly nonlinear waves? Since every wave is determined by two dynamic variables, amplitude and phase, then it is natural to employ the Hamiltonian formalism for waves in conservative medium. Indeed, the Hamiltonian formalism is the most general way to describe systems that satisfy the least action principle, as most closed physical systems do. The main advantage of the Hamiltonian formalism (comparing, say, to its particular case, the Lagrangian formalism) is an ability to use canonical transformations. Those transformations involve both coordinates and momenta and thus are more general than coordinate transformations one uses within the

Lagrangian formalism, which employs coordinates and their time derivatives (do not confuse Lagrangian formalism in mechanics and Lagrangian description in fluid mechanics). Canonical transformations is a powerful tool that allows one to reduce the variety of problems into a few universal problems. So let us try to understand what is a general form of the Hamiltonian of a weakly nonlinear wave system.

As we have seen in Section 1.3.4, Hamiltonian mechanics of continuous systems lives in an even-dimensional space of coordinates $q(\mathbf{r},t)$ and $\pi(\mathbf{r},t)$ that satisfy the equations

$$\frac{\partial q(\mathbf{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta \pi(\mathbf{r},t)}, \qquad \frac{\partial \pi(\mathbf{r},t)}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q(\mathbf{r},t)} .$$

Here the Hamiltonian $\mathcal{H}\{q(\mathbf{r},t),\pi(\mathbf{r},t)\}$ is a functional (simply speaking, function presents a number for every number, while functional presents a number for every function; for example, definite integral is a functional). Variational derivative $\delta/\delta f(r)$ is a generalization of the partial derivative $\partial/\partial f(r_n)$ from discrete to continuous set of variables. The variational derivative of a linear functional of the form $I\{f\} = \int \phi(r')f(r')\,dr'$ is calculated by

$$\frac{\delta I}{\delta f(r)} = \int \phi(r')\frac{\delta f(r')}{\delta f(r)}\,dr' = \int \phi(r')\delta(r-r')\,dr' = \phi(r) .$$

For this, one mentally substitute $\delta/\delta f(r)$ by $\partial/\partial f(r_n)$ and integration by summation.

For example, the Euler and continuity equations for potential flows (in particular, acoustic waves) can be written in a Hamiltonian form:

$$\frac{\partial \rho}{\partial t} = \frac{\delta \mathcal{H}}{\delta \phi}, \quad \frac{\partial \phi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho}, \tag{3.17}$$

$$\mathcal{H} = \int \rho\left[\frac{|\nabla\phi|^2}{2} + E(\rho)\right] d\mathbf{r} .$$

We shall use the canonical variables even more symmetrical than π, q, analogous to what is called creation-annihilation operators in quantum theory. In our case, they are just functions, not operators. Assuming p and q to be of the same dimensionality (which can be always achieved multiplying them by factors) we introduce

$$a = (q + i\pi)/\sqrt{2}, \qquad a^* = (q - i\pi)/\sqrt{2} .$$

Instead of two real equations for p, q we now have one complex equation

$$i\frac{\partial a(\mathbf{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\mathbf{r},t)} . \tag{3.18}$$

The equation for a^* is complex conjugated.

In the linear approximation, waves with different wavevectors do not interact. Normal canonical coordinates in the infinite space are the complex Fourier amplitudes a_k which satisfy the equation

$$\frac{\partial a_k}{\partial t} = -i\omega_k a_k \ .$$

Comparing this to (3.18) we conclude that the Hamiltonian of a linear wave system is quadratic in the amplitudes:

$$\mathcal{H}_2 = \int \omega_k |a_k|^2 \, d\mathbf{k} \ . \tag{3.19}$$

It is the energy density per unit volume. Terms of higher orders describe wave interaction due to nonlinearity, the lowest terms are cubic:

$$\mathcal{H}_3 = \int \Big[\big(V_{123} a_1^* a_2 a_3 + c.c. \big) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) + \big(U_{123} a_1^* a_2^* a_3^* + c.c. \big) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \Big] \, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \ . \tag{3.20}$$

Here c.c. means the complex conjugated terms and we use shorthand notations $a_1 = a(\mathbf{k}_1)$ etc. Delta functions express momentum conservation and appear because of space homogeneity. Indeed, (3.19) and (3.20) are respectively the Fourier representation of the integrals like

$$\int \Omega(\mathbf{r}_1 - \mathbf{r}_2) a(\mathbf{r}_1) a(\mathbf{r}_2) \, d\mathbf{r}_1 d\mathbf{r}_2$$

and

$$\int V(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1 - \mathbf{r}_3) a(\mathbf{r}_1) a(\mathbf{r}_2) a(\mathbf{r}_3) \, d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \ .$$

The Hamiltonian is real and its coefficients have obvious symmetries $U_{123} = U_{132} = U_{213}$ and $V_{123} = V_{132}$. We presume that every next term in the Hamiltonian expansion is smaller than the previous one, in particular, $\mathcal{H}_2 \gg \mathcal{H}_3$, which requires

$$\omega_k \gg V|a_k|k^d \, , U|a_k|k^d \ , \tag{3.21}$$

where d is space dimensionality (2 for surface waves and 3 for sound, for instance).

In a perverse way of people who learn quantum mechanics before fluid mechanics, we may use analogy between a, a^* and the quantum creation-annihilation operators a, a^+ and suggest that the V-term must describe the confluence $2 + 3 \to 1$ and the reverse process of decay $1 \to 2 + 3$. Similarly, the U-term must describe the creation of three waves from vacuum and the opposite process of annihilation. We shall use quantum analogies quite often in this Chapter since the quantum physics is to a large extent a wave physics. To support this quantum-mechanical interpretation and make explicit the physical meaning of different terms in the Hamiltonian, we write a general equation of motion for weakly nonlinear waves:

$$\begin{aligned}\frac{\partial a_k}{\partial t} &= -3i\int U_{k12}a_1^*a_2^*\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})\,d\mathbf{k}_1d\mathbf{k}_2 \\ &-i\int V_{k12}a_1a_2\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,d\mathbf{k}_1d\mathbf{k}_2 \\ &-2i\int V_{1k2}^*a_1a_2^*\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})\,d\mathbf{k}_1d\mathbf{k}_2-(\gamma_k+i\omega_k)a_k\,,\end{aligned} \tag{3.22}$$

where we also included the linear damping γ_k (as one always does when resonances are possible). Delta-functions in the integrals suggest that each respective term describes the interaction between three different waves. To see it explicitly, consider a particular initial condition with two waves, having respectively wavenumbers $\mathbf{k}_1,\mathbf{k}_2$, frequencies ω_1,ω_2 and finite amplitudes A_1,A_2. Then the last nonlinear term in (3.22), $-2ie^{i(\omega_2-\omega_1)t}V_{1k2}^*A_1A_2^*\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})$, provides the wave $\mathbf{k}=\mathbf{k}_1-\mathbf{k}_2$ with the periodic forcing having frequency $\omega_1-\omega_2$, and similarly other terms. The forced solution of (3.22) is then as follows:

$$\begin{aligned}a(\mathbf{k},t) &= -3ie^{i(\omega_1+\omega_2)t}\frac{U_{k12}A_1^*A_2^*\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})}{\gamma_k+i(\omega_1+\omega_2+\omega_k)} \\ &-ie^{-i(\omega_1+\omega_2)t}\frac{V_{k12}A_1A_2\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})}{\gamma_k+i(\omega_1+\omega_2-\omega_k)} \\ &-2ie^{i(\omega_2-\omega_1)t}\frac{V_{1k2}^*A_1A_2^*\delta(\mathbf{k}+\mathbf{k}_2-\mathbf{k}_1)}{\gamma_k+i(\omega_k+\omega_2-\omega_1)}\,.\end{aligned}$$

Here and below, $\omega_{1,2}=\omega(\mathbf{k}_{1,2})$. Because of (3.21) the amplitudes of the secondary waves are small except for the cases of resonances that is when the driving frequency coincides with the eigenfrequency of the wave with the respective k. The amplitude $a(\mathbf{k}_1+\mathbf{k}_2)$ is not small if $\omega(\mathbf{k}_1+\mathbf{k}_2)+\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)=0$ — this usually happens in the non-equilibrium medium where negative-frequency waves are possible. Negative energy means that excitation of the wave decreases the energy of the medium. That may be the case, for instance, when there are currents in the medium and the wave moves against the current. In such cases, already $\mathcal{H}_2$ can be different from (3.19) — see Exercise 3.4. Two other resonances require $\omega(\mathbf{k}_1+\mathbf{k}_2)=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)$ and

$\omega(\mathbf{k}_1 - \mathbf{k}_2) = \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)$ — the dispersion relations that allow for that are called dispersion relations of decay type. For example, the power dispersion relation $\omega_k \propto k^\alpha$ is of decay type if $\alpha \geq 1$ and of non-decay type (that is does not allow for the three-wave resonance) if $\alpha < 1$, see Exercise 3.5.

3.2.2 Hamiltonian normal forms

Intuitively, it is clear that non-resonant processes are unimportant for weak nonlinearity. Technically, one can use the canonical transformations to eliminate the non-resonant terms from the Hamiltonian. Because the terms that we want to eliminate are small, the transformation should be close to identical. Consider some continuous distribution of a_k. If one wants to get rid of the U-term in $\mathcal{H}_3$ one makes the following transformation

$$b_k = a_k - 3\int \frac{U_{k12}a_1^*a_2^*}{\omega_1+\omega_2+\omega_k}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})\big]\, d\mathbf{k}_1 d\mathbf{k}_2\ . \tag{3.23}$$

It is possible when the denominator does not turn into zero in the integration domain, which is the case for most media. The Hamiltonian $\mathcal{H}\{b,b^*\} = \mathcal{H}_2 + \mathcal{H}_3$ does not contain the U-term. The elimination of the V-terms is made by a similar transformation

$$b_k = a_k + \int\left[\frac{V_{k12}a_1a_2\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})}{\omega_1+\omega_2-\omega_k} + \frac{V_{1k2}^*a_1a_2^*\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})}{\omega_1-\omega_2-\omega_k}\right.$$
$$\left. + \frac{V_{2k1}^*a_1^*a_2\delta(\mathbf{k}_2-\mathbf{k}_1-\mathbf{k})}{\omega_2-\omega_1-\omega_k}\right] d\mathbf{k}_1 d\mathbf{k}_2\ , \tag{3.24}$$

possible only for non-decay dispersion relation. One can check that both transformations (3.23,3.24) are canonical that is $i\dot{b}_k = \delta\mathcal{H}\{b,b^*\}/\delta b_k^*$. The procedure described here was invented for excluding non-resonant terms in celestial mechanics and later generalized for continuous systems [6].

We may thus conclude that (3.20) is the proper Hamiltonian of interaction only when all three-wave processes are resonant. When there are no negative-energy waves but the dispersion relation is of decay type (like for capillary waves on a deep water), the proper interaction Hamiltonian contains only V-term. When the dispersion relation is of non-decay type (like for gravity waves on a deep water), all the cubic terms can be excluded and the interaction Hamiltonian must be of the fourth order in wave amplitudes. Moreover, if the dispersion relation does not allow $\omega(\mathbf{k}_1+\mathbf{k}_2)=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)$ then it does not allow $\omega(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3)$ as well. That means that when decays of the type $1\to 2+3$ are non-resonant, the four-wave decays like $1\to 2+3+4$ are non-resonant too. So the proper Hamiltonian in this case describes four-wave scattering $1+2\to 3+4$ which is always resonant:

$$\mathcal{H}_4=\int T_{1234}a_1a_2a_3^*a_4^*\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4\ . \tag{3.25}$$

3.2.3 Wave instabilities.

Wave motion can be unstable as well as steady flow. Let us show that if the dispersion relation is of decay type, then a monochromatic wave of sufficiently high amplitude is subject to a *decay instability*. Consider the medium that contains a finite-amplitude wave $A\exp[i(\mathbf{kr}-\omega_k t)]$ and two small-amplitude waves (initial perturbations) with a_1,a_2 such that $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}$. We leave only resonant terms in (3.22):

$$\dot a_1+(\gamma_1+i\omega_1)a_1+2iV_{k12}Aa_2^*\exp(-i\omega_k t)=0\,,$$
$$\dot a_2^*+(\gamma_2-i\omega_2)a_2^*+2iV_{k12}^*A^*a_1\exp(i\omega_k t)=0\,.$$

The solution can be sought of in the form

$$a_1(t)\propto\exp(\Gamma t-i\Omega_1 t)\,,\quad a_2^*(t)\propto\exp(\Gamma t+i\Omega_2 t)\ .$$

The resonance condition is $\Omega_1+\Omega_2=\omega_k$. The amplitudes of the waves will be determined by the mismatches between their frequencies and the dispersion relation, $\Omega_1-\omega_1$ and $\Omega_2-\omega_2$. The sum of the mismatches is $\Delta\omega=\omega_1+\omega_2-\omega_k$. We have to choose such Ω-s that give maximal real Γ. We expect it to be when 1 and 2 are symmetrical, so that $\Omega_1-\omega_1=\Omega_2-\omega_2=\Delta\omega/2$. We consider for simplicity $\gamma_1=\gamma_2$, then

$$\Gamma=-\gamma\pm\sqrt{4|V_{k12}A|^2-(\Delta\omega)^2/4}\ . \tag{3.26}$$

If the dispersion relation is of non-decay type, then $\Delta\omega\simeq\omega_k\gg|VA|$ and there is no instability. On the contrary, for decay dispersion relations, the resonance is possible $\Delta\omega=\omega_1+\omega_2-\omega_k=0$ so that the growth rate of instability, $\Gamma=2|V_{k12}A|-\gamma$, is positive when the amplitude is larger than the threshold: $A>\gamma/2|V_{k12}|$, i.e. when the nonlinearity overcomes dissipation. The instability first starts for those a_1,a_2 that are in resonance (i.e. $\Delta\omega=0$) and have minimal $(\gamma_1+\gamma_2)/|V_{k12}|$. In the particular case $k=0$, $\omega_0\neq 0$, decay instability is called parametric instability since it corresponds to a periodic change in some system parameter. For example, Faraday discovered that a vertical vibration of the container with a fluid leads to the parametric excitation of a standing surface wave ($k_1 = -k_2$) with half the vibration frequency (the parameter being changed periodically is the gravity acceleration). In a simple case of an oscillator it is called a parametric resonance, known to any kid on a swing who stretches and folds his legs with twice the frequency of the swing.

As with any instability, the usual question is what stops the exponential growth and the usual answer is that further nonlinearity does that. When the amplitude is not far from the threshold, those nonlinear effects can be described in the mean-field approximation as the renormalization of the linear parameters ω_k, γ_k and of the pumping $V_{k12}A$. The renormalization should be such as to put the wave system back to threshold that is to turn the renormalized Γ into zero. The frequency renormalization $\tilde{\omega}_k = \omega_k + \int T_{kk'kk'}|a_{k'}|^2\, d\mathbf{k}'$ (see the next Section) appears due to the four-wave processes and can take waves out of resonance if the set of wavevectors is discrete due to a finite box size (it is a mechanism of instability restriction for finite-dimensional systems like oscillator — swing frequency decreases with amplitude, for instance). If however the box is large enough then frequency spectrum is close to continuous and there are waves to be in resonance for any nonlinearity. In this case, the saturation of instability is due to renormalization of the damping and pumping. The renormalization (increase) of γ_k appears because of the waves of the third generation that take energy from a_1, a_2. The pumping renormalization appears because of the four-wave interaction, for example, (3.25) adds $-ia_2^* \int T_{1234} a_3 a_4 \delta(\mathbf{k} - \mathbf{k}_3 - \mathbf{k}_4)\, d\mathbf{k}_3 d\mathbf{k}_4$ to $\dot{a}_1$.

3.3 Nonlinear Schrödinger Equation (NSE)

This section is devoted to a nonlinear spectrally narrow wave packet. Consideration of a linear propagation of such packet in Sect. 3.1.4 taught us the notions of phase and group velocities. In this section, the account of nonlinearity brings equally fundamental notions of the Bogolyubov spectrum of condensate fluctuations, modulational instability, solitons, self-focusing, collapse and wave turbulence.

3.3.1 Derivation of NSE

Consider a quasi-monochromatic wave packet in an isotropic nonlinear medium. Quasi-monochromatic means spectrally narrow that is the wave amplitudes are nonzero in a narrow region Δk of $\mathbf{k}$-space around some $\mathbf{k}_0$. In this case the processes changing the number of waves (like $1 \to 2+3$ and $1 \to 2+3+4$) are non-resonant because the frequencies of all waves are close. Therefore, all the nonlinear terms can be eliminated from the interaction Hamiltonian except $\mathcal{H}_4$ and the equation of motion has the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i \int T_{k123} a_1^* a_2 a_3 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \ . \quad (3.27)$$

Consider now $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$ with $q \ll k_0$ and expand similar to (3.13)

$$\omega(k) = \omega_0 + (\mathbf{qv}) + \frac{1}{2} q_i q_j \left(\frac{\partial^2 \omega}{\partial k_i \partial k_j} \right)_0 \ ,$$

where $\mathbf{v} = \partial\omega/\partial\mathbf{k}$ at $k = k_0$. In an isotropic medium ω depends only on modulus k and

$$q_i q_j \frac{\partial^2 \omega}{\partial k_i \partial k_j} = q_i q_j \frac{\partial}{\partial k_i} \frac{k_j}{k} \frac{\partial \omega}{\partial k} = q_i q_j \left[\frac{k_i k_j \omega''}{k^2} + \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{v}{k} \right]$$
$$= q_\parallel^2 \omega'' + \frac{q_\perp^2 v}{k} \ .$$

Let us introduce the temporal envelope $a_k(t) = \exp(-i\omega_0 t)\psi(\mathbf{q}, t)$ into (3.27):

$$\left[i\frac{\partial}{\partial t} - (\mathbf{qv}) - \frac{q_\parallel^2 \omega''}{2} - \frac{q_\perp^2 v}{2k} \right] \psi_q = T \int \psi_1^* \psi_2 \psi_3 \delta(\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\, d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 \ .$$

We assumed the nonlinear term to be small, $T|a_k|^2(\Delta k)^{2d} \ll \omega_k$, and took it at $k = k_0$. This result is usually represented in the r-space for $\psi(\mathbf{r}) = \int \psi_q \exp(i\mathbf{qr})\, d\mathbf{q}$. The nonlinear term is local in r-space:

$$\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2)\psi(\mathbf{r}_3) \int d\mathbf{q} d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 \delta(\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)$$
$$\times \exp\Big[i(\mathbf{q}_1\mathbf{r}_1) - i(\mathbf{q}_2\mathbf{r}_2) - i(\mathbf{q}_3\mathbf{r}_3) + i(\mathbf{qr}) \Big]$$
$$= \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2)\psi(\mathbf{r}_3) \delta(\mathbf{r}_1-\mathbf{r})\delta(\mathbf{r}_2-\mathbf{r})\delta(\mathbf{r}_3-\mathbf{r}) = |\psi|^2\psi \ ,$$

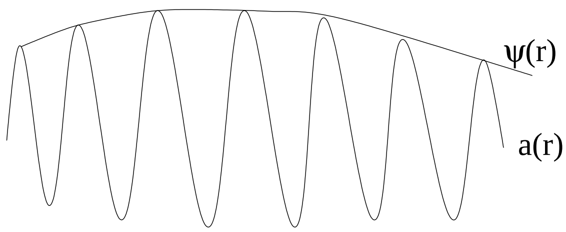
and the equation takes the form

$$\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial z} - \frac{i\omega''}{2}\frac{\partial^2\psi}{\partial z^2} - \frac{iv}{2k}\Delta_\perp\psi = -iT|\psi|^2\psi \ .$$

Here the term $v\partial_z$ is responsible for propagation with the group velocity, $\omega''\partial_{zz}$ for dispersion and $(v/k)\Delta_\perp$ for diffraction. One may ask why in the expansion of $\omega_{\mathbf{k}+\mathbf{q}}$ we kept the terms both linear and quadratic in small q. This is because the linear term (that gives $\partial\psi/\partial z$ in the last equation) can be eliminated by the transition to the moving reference frame $z \to z - vt$. We also renormalize the transversal coordinate by the factor $\sqrt{k_0\omega''/v}$ and obtain the celebrated Nonlinear Schrödinger Equation

$$i\frac{\partial\psi}{\partial t} + \frac{\omega''}{2}\Delta\psi - T|\psi|^2\psi = 0 \ . \tag{3.28}$$

Sometimes (particularly for $T < 0$) it is called Gross-Pitaevsky equation who derived it to describe a quantum condensate. This equation is meaningful at different dimensionalities. It may describe the evolution of three-dimensional packet like in Bose-Einstein condensation of cold atoms. When $\mathbf{r}$ is two-dimensional, it may correspond either to the evolution of the packet in 2d medium (say, for surface waves) or to steady propagation in 3d described by $iv\psi_z + (v/2k)\Delta_\perp\psi = T|\psi|^2\psi$, which turns into (3.28) upon re-labelling $z \to vt$. In a steady case one neglects ψ_{zz} since this term is much less than ψ_z. In a non-steady case, this is not necessary so since ∂_t and $v\partial_z$ might be about to annihilate each other and one is interested in the next terms. And, finishing with dimensionalities, one-dimensional NSE corresponds to a stationary two-dimensional case.



Different media provide for different signs of the coefficients. Apart from hydrodynamic applications, NSE also describes nonlinear optics. Indeed, Maxwell equation for waves takes the form $[\omega^2 - (c^2/n)\Delta]E = 0$. Refraction index depends on the wave intensity: $n = 1 + 2\alpha|E|^2$. There are different reasons for that dependence (and so different signs of α may be realized in different materials), for example: electrostriction, heating and Kerr effect (orientation of non-isotropic molecules by the wave field). We consider waves moving mainly in one direction and pass into the reference frame moving with the velocity c i.e. change $\omega \to \omega - ck$. Expanding

$$ck/\sqrt{n} \approx ck_z(1 - \alpha|E|^2) + ck_\perp^2/2k\,,$$

substituting it into

$$(\omega - ck - ck/\sqrt{n})(\omega - ck + ck/\sqrt{n})E = 0\,,$$

retaining only the first non-vanishing terms in diffraction and nonlinearity we obtain NSE after the inverse Fourier transform. In particular, one-dimensional NSE describes light in optical fibers.

3.3.2 Modulational instability

The simplest effect of the four-wave scattering is the frequency renormalization. Indeed, NSE has a stationary solution as a plane wave with a renormalized frequency $\psi_0(t) = A_0 \exp(-iTA_0^2 t)$ (in quantum physics, this state, coherent across the whole system, corresponds to Bose-Einstein condensate). Let us describe small perturbations of the condensate. We write the perturbed solution as $\psi = Ae^{i\varphi}$ and assume the perturbation one-dimensional (along the direction which we denote ξ). Then,

$$\psi_\xi = (A_\xi + iA\varphi_\xi)e^{i\varphi}\,, \quad \psi_{\xi\xi} = (A_{\xi\xi} + 2iA_\xi\varphi_\xi + iA\varphi_{\xi\xi} - A\varphi_\xi^2)e^{i\varphi}\ .$$

Introduce the current wavenumber $K = \varphi_\xi$. The real and imaginary parts of the linearized NSE take the form

$$\tilde{A}_t + \frac{\omega''}{2}A_0K_\xi = 0\,, \quad K_t = -2TA_0\tilde{A}_\xi + \frac{\omega''}{2A_0}\tilde{A}_{\xi\xi\xi}\ .$$

We look for the solution in the form where both the amplitude and the phase of the perturbation are modulated:

$$\tilde{A} = A - A_0 \propto \exp(ik\xi - i\Omega t)\,, \quad K \propto \exp(ik\xi - i\Omega t)\ .$$

The dispersion relation for the perturbations then takes the form:

$$\Omega^2 = T\omega'' A_0^2 k^2 + \omega''^2 k^4/4\ . \tag{3.29}$$

When $T\omega'' > 0$, it is called Bogolyubov formula for the spectrum of condensate perturbations. We have an instability when $T\omega'' < 0$ (Lighthill criterium). We first explain this criterium using the language of classical waves and at the end of the Section we give an alternative explanation in terms of quantum (quasi)-particles. Classically, define the frequency as minus the time derivative of the phase: $\varphi_t = -\omega$. For a nonlinear wave, the frequency is generally dependent on both the amplitude and the wave number. Factors ω'' and T are the second derivatives of the frequency with respect to the amplitude and the wave number respectively. Intuitively, one can explain the modulational instability in the following way: Consider, for instance, $\omega'' > 0$ and $T < 0$. If the amplitude acquires a local minimum as a result of perturbation then the frequency has a maximum there because $T < 0$. The time derivative of the current wavenumber is as follows: $K_t = \varphi_{\xi t} = -\omega_\xi$. The local maximum in ω means that K_t changes sign that is K will grow to the right of the omega maximum and decrease to the left of it. The group velocity ω' grows with K since $\omega'' > 0$. Then the group velocity grows to the right and decreases to the left so the parts separate (as arrows show) and the perturbation deepens as shown in the Figure 3.5.

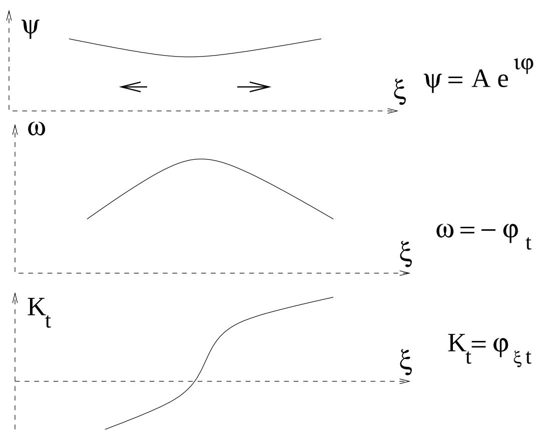


Figure 3.5 Space dependencies of the wave amplitude, frequency and the time derivative of the wavenumber, which demonstrate the mechanism of the modulational instability for $\omega'' > 0$ and $T < 0$.

The result of this instability you all can see on the beach where waves coming to the shore are modulated. Indeed, for long water waves $\omega_k \propto \sqrt{k}$ so that $\omega'' < 0$. As opposite to pendulum and somewhat counterintuitive, the frequency grows with the amplitude and $T > 0$; it is related to the change of wave shape from sinusoidal to that forming a sharpened crest which reaches 120° for sufficiently high amplitudes.

Long water wave is thus unstable with respect to longitudinal modulations (Benjamin-Feir instability, 1967). The growth rate is maximal for $k = A_0\sqrt{-2T/\omega''}$ which depends on the amplitude. Still, folklore keeps it that approximately every ninth wave is the largest.

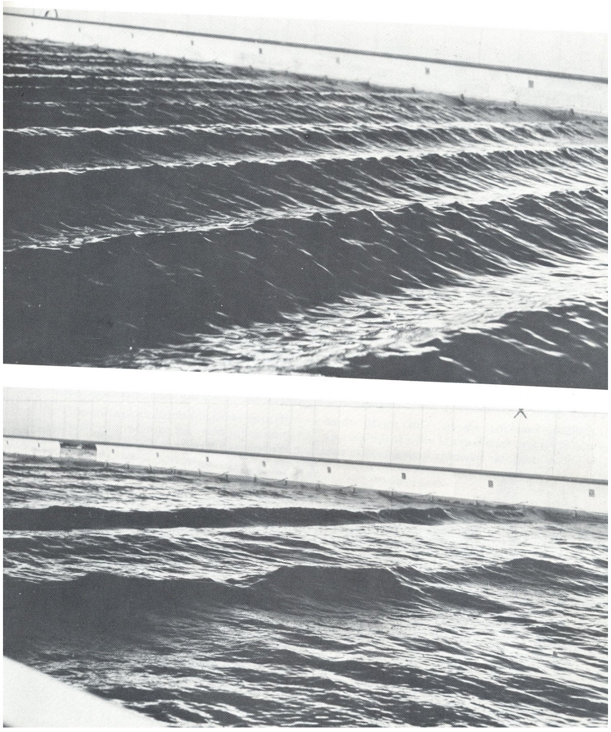


Figure 3.6 Disintegration of the periodic wave due to modulational instability as demonstrated experimentally by Benjamin and Feir (1967). The upper photograph shows a regular wave pattern close to a wavemaker. The lower photograph is made some 60 meters (28 wavelengths) away, where the wave amplitude is comparable, but spatial periodicity is lost. The instability was triggered by imposing on the periodic motion of the wavemaker a slight modulation at the unstable side-band frequency; the same disintegration occurs naturally over longer distances.

For transverse propagation of perturbations, one has to replace ω'' by v/k which is generally positive so the criterium of instability is $T < 0$ or $\partial\omega/\partial|a|^2 < 0$, which also means that for instability the wave velocity has to decrease with the amplitude. That can be easily visualized: if the wave is transversely modulated then the parts of the front where amplitude is larger will move slower and further increase the amplitude because of focusing from neighboring parts, as shown in Figure 3.7

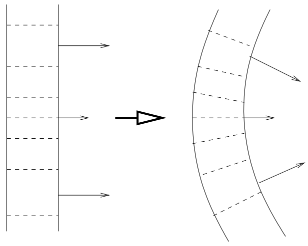


Figure 3.7 Transverse instability for the velocity decreasing with the amplitude.

Let us now give a quantum explanation for the modulational instability. Remind that NSE (3.28) is a Hamiltonian system ($i\psi_t = \delta H/\delta\psi^*$) with

$$H = \frac{1}{2}\int\left(\omega''|\nabla\psi|^2 + T|\psi|^4\right)d\mathbf{r}\ . \tag{3.30}$$

The Lighthill criterium means that the modulational instability happens when the Hamiltonian is not sign-definite. The overall sign of the Hamiltonian is unimportant as one can always change $H \to -H$, $t \to -t$, it is important whether the Hamiltonian can have different signs for different configuration of $\psi(\mathbf{r})$. Consider $\omega'' > 0$. Using the quantum language one can interpret the first term in the Hamiltonian as the kinetic energy of (quasi)-particles and the second term as their potential energy. For $T < 0$, the interaction is attractive, which leads to the instability: For the condensate, the kinetic energy (or pressure) is balanced by the interaction; local perturbation with more particles (higher $|\psi|^2$) will make interaction stronger which leads to the contraction of perturbation and further growth of $|\psi|^2$.

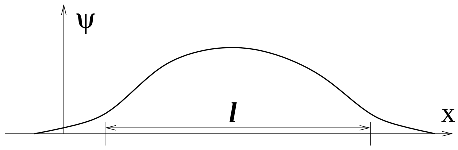
3.3.3 Soliton, collapse and turbulence

The outcome of the modulational instability depends on space dimensionality. Breakdown of a homogeneous state may lead all the way to small-scale fragmentation or creation of singularities. Alternatively, stable finite-size objects may appear as an outcome of instability. As it often happens, the analysis of conservation laws helps to understand the destination of a complicated process. Since the equation NSE (3.28) describes wave propagation and four-wave scattering, apart from the Hamiltonian (3.30), it conserves also the wave action $N = \int |\psi|^2 \, d\mathbf{r}$ which one may call the number of waves. The conservation follows from the continuity equation

$$2i\partial_t |\psi|^2 = \omega'' \nabla(\psi^* \nabla \psi - \psi \nabla \psi^*) \equiv -2\mathrm{div}\, \mathbf{J} \ . \tag{3.31}$$

Note also conservation of the momentum or total current, $\int \mathbf{J} \, d\mathbf{r}$, which does not play any role in this Section but is important for Exercise 3.7.

Consider a wave packet characterized by the generally time-dependent size l and the constant value of N.



Since one can estimate the typical value of the envelope in the packet as $|\psi|^2 \simeq N/l^d$, then $H \simeq \omega'' N l^{-2} + T N^2 l^{-d}$ — remind that the second term is negative here. We consider the conservative system so the total energy is conserved yet we expect the radiation from the wave packet to bring it to the minimum of energy. In the process of weak radiation, wave action is conserved since it is an adiabatic invariant. This is particularly clear for a quantum system like a cloud of cold atoms where N is their number. Whether this minimum corresponds to $l \to 0$ (which is called self-focusing or collapse) is determined by the balance between $|\nabla\psi|^2$ and $|\psi|^4$. The Hamiltonian H as a function of l in three different dimensionalities is shown in the Figure 3.8.

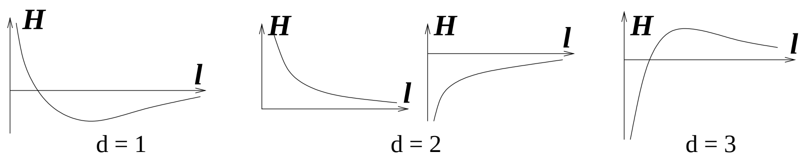


Figure 3.8 The Hamiltonian H as a function of the packet size l under the fixed N.

i) $d = 1$. At small l repulsion dominates with $H \simeq \omega'' N l^{-2}$ while attraction dominates at large l with $H \simeq -TN^2 l^{-1}$. It is thus clear that a stationary solution must exist with $l \sim \omega''/TN$ which minimizes the energy. Physically, the pressure of the waves balances attraction force. Such a stationary solution is called *soliton*, short for solitary wave. It is a traveling-wave solution of (3.28) with the amplitude function just moving, $A(x,t) = A(x-ut)$, and the phase having both a space-dependent traveling part and a uniform nonlinear part linearly growing with time: $\varphi(x,t) = f(x-ut) - TA_0^2 t$. Here A_0 and u are soliton parameters. We substitute travel solution into (3.28) and separate real and imaginary parts:

$$A'' = \frac{2T}{\omega''}\left(A^3 - A_0^2 A\right) + Af'\left(f' - \frac{2u}{\omega''}\right), \quad \omega''\left(A'f' + \frac{Af''}{2}\right) = uA'. \tag{3.32}$$

For the simple case of the standing wave ($u = 0$) the second equation gives $f = \text{const}$ which can be put zero. The first equation can be considered as a Newton equation $A'' = -dU/dA$ for the particle with the coordinate A in the potential $U(A) = -(T/2\omega'')(A^4 - 2A^2A_0^2)$ and the space coordinate x playing particle's time. The soliton is a separatrix that is a solution which requires for particle an infinite time to reach zero, or in original terms where $A \to 0$ as $x \to \pm\infty$. The upper part of Figure 3.9 presumes $T/\omega'' < 0$ that is the case of modulational instability. Mention in passing that the separatrix exists also for $T/\omega'' > 0$ but in this case the running wave is a kink that is a transition between two different values of the stable condensate (the lower part of the Figure). Kink is seen as a dip in intensity $|\psi|^2$.

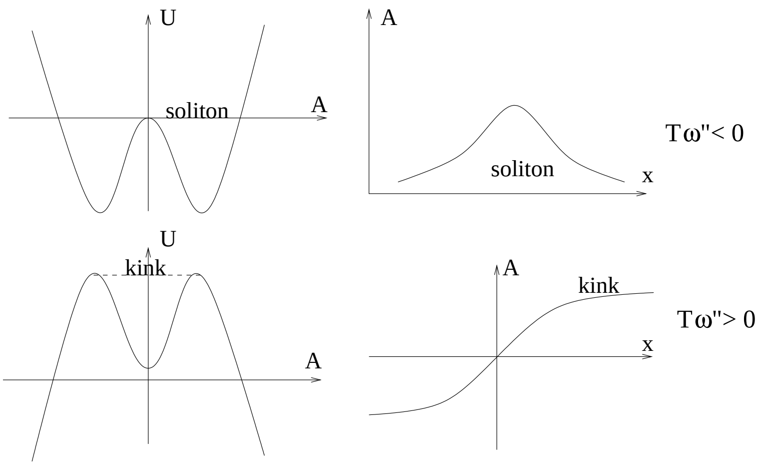


Figure 3.9 Energy as a function of the amplitude of a running wave and the profile of the wave. Upper part corresponds to the case of an unstable condensate where a steady solution is a soliton, lower part to a stable condensate where it is a kink.

Considering a general case of a traveling soliton (at $T\omega'' < 0$), one can multiply the second equation by A and then integrate: $\omega'' A^2 f' = u(A^2 - A_0^2)$ where by choosing the constant of integration we defined A_0 as A at the point where $f' = 0$. We can now substitute f' into the first equation and get the closed equation for A. The soliton solution has the form:

$$\psi(x,t) = \sqrt{2}A_0 \cosh^{-1}\left[\left(\frac{-2T}{\omega''}\right)^{1/2} A_0(x-ut)\right] e^{i(2x-ut)u/2\omega'' - iTA_0^2 t} .$$

Note that the Galilean transformation for the solutions of the NSE looks as follows: $\psi(x,t) \to \psi(x-ut,t)\exp[iu(2x-ut)/2\omega'']$. In the original variable $a(\mathbf{r})$, our envelope solitons look as shown in Figure 3.10.

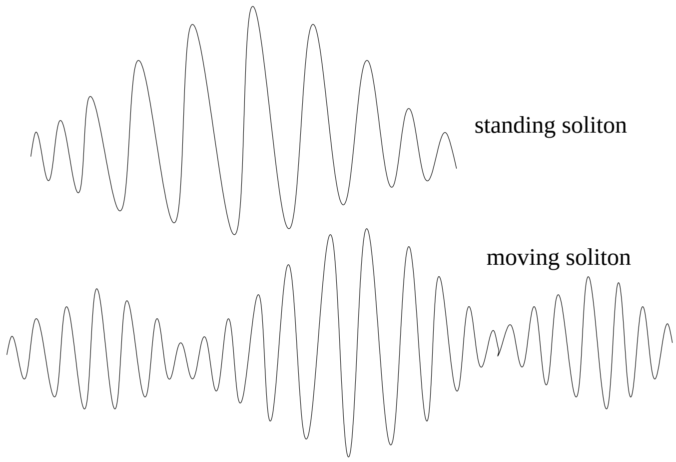


Figure 3.10 Standing and traveling solitons of the envelope of an almost monochromatic wave.

ii) $d = 2, 3$. When the condensate is stable, there exist stable solitons analogous to kinks which are localized minima in the condensate intensity. In optics they can be seen as grey and dark filaments in a laser beam propagating through a nonlinear medium. The wave (condensate) amplitude turns into zero in a dark filament which means that it is a vortex i.e. a singularity of the wave phase, see Exercise 3.6.

When the condensate is unstable, there are no stable stationary solutions for $d = 2, 3$. From the dependence $H(l)$ shown in Figure 3.8 we expect that the character of evolution is completely determined by the sign of the Hamiltonian at $d = 2$: the wave packets with positive Hamiltonian spread because the wave dispersion (kinetic energy or pressure, in other words) dominates while the wave packets with negative

Hamiltonian shrink and collapse. Let us stress that the way of arguing based on the dependence $H(l)$ is non-rigorous and suggestive at best. A rigorous proof of the fact that the Hamiltonian sign determines whether the wave packet spreads or collapses in 2d is called Talanov's theorem which is the expression for the second time derivative of the packet size squared, $l^2(t) = \int |\psi|^2 r^2 \, d\mathbf{r}$. To obtain that expression, differentiate over time using (3.31) then integrate by parts then differentiate again:

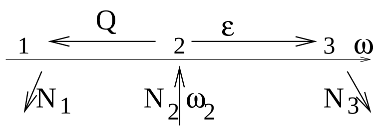
$$\begin{aligned}\frac{d^2 l^2}{dt^2} &= \frac{i\omega''}{2}\partial_t \int r^2 \nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) d\mathbf{r} \\ &= i\omega''\partial_t \int r_\alpha\left(\psi^*\nabla_\alpha\psi - \psi\nabla_\alpha\psi^*\right) d\mathbf{r} = 2\omega''^2 \int |\nabla\psi|^2 \, d\mathbf{r} \\ &+ d\omega'' T \int |\psi|^4 \, d\mathbf{r} = 4H + 2(d-2)\omega'' T \int |\psi|^4 \, d\mathbf{r} \, .\end{aligned}$$

Consider an unstable case with $T\omega'' < 0$. We see that indeed for $d \geq 2$ one has an inequality $\partial_{tt} l^2 \leq 4\omega'' H$ so that

$$l^2(t) \leq 2\omega'' H t^2 + C_1 t + C_2$$

and for $\omega'' H < 0$ the packet shrinks to singularity in a finite time (this is the singularity in the framework of NSE which is itself valid only for the scales much larger than the wavelength of the carrier wave $2\pi/k_0$). This, in particular, describes self-focusing of light in nonlinear media. For $d = 2$ and $\omega'' H > 0$, on the contrary, one has dispersive expansion and decay.

Turbulence with two cascades. As mentioned, any equation (3.27) that describes only four-wave scattering necessarily conserves two integrals of motion, the energy H and the number of waves (or wave action) N. For waves of small amplitude, the energy is approximately quadratic in wave amplitudes, $H \approx \int \omega_k |a_k|^2 \, d\mathbf{k}$, as well as $N = \int |a_k|^2 \, d\mathbf{k}$. The existence of two quadratic positive integrals of motion in a closed system means that if such system is subject to external pumping and dissipation, it may develop turbulence that consists of two cascades.



Indeed, imagine that the source at some ω_2 pumps N_2 waves per unit time. It is then clear that for a steady state one needs two dissipation regions in ω-space (at some ω_1 and ω_3) to absorb inputs of both N and E. Conservation laws allow one to determine the numbers of waves, N_1 and N_3, absorbed per unit time respectively in the regions of lower and high frequencies. Schematically, solving $N_1 + N_3 = N_2$ and $\omega_1 N_1 + \omega_3 N_3 = \omega_2 N_2$ we get

$$N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}\,, \qquad N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1}\,. \tag{3.33}$$

We see that for a sufficiently large left interval (when $\omega_1 \ll \omega_2 < \omega_3$) most of the energy is absorbed by the right sink: $\omega_2 N_2 \approx \omega_3 N_3$. Similarly at $\omega_1 < \omega_2 \ll \omega_3$ most of the wave action is absorbed at small ω: $N_2 \approx N_1$. When $\omega_1 \ll \omega_2 \ll \omega_3$ we have two cascades with the fluxes of energy ϵ and wave action Q. The Q-cascade towards large scales is called inverse cascade (Kraichnan 1967, Zakharov 1967), it corresponds (somewhat counter-intuitively) to a kind of self-organization i.e. creation of larger and slower modes out of small-scale fast fluctuations [7]. The limit $\omega_1 \to 0$ is well-defined, in this case the role of the left sink can be actually played by a condensate which absorbs an inverse cascade. Note in passing that consideration of thermal equilibrium in a finite-size system with two integrals of motion leads to the notion of negative temperature [8].

An important hydrodynamic system with two quadratic integrals of motion is two-dimensional ideal fluid. In two dimensions, the velocity $\mathbf{u}$ is perpendicular to the vorticity $\omega = \nabla \times \mathbf{u}$ so that the vorticity of any fluid element is conserved by virtue of the Kelvin theorem. That means that the space integral of any function of vorticity is conserved, including $\int \omega^2 \, d\mathbf{r}$ called enstrophy. We can write the densities of the two quadratic integrals of motion, energy and enstrophy, in terms of velocity spectral density: $E = \int |\mathbf{v_k}|^2 d\mathbf{k}$ and $\Omega = \int |\mathbf{k} \times \mathbf{v_k}|^2 d\mathbf{k}$. Assume now that we excite turbulence with a force having a wavenumber k_2 while dissipation regions are at k_1, k_3. Applying the consideration similar to (3.33) we obtain

$$E_1 = E_2 \frac{k_3^2 - k_2^2}{k_3^2 - k_1^2}\,, \qquad E_3 = E_2 \frac{k_2^2 - k_1^2}{k_3^2 - k_1^2}\,. \tag{3.34}$$

We see that for $k_1 \ll k_2 \ll k_3$, most of the energy is absorbed by the left sink, $E_1 \approx E_2$, and most of the enstrophy is absorbed by the right one, $\Omega_2 = k_2^2 E_2 \approx \Omega_3 = k_3^2 E_3$. We conclude that conservation of both energy and enstrophy in two-dimensional flows requires two cascades: that of the enstrophy towards small scales and that of the energy towards large scales (opposite to the direction of the energy cascade in three dimensions). Large-scale motions of the ocean and planetary atmospheres can be considered approximately two-dimensional; creation and persistence of large-scale flow patterns in these systems is likely related to inverse cascades [9].

3.4 Korteveg - de Vries Equation (KdV)

Shallow-water waves and KdV equation. Soliton. Evolution of arbitrary initial distribution in the framework of KdV equation. Elastic scattering of solitons.

3.4.1 Weakly nonlinear waves in shallow water

Linear gravity-capillary waves have $\omega_k^2 = (gk + \alpha k^3/\rho)\tanh kh$, see (3.12). That is for sufficiently long waves (when the wavelength is larger than both h and $\sqrt{\alpha/\rho g}$) their dispersion relation is close to linear:

$$\omega_k = \sqrt{gh}\, k - \beta k^3 \ , \qquad \beta = \frac{\sqrt{gh}}{2}\left(\frac{h^2}{3} - \frac{\alpha}{\rho g}\right) \ . \tag{3.35}$$

Therefore, one can expect a quasi-simple plane wave propagating in one direction like that described in Sections 2.3.2 and 2.3.3. Let us derive the equation satisfied by such a wave. From the dispersion relation, we get the linear part of the equation: $u_t + \sqrt{gh}u_x = -\beta u_{xxx}$ or in the reference frame moving with the velocity $\sqrt{gh}$ one has $u_t = -\beta u_{xxx}$. To derive the nonlinear part of the equation in the long-wave limit, it is enough to consider the first non-vanishing derivative (i.e. first). The motion is close to one-dimensional so that $u = v_x \gg v_y$. The z-component of the Euler equation gives $\partial p/\partial z = -\rho g$ and $p = p_0 + \rho g(\zeta - z)$ which we substitute into the x-component

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x} \ .$$

In the continuity equation, $h + \zeta$ now plays the role of density:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(h + \zeta)u = 0 \ .$$

We differentiate it with respect to time, substitute the Euler equation and neglect the cubic term:

$$\begin{aligned}\frac{\partial^2\zeta}{\partial t^2} &= \partial_x(h+\zeta)(uu_x+g\zeta_x)+\partial_x u\partial_x(h+\zeta)u \\ &\approx \frac{\partial^2}{\partial x^2}\left(gh\zeta+\frac{g}{2}\zeta^2+hu^2\right) . \end{aligned} \tag{3.36}$$

Apparently, the right-hand-side of this equation contains terms of different orders. The first term describes the linear propagation with the velocity $\sqrt{gh}$ while the rest describe small nonlinear effect. Such equations are usually treated by the method which is called multiple time (or multiple scale) expansion. We actually applied this method in deriving Burgers and Nonlinear Schrödinger equations. We assume that u and ζ depend on two arguments, namely $u(x-\sqrt{gh}\,t,t)$, $\zeta(x-\sqrt{gh}\,t,t)$ and the dependence of the second argument is slow, that is derivative with respect to it is small. In what follows, we write the equation for u. Then at the main order $\partial_t u = -\sqrt{gh}u_x = -g\zeta_x$ that is $\zeta = u\sqrt{h/g}$ which is a direct analog of $\delta\rho/\rho = u/c$ for acoustics. From now on, u_t denotes the derivative with respect to the slow time (or simply speaking, in the reference frame moving with the velocity $\sqrt{gh}$). We obtain now from (3.36):

$$(\partial_t-\sqrt{gh}\partial_x)(\partial_t+\sqrt{gh}\partial_x)u \approx -2\sqrt{gh}u_{xt} = (3/2)\sqrt{gh}(u^2)_{xx}$$

that is nonlinear contribution into u_t is $-3uu_x/2$ — comparing this with the general acoustic expression (2.23), which is $-(\gamma+1)uu_x/2$, we see that shallow-water waves correspond to $\gamma=2$. This is also clear from the fact that the local "sound" velocity is $\sqrt{g(h+\zeta)} \approx \sqrt{gh}+u/2 = c+(\gamma-1)u/2$, see (2.22). In nineteenth century, J. S. Russel used this formula to estimate the atmosphere height observing propagation of weather changes that is of pressure waves. Analogy between shallow-water waves and sound means that there exist shallow-water shocks called bores [10] and hydraulic jumps. The Froude number u^2/gh plays the role of the (squared) March number in this case.

Hydraulic jumps can be readily observed in the kitchen sink when water from the tap spreads radially with the speed exceeding the linear "sound" velocity $\sqrt{gh}$: fluid layer thickness suddenly increases, which corresponds to a shock, see Figure 3.11 and Exercise 3.8. That shock is sent back by the sink sides that stop the flow; the jump position is where the shock speed is equal to the flow velocity [11]. For a weak shock, the shock speed is the speed of "sound" so that the flow is "supersonic" inside and "subsonic" outside. Long surface waves cannot propagate into the interior region which thus can be called a white hole (as opposite to a black hole in general relativity) with the hydraulic jump playing the role of a horizon. In Figure 3.11 one sees circular capillary ripples propagating inside; they are to be distinguished from the jump itself which is non-circular.

Figure 3.11 Hydraulic jump in a kitchen sink.

3.4.2 KdV equation and soliton

Now we are ready to combine both the linear term from the dispersion relation and the nonlinear term just derived. Making a change $u \to 2u/3$ we turn the coefficient at the nonlinear term into unity. The equation

$$u_t + uu_x + \beta u_{xxx} = 0 \tag{3.37}$$

has been derived by Korteveg and de Vries equation in 1895 and is called KdV. Together with the nonlinear Schrödinger and Burgers equations, it is a member of an exclusive family of universal nonlinear models. It is one-dimensional as Burgers and has the same degree of universality. Indeed, most of the systems (namely those with a continuous symmetry spontaneously broken) allow for what is called Goldstone mode that is having $\omega \to 0$ when $k \to 0$. Most of such systems have an acoustic branch of excitations — this can be simply argued that in conservative time-reversible center-symmetrical systems one has to have $\omega^2(k^2)$ and the expansion of this function in the small-k limit generally starts from the first term that is $\omega^2 \propto k^2$. The next term is k^4 that is for long waves moving in one direction one has $\omega_k = ck[1+C(kl_0)^2]$ where C is dimensionless coefficient of order unity and l_0 is some typical internal scale in the system. For gravity water waves, l_0 is the water depth; for capillary waves, it is $\sqrt{\alpha/\rho g}$. We see from (3.35) that, depending on which scale is larger, β can be either positive or negative, which corresponds to the waves with finite k moving respectively slower or faster than "sound" velocity $\sqrt{gh}$. Adding surface tension to the restoring force, one increases the frequency. On the other hand, finite depth-to-wavelength ratio means that fluid particles move in ellipses (rather than in straight lines) which decreases the frequency. Quadratic nonlinearity $\partial_x u^2$ (which stands both in Burgers and KdV and means simply the renormalization of the speed of sound) is also pretty general, indeed, it has to be zero for a uniform velocity so it contains a derivative. An incomplete list of the excitations described by KdV contains acoustic perturbations in plasma (then l is either the Debye radius of charge screening or Larmor cyclotron radius in the magnetized plasma), phonons in solids (then l is the distance between atoms) and phonons in helium (in this case, amazingly, the sign of β depends on pressure).

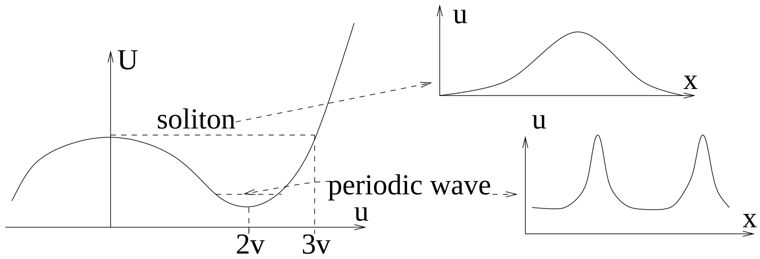
KdV has a symmetry $\beta \to -\beta$, $u \to -u$ and $x \to -x$ which makes it enough to consider only positive β. Let us first look for the traveling waves, that is substitute $u(x-vt)$ into (3.37):

$$\beta u_{xxx} = vu_x - uu_x \ .$$

This equation has a symmetry $u \to u+w$, $v \to v+w$ so that integrating it once we can put the integration constant zero choosing an appropriate constant w (this is the trivial renormalization of sound velocity due to uniformly moving fluid). We thus get the equation

$$\beta u_{xx} = -\frac{\partial U}{\partial u} \ , \qquad U(u) = \frac{u^3}{6} - \frac{vu^2}{2} \ .$$

A general solution of this ordinary differential equations can be written in elliptic functions. We don't need it though to understand the general properties of the solutions and to pick up the special one (i.e. soliton). Just like in Sect. 3.3.3, we can treat this equation as a Newton equation for a particle in a potential. We consider positive v since it is a matter of choosing proper reference frame. We are interested in the solution with finite $|u(x)|$; unrestricted growth would violate the assumption on weak nonlinearity. Such finite solutions have u in the interval $(0, 3v)$.



We see that linear periodic waves exist near the bottom at $u \approx 2v$. Their amplitude is small in the reference frame moving with $-2v$ — in that reference frame they have negative velocity as it must be for positive β. Indeed, the sign of β is minus the sign of the dispersive correction $-3\beta k^2$ to the group velocity $d\omega_k/dk$. Soliton, on the contrary, moves with positive velocity in the reference frame where there is no perturbation at infinity (it is precisely the reference frame used in the above picture). In physical terms, solitons are supersonic if the periodic waves are subsonic and vise versa (the physical reason for this is that soliton should not be able to radiate linear waves by Cherenkov radiation).

As usual, the soliton solution is a separatrix:

$$u(x,t) = 3v \cosh^{-2}\left[\sqrt{\frac{v}{4\beta}}\,(x - vt)\right] . \tag{3.38}$$

The higher the amplitude, the faster it moves (for $\beta > 0$) and the more narrow is. Similarly to the argument at the end of (3.3.2), one can realize that 1d KdV soliton is unstable with respect to the perpendicular perturbations if its speed decreases with the amplitude that is for $\beta < 0$ when linear waves are supersonic and soliton is subsonic.

Without friction or dispersion, nonlinearity breaks acoustic perturbation. We see that wave dispersion stabilizes the wave similar to the way the viscous friction stabilizes shock front, but the waveform is, of course, different. The ratio of nonlinearity to dispersion, $\sigma = uu_x/\beta u_{xxx} \sim ul^2/\beta$, is an intrinsic nonlinearity parameter within KdV (in addition to the original "external" nonlinearity parameter, Mach number u/c, assumed to be always small). For soliton, $\sigma \simeq 1$ that is nonlinearity balances dispersion as that was the case with the NSE soliton from Sect. 3.3.3. That also shows that soliton is a non-perturbative object, one cannot derive it starting from a linear traveling waves and treating nonlinearity perturbatively. For Burgers equation, both finite-M and traveling wave solutions depended smoothly on the respective intrinsic parameter Re and existed for any Re. What can one say about the evolution of a perturbation with $\sigma \gg 1$ within KdV? Can we assert that any perturbation with $\sigma \ll 1$ corresponds to linear waves?

3.4.3 Inverse scattering transform

It is truly remarkable that the evolution of arbitrary initial perturbation can be studied analytically within KdV. It is done in somewhat unexpected way by considering a linear stationary Schrödinger equation with the function $-u(x,t)/6\beta$ as a potential depending on time as a parameter:

$$\left[-\frac{d^2}{dx^2} - \frac{u(x,t)}{6\beta}\right]\Psi = E\Psi \ . \tag{3.39}$$

Positive $u(x)$ could create bound states that is a discrete spectrum. As has been noticed by Gardner, Green, Kruskal and Miura in 1967, the spectrum E does not depend on time if $u(x,t)$ evolves according to KdV equation. In other words, KdV describes the iso-spectral transformation of the quantum potential. To show that, express u via Ψ

$$u = -6\beta\left(E + \frac{\Psi_{xx}}{\Psi}\right) \ . \tag{3.40}$$

Notice similarity to the Hopf substitution one uses for Burgers, $v = -2\nu\phi_\xi/\phi$, one more derivative in (3.40) because there is one more derivative in KdV — despite the seemingly naive and heuristic way of such thinking, it is precisely the way Miura came to suggest (3.40). Now, substitute (3.40) into KdV and derive

$$\Psi^2\frac{dE}{dt} = 6\beta\partial_x\left[\left(\Psi\partial_x - \Psi_x\right)\left(\Psi_t + \Psi_{xxx} - \Psi_x(u+E)/2\right)\right] \ . \tag{3.41}$$

Integrating it over x we get $dE/dt = 0$ since $\int_{-\infty}^{\infty}\Psi^2\,dx$ is finite for bound state. The eigenfunctions evolve according to the equation that can be obtained twice integrating (3.41) and setting integration constant zero because of the normalization:

$$\Psi_t + \Psi_{xxx} - \Psi_x\frac{u+E}{2} = 0 \ . \tag{3.42}$$

From the viewpoint of (3.39), soliton is the well with exactly one level, $E = v/8\beta$, which could be checked directly. For distant solitons, one can define energy levels independently. For different solitons, velocities are different and they generally will have collisions. Since the spectrum is conserved, after all the collisions we have to have the same solitons. Since the velocity is proportional to the amplitude the final state of the perturbation with $\sigma \gg 1$ must look like a linearly ordered sequence of solitons: The quasi-linear waves that correspond to a continuous spectrum are

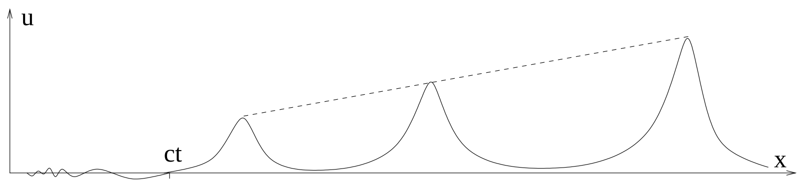


Figure 3.12 Asymptotic form of a localized perturbation.

left behind and eventually spread. One can prove that and analyze the evolution of an arbitrary initial perturbation by the method of *Inverse Scattering Transform* (IST):

$$u(x,0) \;\to\; \Psi(x,0) = \sum a_n \Psi_n(x,0) + \int a_k \Psi_k(x,0)\, dk$$

$$\to\; \Psi(x,t) = \sum a_n \Psi_n(x,t) + \int a_k \Psi_k(x,0) e^{-i\omega_k t}\, dk \;\to\; u(x,t)\ .$$

The first step is to find the eigenfunctions and eigenvalues in the potential $u(x,0)$. The second (trivial) step is to evolve the discrete eigenfunctions according to (3.42) and continuous according to frequencies. The third (nontrivial) step is to solve an inverse scattering problem that is to restore potential $u(x,t)$ from it's known spectrum and the set of (new) eigenfunctions [12].

Considering weakly nonlinear initial data ($\sigma \ll 1$), one can treat the potential energy as perturbation in (3.39) and use the results from the quantum mechanics of shallow wells. Remind that the bound state exists in 1d if the integral of the potential is negative, that is in our case when the momentum $\int u(x)\, dx > 0$ is positive i.e. the perturbation is supersonic. The subsonic small perturbation (with a negative momentum) do not produce solitons, it produces subsonic quasi-linear waves. On the other hand, however small is the nonlinearity σ of the initial perturbation with positive momentum, the soliton (an object with $\sigma \sim 1$) will necessarily appear in the course of evolution. The amplitude of the soliton is proportional to the energy of the bound state which is known to be proportional to $(\int u\, dx)^2$ in the shallow well.

The same IST method was applied to the 1d Nonlinear Schrodinger

Equation by Zakharov and Shabat in 1971. Now, the eigenvalues E of the system

$$i\partial_t\psi_1 + \psi\psi_2 = E\psi_1$$
$$-i\partial_t\psi_2 - \psi^*\psi_1 = E\psi_2$$

are conserved when ψ evolves according to 1d NSE. Similarly to KdV, one can show that within NSE arbitrary localized perturbation evolves into a set of solitons and diffusing quasi-linear wave packet.

The reason why universal dynamic equations in one space dimension happen to be integrable may be related to their universality. Indeed, in many different classes of systems, weakly-nonlinear long-wave perturbations are described by Burgers or KdV and quasi-monochromatic perturbations by NSE. Those classes may happen to contain degenerate integrable cases, then integrability exists for the limiting equations as well. These systems are actually two-dimensional (space plus time) and their integrability can be related to a unique role of the conformal group (which is infinite in 2d). From potential flows described in Section 1.2.4 to nonlinear waves described in this Chapter, it seems that the complex analysis and the idea of analyticity is behind most of the solvable cases in fluid mechanics, as well as in other fields of physics.

Exercises

3.1 Why beachcomber (a long water wave rolling upon the beach) usually comes to the coast being almost parallel to the coastline even while the wind blows at an angle?

3.2 The quasi-monochromatic packet of waves contains N crests and wells propagating along the fluid surface. How many "up and down" motions undergoes a light float while the packet passes? Consider two cases: i) gravity waves on deep water, ii) capillary waves on a deep water.

3.3 Dropping stone into the deep water, one could see, after a little while, waves propagating outside an expanding circle of a rest fluid. Sketch a snapshot of the wave crests. What is the velocity of the boundary of quiescent fluid circle?

3.4 The existence of the stable small-amplitude waves that are described by (3.19) cannot be taken for granted. Consider a general form of the quadratic Hamiltonian

$$\mathcal{H}_2 = \int \Big[A(k)|b_k|^2 + B(k)(b_k b_{-k} + b_k^* b_{-k}^*) \Big] \, d\mathbf{k} \, . \tag{3.43}$$

1a. Find the linear transformation (Bogolyubov u-v transformation) $b_k = u_k a_k + v_k a_{-k}^*$ that turns (3.43) into (3.19).

1b. Consider the case when the even part $A(k) + A(-k)$ changes sign on some surface (or line) in k-space while $B(k) \neq 0$ there. What does it mean physically? In this case, what simplest form the quadratic Hamiltonian $\mathcal{H}_2$ can be turned into?

3.5 Show that the power dispersion relation $\omega_k \propto k^\alpha$ is of decay type if $\alpha \geq 1$, i.e. it is possible to find such $\mathbf{k}_1, \mathbf{k}_2$ that $\omega(\mathbf{k}_1 + \mathbf{k}_2) = \omega_1 + \omega_2$. Consider two-dimensional space $\mathbf{k} = \{k_x, k_y\}$. Hint: ω_k is a concave surface and the resonance condition can be thought of as an intersection of some two surfaces.

3.6 Consider the case of a stable condensate in 3d and describe a solution of the NSE equation (3.28) having the form

$$\psi = A e^{-iTA^2 t + i\varphi} f(r/r_0) \, ,$$

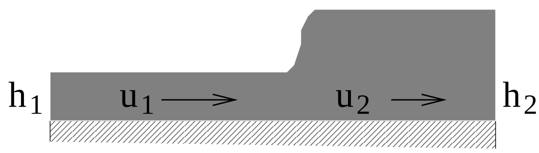
where r is the distance from the vortex axis and φ is a polar angle. Are the parameters A and r_0 independent? Describe the asymptotics of f for small and large distances. Why it is called a vortex?

3.7 Consider a discrete spectral representation of the Hamiltonian of the 1d NSE in a finite medium:

$$H = \sum_m \beta m^2 |a_m|^2 + (T/2) \sum_{ikm} a_i a_k a_{i+k-m}^* a_m^* \, .$$

Take only three modes $m = 0, 1, -1$. Describe dynamics of such 3-mode system.

3.8 Calculate the energy dissipation rate per unit length of the hydraulic jump. Fluid flows with the velocity u_1 in a thin layer of the height h_1 such that the Froude number slightly exceeds unity: $u_1^2/gh_1 = 1 + \epsilon$, $\epsilon \ll 1$.

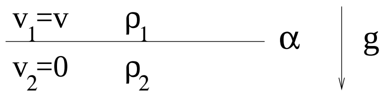


3.9 If one ought to take into account both dissipation and dispersion of sound wave, then the so-called KdV-Burgers equation arises:

$$u_t + uu_x + \beta u_{xxx} - \mu u_{xx} = 0\,. \qquad (3.44)$$

For example, it allows to describe the influence of dispersion on the structure of weak shock wave. Consider a traveling solution $u_0(x - vt)$ of this equation, assuming zero conditions at $+\infty$: $u_0 = u_{0x} = u_{0xx} = 0$. Sketch the form of $u_0(x)$ for $\mu \ll \sqrt{\beta v}$ and for $\mu \gg \sqrt{\beta v}$.

3.10 Find the dispersion relation of the waves on the boundary between two fluids, one flowing and one still, at the presence of gravity g and the surface tension α. Describe possible instabilities. Consider, in particular, the cases $\rho_1 > \rho_2$, $v = 0$ (inverted gravity) and $\rho_1 \ll \rho_2$ (wind upon water).



4
Epilogue

Now that we have learnt basic mechanisms and elementary interplay between nonlinearity, dissipation and dispersion in fluid mechanics, where one can go from here? It is important to recognize that this book describes only few basic types of flows and leaves whole sets of physical phenomena outside of its scope. It is yet impossible to fit all of fluid mechanics into the format of a single story with few memorable protagonists. Here is the brief guide to further reading, more details can be found in Endnotes.

Comparable elementary textbook (which is about twice larger in size) is that of Acheson [1], it provides extra material and some alternative explanations on the subjects described in our Chapters 1 and 2. On the subjects of Chapter 3, timeless classics is the book by Lighthill [11]. For a deep and comprehensive study of fluid mechanics as a branch of theoretical physics one cannot do better than using another timeless classics, volume VI of the Landau-Lifshits course [10]. Apart from a more detailed treatment of the subjects covered here, it contains variety of different flows, the detailed presentation of the boundary layer theory, the theory of diffusion and thermal conductivity in fluids, relativistic and superfluid hydrodynamics etc. In addition to reading about fluids, it is worth looking at flows, which is as appealing aesthetically as it is instructive and helpful in developing physicist's intuition. Plenty of visual material, both images and videos, can be found in [9, 19] and Galleries at http://www.efluids.com/. And last but not least: the beauty of fluid mechanics can be revealed by simply looking at the world around us and doing simple experiments in a kitchen sink, bath tube, swimming pool etc. It is likely that fluid mechanics is the last frontier where fundamental discoveries in physics can still be made in such a way.

After learning what fluid mechanics can do for you, some of you may be interested to know what you can do for fluid mechanics. Let me briefly mention several directions of the present-day action in the physics of fluids. Considerable analytical and numerical work continues to be devoted to the fundamental properties of the equations of fluid mechanics, particularly to the existence and uniqueness of solutions. The subject of a finite-time singularity in incompressible flows particularly stands out.

Those are not arcane subtleties of mathematical description but the questions whose answers determine important physical properties, for instance, statistics of large fluctuations in turbulent flows. On the one hand, turbulence is a paradigmatic far-from-equilibrium state where we hope to learn general laws governing non-equilibrium systems; on the other hand, its ubiquity in nature and industry requires knowledge of many specific features. Therefore, experimental and numerical studies of turbulence continue towards both deeper understanding and wider applications in geophysics, astrophysics and industry, see e.g. [4, 8, 21]. At the other extreme, we have seen that flows at very low Reynolds number are far from being trivial; needs of biology and industry triggered an explosive development of micro-fluidics bringing new fundamental phenomena and amazing devices. Despite a natural tendency of theoreticians towards limits (of low and high $Re, Fr, \mathcal{M}$), experimentalists, observers and engineers continue to discover fascinating phenomena for the whole range of flow control parameters.

The domain of quantum fluids continues to expand including now superfluid liquids, cold gases, superconductors, electron droplets and other systems. Quantization of vorticity and a novel factor of disorder add to the interplay of nonlinearity, dispersion, dissipation. Many phenomena in plasma physics also belong to a domain of fluid mechanics. Quantum fluids and plasma can often be described by two interconnected fluids (normal and superfluid, electron and ion) which allows for rich set of phenomena.

Another booming subject is the studies of complex fluids. One important example is a liquid containing long polymer molecules that are able to store elastic stresses providing fluid with a memory. That elastic memory provides for inertia (and nonlinearity) of its own and introduces the new dimensionless control parameter, Weissenberg number, which is the product of fluid strain and the polymer response time. When the Weissenberg number increases, instabilities takes place (even at very low Reynolds number) culminating in so-called elastic turbulence [17]. Another example is a two-phase flow with numerous applications, from clouds to internal combustion engines; here a lot of interesting physics is related to relative inertia of two phases and very inhomogeneous distribution of droplets, particles or bubbles in a flow.

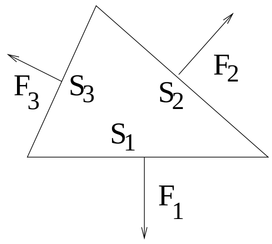
And coming back to basics: our present understanding of how fish and microorganisms swim and how birds and insects fly is so poor that further research is bound to bring new fundamental discoveries and new engineering ideas.

5

Solutions of exercises

"... a lucky guess is never merely luck."
Jane Austen

1.1.
Consider a prism inside the fluid.



Forces must sum into zero in equilibrium which means that after being rotated by $\pi/2$ force vectors form a closed triangle similar to that of the prism. Therefore, the forces are proportional to the areas of the respective faces and the pressures are equal.

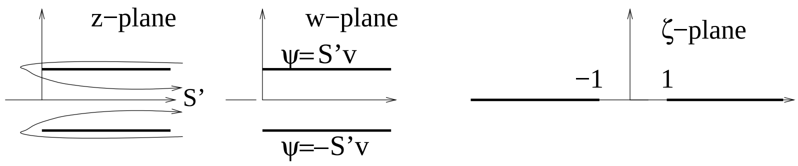
1.2.
The force $-\nabla\psi$ must balance the gradient of pressure

$$\frac{d}{dr}\left(\frac{r^2 dp}{\rho dr}\right) = -4\pi G r^2 \rho\,, \qquad p = \frac{2}{3}\pi G \rho^2 (R^2 - r^2)\ . \tag{5.1}$$

1.3.
The discharge rate is $S'\sqrt{2gh}$. Energy conservation gives us $v = \sqrt{gh}$ at the vena contracta. This velocity squared times density times area S' gives the momentum flux which must be determined by the force exerted by the walls on the fluid. The difference between the orifice, drilled directly in the wall as in Fig. 1.3, and the tube, projecting inward

(called Borda's mouthpiece), is that in the latter case one can neglect the motion near the walls so that the force imbalance is the pressure $p = \rho gh$ times the hole area S. We thus get $\rho v^2 S' = \rho ghS$ and $S' = S/2$. Generally, the motion near the walls diminishes the pressure near the exit and makes the force imbalance larger. The reaction force is therefore greater and so is the momentum flux. Since the jet exits with the same velocity it must have a larger cross-section, so that $S'/S \geq 1/2$ (for a round hole in a thin wall it is empirically known that $S'/S \simeq 0.62$).

The above general argument based on the conservation laws of energy and momentum works in any dimension. For Borda's mouthpiece in two dimensions, one can describe the whole flow neglecting gravity and assuming the (plane) tube infinite, both assumptions valid for a flow not far from the corner. This is a flow along the tube wall on the one side and detached from it on the other side in distinction from a symmetric flow shown in Figure 1.8 with $n = 1/2$. Flow description can be done using conformal maps shown in the following figure:



The tube walls coincide with the streamlines and must be cuts in ζ-plane because of jumps in the potential. That corresponds to $w \propto \ln \zeta$, the details can be found in Section 11.51 of [13]. For a slit in a thin wall, 2d solution can be found in Section 11.53 of [13] or Section 10 of [10], which gives the coefficient of contraction $\pi/(\pi + 2) \approx 0.61$.

1.4.

Simply speaking, vorticity is the velocity circulation (=vorticity flux) divided by the area while the angular velocity Ω is the velocity circulation (around the particle) divided by the radius a and the circumference $2\pi a$:

$$\Omega = \int u\, dl/2\pi a^2 = \int \omega\, df/2\pi a^2 = \omega/2 \ .$$

A bit more formally, place the origin inside the particle and consider the velocity of a point of the piece with radius vector $\mathbf{r}$. Since the particle is small we use Taylor expansion $v_i(\mathbf{r}) = S_{ij} r_j + A_{ij} r_j$ where $S_{ij} = (\partial_i r_j + \partial_j r_i)/2$ and $A_{ij} = (\partial_i r_j - \partial_j r_i)/2$. Rigid body can not be deformed thus $S_{ij} = 0$. The only isometries that do not deform the body and do not shift its center of mass are the rotations. The rotation with the angular velocity $\mathbf{\Omega}$ gives $A_{ik}r_k = \epsilon_{ijk}\Omega_j r_k$. On the other hand, the vorticity component

$$\omega_i \equiv [\nabla \times \mathbf{v}]_\mathbf{i} = \epsilon_\mathbf{ijk}\partial_\mathbf{j}\mathbf{v_k} = \frac{\mathbf{1}}{\mathbf{2}}\epsilon_\mathbf{ijk}(\partial_\mathbf{j}\mathbf{v_k} - \partial_\mathbf{k}\mathbf{v_j}) \ .$$

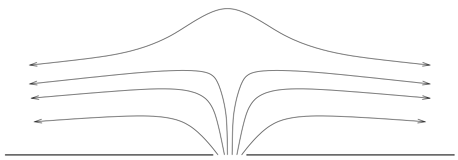
Using the identity $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$ and

$$\epsilon_{imn}\omega_i = \frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk})(\partial_j v_k - \partial_k v_j) = \partial_m v_n - \partial_n v_m \ ,$$

we derive $A_{ik}r_k = \epsilon_{ijk}\omega_j r_k/2$ and $\Omega = \omega/2$.

1.5.

Use Bernoulli equation, written for the point of maximal elevation (when $v = 0$ and the height is H) and at infinity: $2gH = 2gh + v_\infty^2$.



i) Flow is two-dimensional and far from the slit has only a horizontal velocity which does not depend on the vertical coordinate because of potentiality. Conservation of mass requires $v_\infty = q/2\rho g$ and the elevation $H - h = q^2/8g\rho^2h^2$.
ii) There is no elevation for a potential flow in this case since the velocity goes to zero at large distances (as an inverse distance from the source). A fountain with an underwater source is surely due to a non-potential flow.

1.6.

In the reference frame of the sphere, the velocity of the inviscid potential flow is as follows:

$$v_r = u\cos\theta\left(1 - (R/r)^3\right) \ ,$$

$$v_\theta = -u\sin\theta\left(1 + \left(R^3/2r^3\right)\right) \ .$$

Streamlines by definition are the lines where

$$\mathbf{v} \times d\mathbf{f} = \left(v_r\hat{\mathbf{r}} + v_\theta\hat{\theta}\right) \times \left(dr\hat{\mathbf{r}} + d\theta\hat{\theta}\right) = rv_r d\theta - v_\theta dr = 0 \, ,$$

which gives the equation

$$\frac{d\theta}{dr} = -\frac{2r^3 + R^3}{2r\left(r^3 - R^3\right)} \tan\theta \,,$$

whose integration gives the streamlines

$$-\int_{\theta_1}^{\theta_2} d\theta \frac{2}{\tan\theta} = \int_{r_1}^{r_2} dr \frac{2r^3 + R^3}{r\left(r^3 - R^3\right)}$$
$$\left(\frac{\sin\theta_1}{\sin\theta_2}\right)^2 = \frac{r_2(r_1 - R)\left(r_1^2 + r_1R + R^2\right)}{r_1(r_2 - R)\left(r_2^2 + r_2R + R^2\right)} \,.$$

It corresponds to the the stream function in the sphere reference frame as follows: $\psi = -ur^2 \sin^2\theta(1/2 - R^3/r^3)$. The streamlines relative to the sphere are in the right part of Figure 5.1.

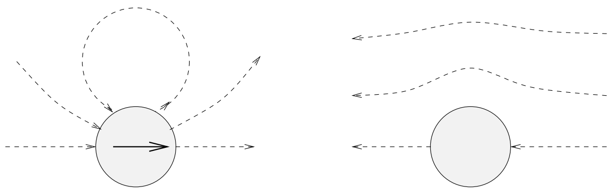


Figure 5.1 Streamlines of the potential flow around a sphere in the reference frame where the fluid is at rest at infinity (left) and in the reference frame moving with the sphere (right).

In the reference frame where the fluid is at rest at infinity,

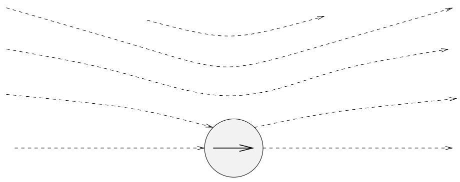
$$v_r = -u\cos\theta\,(R/r)^3 \;, \quad v_\theta = -(1/2)u\sin\theta\,(R/r)^3 \;.$$

Integrating

$$\frac{d\theta}{dr} = -\frac{\tan\theta}{2r} \,,$$

one obtains the stream function $\psi = -uR^3 \sin^2\theta/2r$ whose streamlines are shown in the left part of Figure 5.1.

From the velocity of the viscous Stokes flow given by (1.48), one can obtain the stream function. In the reference frame where the fluid is at rest at infinity, $\psi = urR\sin^2\theta(3/4 - R^3/3r^3)$, the streamlines are shown in the figure below.



Apparently, the main difference is that inviscid streamlines are loops (compare with the loops made by trajectories as shown in Fig. 1.9), while viscous flow is one-way.

In the sphere reference frame, the Stokes stream function is $\psi = -ur^2 \sin^2\theta(1/2 - 3R/4r + R^3/4r^3)$ and the streamlines are qualitatively similar to those in the right part of Figure 5.1 .

1.7.

The equation of motion for the ball on a spring is $m\ddot{x} = -kx$ and the corresponding frequency is $\omega_a = \sqrt{k/m}$. In a fluid,

$$m\ddot{x} = -kx - \tilde{m}\ddot{x}\,, \tag{5.2}$$

where $\tilde{m} = \rho V/2$ is the induced mass of a sphere. The frequency of oscillations in an ideal fluid is

$$\omega_{a,\text{fluid}} = \omega_a \sqrt{\tfrac{2\rho_0}{2\rho_0+\rho}}\,, \tag{5.3}$$

here ρ is fluid density and ρ_0 is the ball's density.

The equation of motion for the pendulum is $ml\ddot{\theta} = -mg\theta$. In a fluid, it is

$$ml\ddot{\theta} = -mg\theta + \rho Vg\theta - \tilde{m}l\ddot{\theta} \tag{5.4}$$

where $\rho Vg\theta$ is the Archimedes force, $-\tilde{m}l\ddot{\theta}$ inertial force. From $ml\ddot{\theta} = -mg\theta$ we get $\omega_b = \sqrt{g/l}$, while when placed in the fluid we have that frequency of oscillations is

$$\omega_{b,\text{fluid}} = \omega_b \sqrt{\tfrac{2(\rho_0-\rho)}{2\rho_0+\rho}} \tag{5.5}$$

Fluid viscosity would lead to some damping. When the viscosity is small, $\nu \ll \omega_{a,b}a^2$, then the width of the boundary layer is much less then the size of the body: $\nu/\omega_{a,b}a \ll a$. We then can consider the boundary layer as locally flat and for a small piece near a flat surface we derive

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} ,$$
$$v_x(y,t) = u \exp\{-[(1+i)y/\delta + i\omega t]\} , \quad \delta = \sqrt{2\eta/\rho_0\omega} . \quad (5.6)$$

Such fluid motion provides for the viscous stress on the body body surface

$$\sigma_{yx} = \eta \frac{\partial v_v(0,t)}{\partial y} = (i-1) v_x(0,t) \sqrt{\omega\eta\rho/2} ,$$

which leads to the energy dissipation rate per unit area

$$-\sigma_{yx} v_y = u^2 \sqrt{\omega\eta\rho/8} .$$

An estimate of the energy loss one obtains multiplying it by the surface area. To get an exact answer for the viscous dissipation by the oscillating sphere, one needs to find the velocity distribution around the surface, see e.g. Sect. 24 of [10].

1.8.

Dimensional analysis and simple estimates. In the expression $T \propto E^\alpha p^\beta \rho^\gamma$, three unknowns α, β, γ can be found from considering three dimensionalities (grams, meters and seconds), which gives

$$T \propto E^{1/3} p^{-5/6} \rho^{1/2} .$$

Analogously, $T \propto a p^{-1/2} \rho^{1/2}$. Note that here $c \propto \sqrt{p/\rho}$ is the sound velocity so that the period is a/c. The energy is pressure times volume: $E = 4\pi a^3 p/3$. That way people measure the energy of the explosions underwater: wait until the bubble is formed and then relate the bubble size, obtained by measuring bubble oscillations, to the energy of the explosion.

Sketch of a theory. The radius of the bubble varies like: $r_0 = a + b\exp(-i\omega t)$, where a is the initial radius and $b \ll a$ is a small amplitude of oscillations with the period $T = 2\pi/\omega$. The induced fluid flow is radial, if we neglect gravity, $v = v_r(r,t)$. Incompressibility requires $v(r,t) = A\exp(-i\omega t)/r^2$. On the surface of the bubble, $dr_0/dt = v(r,t)$, which gives $A = -iba^2\omega$. So the velocity is as follows

$$v(r,t) = -ib(a/r)^2 \omega \exp(-i\omega t) \quad (5.7)$$

Note that $(\mathbf{v} \cdot \nabla)\mathbf{v} \simeq b^2\omega^2/a \ll \partial_t \mathbf{v} \simeq b\omega^2$, since it corresponds to the assumption $b \ll a$ ($|(\mathbf{v}\cdot\nabla)\mathbf{v}| \propto$). Now we use the linearized Navier-Stokes equations and spherical symmetry and get:

$$p_{\text{water}} = p_{\text{static}} - \rho\omega^2 b\left(\frac{a^2}{r}\right)e^{-i\omega t} \tag{5.8}$$

where p_{static} is the static pressure of water - unperturbed by oscillations. Since $\rho_{\text{air}}/\rho_{\text{water}} = 10^{-3}$, then the bubble compressions and expansions can be considered quasi-static, $p_{\text{bubble}}{r_0}^{3\gamma} = p_{\text{static}}a^{3\gamma}$, which gives:

$$p_{\text{bubble}} = p_{\text{static}}\left(1 - 3\gamma(b/a)e^{-i\omega t}\right) \tag{5.9}$$

Now use the boundary condition for the bubble-water interface at $r = a$

$$-p_{\text{bubble}}\delta_{ik} = -p_{\text{water}}\delta_{ik} + \eta(\partial_k v_i + \partial_i v_k)\,, \tag{5.10}$$

where η is the water dynamic viscosity. The component σ_{rr} gives $\rho(a\omega)^2 + 4i\eta - 3\gamma p = 0$ with the solution

$$\omega = \left(-\frac{2\eta i}{a^2\rho} \pm \sqrt{\frac{3\gamma p}{\rho a^2} - \frac{4\eta^2}{a^4\rho^2}}\right)\ . \tag{5.11}$$

For large viscosity, it describes aperiodic decay; for $\eta^2 < 3\gamma p\rho a^2/4$ the frequency of oscillations is as follows:

$$\omega = 2\pi/T(a,p,\rho) = \sqrt{\frac{3\gamma p}{\rho a^2} - \frac{4\eta^2}{a^4\rho^2}}\ . \tag{5.12}$$

Viscosity increases the period which may be relevant for small bubbles, see [20] for more details.

1.9.

The Navier-Stokes equation for the vorticity in an incompressible fluid,

$$\partial_t\omega + (\mathbf{v}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{v} = \nu\mathbf{\Delta}\omega$$

in the cylindrically symmetric case is reduced to the diffusion equation,

$$\partial_t\omega = \nu\mathbf{\Delta}\omega\,,$$

since $(\mathbf{v}\cdot\nabla)\omega = (\omega\cdot\nabla)\mathbf{v} = \mathbf{0}$. The diffusion equation with the delta-function initial condition has the solution

$$\omega(r,t) = \frac{\Gamma}{4\pi\nu t}\exp\!\left(-\frac{r^2}{4\nu t}\right),$$

which conserves the total vorticity:

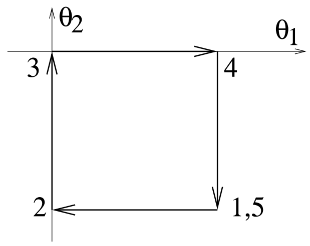
$$\Omega(t) = 2\pi\int_0^\infty \omega(r,t)rdr = \frac{\Gamma}{2\nu t}\int_0^\infty \exp\left(-\frac{r^2}{4\nu t}\right)rdr = \Gamma\ .$$

Generally, for any two-dimensional incompressible flow, the Navier-Stokes equation takes the form $\partial_t\omega + (\mathbf{v}\cdot\nabla)\omega = \nu\boldsymbol{\Delta}\omega$, which conserves the vorticity integral as long as it is finite.

1.10.

The shape of the swimmer is characterized by the angles between the arms and the middle link. Therefore, the configuration space is two-dimensional. Our swimmer goes around a loop in this space with the displacement proportional to the loop area which is θ^2. Transformation $y \to -y$, $\theta_1 \to -\theta_1$, $\theta_2 \to -\theta_2$ produces the same loop, therefore y-displacement must be zero. Since it is easier to move when the non-moving arm is aligned with the body (i.e. either θ_1 or θ_2 is zero), then it is clear that during $1 \to 2$ and $4 \to 5$ the swimmer shifts to the left less than it shifts to the right during $2 \to 3$ and $3 \to 4$, at least when $\theta \ll 1$. Therefore the total displacement is to the right or generally in the direction of the arm that moved first. Further reading: Sect. 7.5 of [1]; Purcell, E. M. (1977) Life at low Reynolds number, *American Journal of Physics*, vol 45, p. 3; Childress, S. (1981) *Mechanics of swimming and flying* (Cambridge Univ. Press).

Anchoring the swimmer one gets a pump. Geometrical nature of swimming and pumping by micro-organisms makes them a subject of a non-Abelian gauge field theory with rich connections to many other phenomena, see Wilczek, F. and Shapere, A. (1989) Geometry of self-propulsion at low Reynolds number, Journal of Fluid Mechanics, vol 198, p. 557.



1.11.

Simple estimate. The lift force can be estimated as $\rho v\Omega RR^2 \simeq 3$ N. The rough estimate of the deflection can be done by neglecting the ball deceleration and estimating the time of flight as $T \simeq L/v_0 \simeq 1$ s. Further, neglecting the drag in the perpendicular direction we estimate that the acceleration $\rho v_0 \Omega R^3/m \simeq 6.7\,\mathrm{m\cdot s^{-2}}$ causes the deflection

$$y(T) = \frac{\rho v_0 \Omega R^3 T^2}{2m} \simeq \frac{\rho \Omega R^3 L^2}{2 m v_0} = L\frac{\Omega R}{v_0}\frac{\rho R^2 L}{2m} \simeq 3.35\ \mathrm{m}\ . \quad (5.13)$$

Sketch of a theory. It is straightforward to account for the drag force, $C\rho v^2 \pi R^2/2$, which leads to the logarithmic law of displacement:

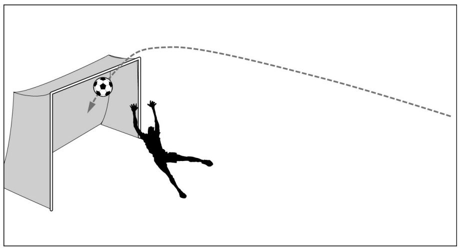
$$\dot v = -\frac{v^2}{L_0}\,, \quad v(t) = \frac{v_0}{1 + v_0 t/L_0}\,,$$
$$x(t) = L_0 \ln(1 + v_0 t/L_0)\ . \quad (5.14)$$

Here $L_0 = 2m/C\rho\pi R^2 \simeq 100$ m with $C \simeq 0.25$ for $Re = v_0 R/\nu \simeq 2\cdot 10^5$. The meaning of the parameter L_0 is that this is the distance at which drag substantially affects the speed; non-surprisingly, it corresponds to the mass of the air displaced, $\rho\pi R^2 L$, being of the order of the ball mass. We get the travel time T from (5.14): $v_0 T/L_0 = \exp(L/L_0) - 1 > L/L_0$. We can now account for the time-dependence $v(t)$ in the deflection. Assuming that the deflection in y-direction is small comparing to the path travelled in x direction, we get

$$\frac{d^2 y(t)}{dt^2} = \frac{\rho\Omega R^3 v(t)}{m} = \frac{\rho \Omega R^3 v_0}{m(1 + v_0 t/L_0)}\,,$$
$$y(0) = \dot y(0) = 0\,,$$
$$y(t) = L_0 \frac{\Omega R}{v_0}\frac{2}{\pi C}\left[(1 + v_0 t/L_0)\ln(1 + v_0 t/L_0) - v_0 t/L_0\right]\ . \quad (5.15)$$

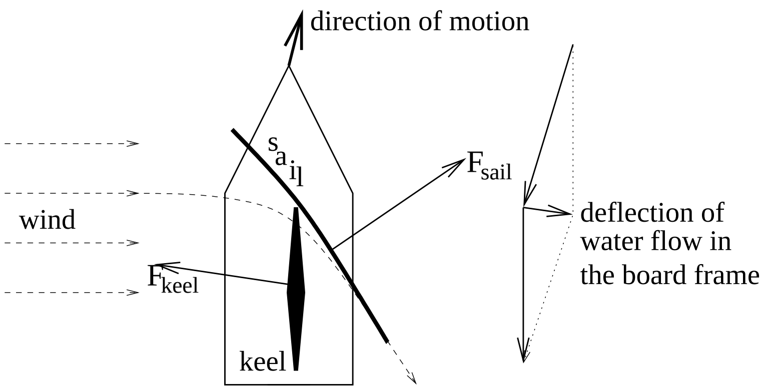
It turns into (5.13) in the limit $L \ll L_0$ (works well for a penalty kick). For longer L one also needs to account for the drag in the y-direction which leads to the saturation of $\dot y$ at the value $\sim \sqrt{\Omega R v}$. Still, such detailed consideration does not make much sense because we took a very crude estimate of the lift force and neglected vertical displacement $gT^2/2$, which is comparable with the deflection.

Remark. Great soccer players are able also to utilize the drag crisis which is a sharp increase of the drag coefficient C from 0.15 to 0.5 when Re decreases from $2.5\cdot 10^5$ to $1.5\cdot 10^5$ (ball velocity drops from 37.5 m/s to 22.5 m/s). As a result, some way into its path the ball sharply decelerates and the Magnus force comes even more into effect. The phenomenon of drag crisis is also used for making a long shot over the goalkeeper who came out too far from the goal; in this case, the ball smoothly rises up and then falls down steeply (see Figure). A topspin adds Magnus force which enhances this effect.



1.12.

Lift force on the keel exists only if the board moves not exactly in the direction in which it is pointed. The direction of the force acting on the keel can be understood considering the deflection of water in the reference frame of the keel — water comes from the direction of motion and leaves along the keel. The direction of the force acting on the keel is opposite to the direction of deflection of water by the keel. That force, $\mathbf{F}_{keel}$, acting mainly to the side (left in the Figure), must be counteracted by the force acting by the wind on the sail. Wind leaves along the sail and its deflection determines $\mathbf{F}_{sail}$. For a board in a steady motion, the vector $\mathbf{F}_{keel} + \mathbf{F}_{sail}$ points in the direction of motion and it is balanced by the drag force. Decreasing the drag one can in principle move faster than the wind since $\mathbf{F}_{sail}$ does not depend on the board speed (as long as one keeps the wind perpendicular *in the reference frame of the board*). On the contrary, when the board moves downwind, it cannot move faster than the wind.



For details, see B.D. Anderson, The physics of sailing, Physics Today, February 2008, 38-43.

1.13.

Simple answer. If droplet was a solid sphere, one uses the Stokes force and gets the steady fall velocity from the force balance

$$6\pi R\eta_a u = mg\,, \quad u = \frac{2\rho_w g R^2}{9\eta_a} \simeq 1.21\,\mathrm{cm/s}\,. \tag{5.16}$$

Justification and correction. The Reynolds number is Re $\simeq 0.008$, which justifies our approach and guarantees that we can neglect finite-*Re* corrections. Note that Re $\propto vR \propto R^3$, so that Re $\simeq 1$ already for $R = 0.05\,\mathrm{mm}$. Sphericity is maintained by surface tension, the relevant parameter is the ratio of the viscous stress $\eta_w u/R$ to the surface tension stress α/R, that ratio is $\eta_w u/\alpha \simeq 0.00017$ for $\alpha = 70\,g/s^2$. Another unaccounted phenomenon is an internal circulation in a liquid droplet. Viscous stress tensor $\sigma_{r\theta}$ must be continuous through the droplet surface, so that the velocity inside can be estimated as the velocity outside times the small factor $\eta_a/\eta_w \simeq 0.018 \ll 1$, which is expected to give 2% correction to the force and to the fall velocity. Let us calculate this. The equation for the motion of the fluid inside is the same as outside: $\Delta^2\nabla f = 0$, see (1.46). The solution regular at infinity is (1.47) i.e.

$$\mathbf{v}_a = \mathbf{u} - a\frac{\mathbf{u} + \hat{\mathbf{n}}(\mathbf{u}\cdot\hat{\mathbf{n}})}{r} + b\frac{3\hat{\mathbf{n}}(\mathbf{u}\cdot\hat{\mathbf{n}} - \mathbf{u})}{r^3}$$

while the solution regular at zero is $f = Ar^2/4 + Br^4/8$, which gives

$$\mathbf{v_w} = -\mathbf{Au} + \mathbf{Br^2}(\hat{\mathbf{n}}(\mathbf{u}\cdot\hat{\mathbf{n}}) - \mathbf{2u}\,.$$

Four boundary conditions on the surface (zero normal velocities and continuous tangential velocities and stresses) fix the four constants A, A, a, b and gives the drag force

$$F = 8\pi a\eta u = 2\pi u\eta_a R\frac{2\eta_a + 3\eta_w}{\eta_a + \eta_w}\,. \tag{5.17}$$

which leads to

$$u = \frac{2\rho R^2 g}{3\eta_a}\left(\frac{3\eta_a + 3\eta_w}{2\eta_a + 3\eta_w}\right) \simeq \frac{2\rho R^2 g}{9\eta_a}\left(1 + \frac{1}{3}\frac{\eta_a}{\eta_w}\right) \simeq 1.22\,\frac{\mathrm{cm}}{\mathrm{s}}$$

Inside circulation acts as a lubricant decreasing drag and increasing fall velocity. In reality, however, water droplets often fall as solid spheres because of a dense "coat" of dust particles accumulated on the surface.

1.14.

Basic Solution. Denote the droplet radius r and its velocity v. We need to write conservation of mass $\dot{r} = Av$ and the equation of motion, $dr^3v/dt = gr^3 - Bvr$, where we assumed a low Reynolds number and used the Stokes expression for the friction force. Here A, B are some constants. One can exclude v from here but the resulting second-order differential equation is complicated. To simplify, we assume that the motion is quasi-steady so that gravity and friction almost balance each other. That requires $\dot{v} \ll g$ and gives $v \approx gr^2/B$. Substituting that into the mass conservation gives $dr/dt = Agr^2/B$. The solution of this equation gives an explosive growth of the particle radius and velocity: and $r(t) = r_0/(1 - r_0Agt/B)$. This solution is true as long as $\dot{v} \ll g$ and $Re = vr/\nu \ll 1$.

Detailed Solution. Denote ρ_w, ρ, ρ_v respectively densities of the liquid water, air and water vapor. Assume $\rho_w \gg \rho \gg \rho_v$. Mass conservation gives $dm = \rho_v \pi r^2 v dt = \rho_w 4\pi r^2 dr$ so that $dr/dt = v\rho_v/4\rho_w$. Initially, we may consider low-Re motion so that the equation of motion is as follows: $dr^3v/dt = gr^3 - 9\nu\rho vr/2\rho_0$. Quasi-static approximation is $v \approx 2gr^2\rho_w/9\nu\rho$ according to (5.16), which gives the equation $dr/dt = gr^2\rho_v/18\nu\rho$ independent of ρ_w. The solution of this equation gives an explosive growth of the particle radius and velocity:

$$r(t) = r_0 \left(1 - \frac{\rho_v}{\rho}\frac{r_0 gt}{18\nu}\right)^{-1}, \quad v(t) = \frac{\rho_w}{\rho}\frac{2gr_0^2}{9\nu}\left(1 - \frac{\rho_v}{\rho}\frac{r_0 gt}{18\nu}\right)^{-2}.$$

This solution is true as long as $\dot{v} = 4gr\dot{r}\rho_0/9\nu\rho = 2g^2r^3\rho_v\rho_0/81\nu^2\rho^2 \ll g$. Also, when $r(t)v(t) \simeq \nu$ the regime changes so that $mg = C\rho\pi r^2v^2$, $v \propto \sqrt{r}$ and $r \propto t^2$, $v \propto t$.

1.15

Pressure is constant along the free jet boundaries and so the velocity is constant as well. Therefore, the asymptotic velocities in the outgoing jets are the same as in the incoming jets. Conservation of mass, energy and horizontal momentum give for the left/right jets respectively

$$h_l = h(1 + \cos\theta_0), \quad h_r = h(1 - \cos\theta_0),$$

where h is the width of the incoming jet. Therefore, a fraction $(1 - \cos\theta_0)/2$ of the metal cone is injected into the forward jet.

One can describe the whole flow field in terms of the complex velocity $\mathbf{v}$ which changes inside a circle whose radius is the velocity at infinity, u (see, e.g. Chapter XI of [13]). On the circle, the stream function is piecewise constant with the jumps equal to the jet fluxes:

$$\psi = 0 \quad \text{for} \quad 0 \le \theta \le \theta_0\,, \qquad \psi = -hu \quad \text{for} \quad \theta_0 \le \theta \le \pi\,,$$
$$\psi = (h_l - h)u \quad \text{for} \quad \pi \le \theta \le 2\pi - \theta_0\,, \quad \text{etc.}$$

We can now find the complex potential everywhere in the circle by using the Schwartz formula:

$$\begin{aligned} w(\mathbf{v}) &= \frac{i}{2\pi}\int_0^{2\pi} \psi(\theta)\frac{u\exp(i\theta)+\mathbf{v}}{u\exp(i\theta)-\mathbf{v}}\,d\theta \\ &= \frac{u}{\pi}\Bigg\{h_r \ln\left(1-\frac{\mathbf{v}}{u}\right) + h_l \ln\left(1+\frac{\mathbf{v}}{u}\right) \\ &\quad - h\ln\left[\left(1-\frac{\mathbf{v}}{u}e^{i\theta_0}\right)\left(1-\frac{\mathbf{v}}{u}e^{-i\theta_0}\right)\right]\Bigg\}\,. \end{aligned}$$

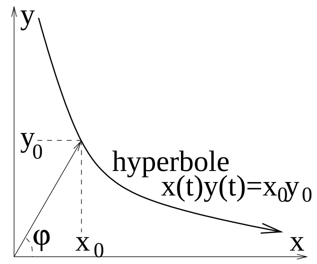
To relate the space coordinate z and the velocity $\mathbf{v}$ we use $\mathbf{v} = -dw/dz$ so that one needs to differentiate $w(\mathbf{v})$ and then integrate once the relation $dz/d\mathbf{v} = -\mathbf{v}^{-1}dw/d\mathbf{v}$ using $z = 0$ at $\mathbf{v} = 0$:

$$\begin{aligned} \frac{\pi z}{h} &= (1-\cos\theta_0)\ln\left(1-\frac{\mathbf{v}}{u}\right) - (1+\cos\theta_0)\ln\left(1+\frac{\mathbf{v}}{u}\right) \\ &\quad + e^{i\theta_0}\ln\left(1-\frac{\mathbf{v}}{u}e^{i\theta_0}\right) + e^{-i\theta_0}\ln\left(1-\frac{\mathbf{v}}{u}e^{-i\theta_0}\right)\,. \end{aligned}$$

2.1.

We have seen in Sect. 1.2.1 that in a locally smooth flow, fluid elements either stretch/contract exponentially in a strain-dominated flow or rotate in a vorticity-dominated flow. This is true also for the flows in phase space, discussed at the beginning of Sect. 2.2.

i) Since $x(t) = x_0\exp(\lambda t)$ and $y(t) = y_0\exp(-\lambda t) = x_0y_0/x(t)$ then every streamline (and trajectory) is a hyperbole. A vector initially forming an angle φ with the x axis will be stretched after time T if $\cos\varphi \ge [1+\exp(2\lambda T)]^{-1/2}$, i.e. the fraction of stretched directions is larger than half. That means, in particular, that if one randomly changes directions after some times, still the net effect is stretching.



ii) The eigenvectors of this problem evolve according to $\exp(\pm i\Omega t)$ where

$$\Omega^2 = 4\omega^2 - \lambda^2 \ . \tag{5.18}$$

We see that fluid rotates inside vorticity-dominated (elliptic) regions and is monotonically deformed in strain-dominated (hyperbolic) regions. The marginal case is a shear flow (see Figure 1.6) where $\lambda = 2\omega$ and the distances grow linearly with time.

In a random flow (either in real space or in phase space), a fluid element visits on its way many different elliptic and hyperbolic regions. After a long random sequence of deformations and rotations, we find it stretched into a thin strip. Of course, this is a statistical statement meaning that the probability to find a ball turning into an exponentially stretching ellipse goes to unity as time increases. The physical reason for it is that substantial deformation appears sooner or later. To reverse it, one needs to meet an orientation of stretching/contraction directions in a narrow angle (defined by the ellipse eccentricity), which is unlikely. Randomly oriented deformations on average continue to increase the eccentricity. After the strip length reaches the scale of the velocity change (when one already cannot approximate the velocity by a linear profile), strip starts to fold continuing locally the exponential stretching. Eventually, one can find the points from the initial ball everywhere which means that the flow is mixing.

2.2.

Dimensional reasoning. With six parameters, g,β,Θ,h,ν, χ and three independent dimensions, *cm*, *sec* and Kelvin degree, one can combine three different dimensionless parameters, according to the π-theorem of Sect. 1.4.4. That is too many parameters for any meaningful study.

Basic physical reasoning suggests that the first three parameters can come up only as a product, $g\beta\Theta$, which is a buoyancy force per unit mass (the density cannot enter because there is no other parameter having mass units). We now have four parameters, $\beta g\Theta, h, \nu, \chi$ and two independent dimensions, *cm*, *sec*, so that we can make two dimensionless parameters. The first one characterizes the medium and is called the Prandtl number:

$$Pr = \nu/\chi \ . \tag{5.19}$$

The same molecular motion is responsible for the diffusion of momentum by viscosity and the diffusion of heat by thermal conductivity. Nevertheless, the Prandtl number varies greatly from substance to substance. For gases, one can estimate χ as the thermal velocity times the mean free path, exactly like for viscosity in Section 1.4.3, so that the Prandtl number is always of order unity. For liquids, Pr varies from 0.044 for mercury to 6.75 for water and 7250 for glycerol.

The second parameter can be constructed in infinitely many ways as it can contain an arbitrary function of the first parameter. One may settle on any such parameter claiming that it is a good control parameter for a given medium (for fixed Pr). However, one can do better than that and find the control parameter which is the same for all media (i.e. all Pr). The physical reasoning helps one to choose the right parameter. It is clear that convection can occur when the buoyancy force, $\beta g\Theta$, is larger than the friction force, $\nu v/h^2$. It may seem that taking velocity v small enough, one can always satisfy that criterium. However, one must not forget that as the hotter fluid rises it looses heat by thermal conduction and gets more dense. Our estimate of the buoyancy force is valid as long as the conduction time, h^2/χ, exceeds the convection time, h/v, so that the minimal velocity is $v \simeq \chi/h$. Substituting that velocity into the friction force, we obtain the correct dimensionless parameter as the force ratio which is called the Rayleigh number:

$$Ra = \frac{g\beta\Theta h^3}{\nu\chi} \ . \tag{5.20}$$

Sketch of a theory. The temperature T satisfies the linear convection-conduction equation

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \chi\Delta T \ . \tag{5.21}$$

For the perturbation $\tau = (T - T_0)/T_0$ relative to the steady profile $T_0(z) = \Theta z/h$, we obtain

$$\frac{\partial \tau}{\partial t} - v_z\Theta/h = \chi\Delta\tau \ . \tag{5.22}$$

Since the velocity is itself a perturbation, so that it satisfies the incompressibility condition, $\nabla \cdot \mathbf{v} = 0$, and the linearized Navier-Stokes equation with the buoyancy force:

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla W + \nu \Delta \mathbf{v} + \beta \tau \mathbf{g} \ , \tag{5.23}$$

where W is the enthalpy perturbation. Of course, the properties of the convection above the threshold depend on both parameters, Ra and Pr, so that one cannot eliminate one of them from the system of equations. If, however, one considers the convection threshold where $\partial \mathbf{v}/\partial t = \partial \tau/\partial t = 0$, then one can choose the dimensionless variables $\mathbf{u} = \mathbf{v}h/\chi$ and $w = Wh^2/\nu\chi$ such that the system contains only Ra:

$$-u_z = \Delta \tau \ , \quad \nabla \cdot \mathbf{v} = 0, \quad \frac{\partial w}{\partial z} = \Delta u_z + \tau Ra \ , \frac{\partial w}{\partial x} = \Delta u_x \ . \tag{5.24}$$

Solving this with proper boundary conditions, for eigenmodes built out of $\sin(kx), \cos(kx)$ and $\sinh(qz), \cosh(qz)$ (which describe rectangular cells or rolls), one obtains Ra_{cr} as the lowest eigenvalue, see e.g. [10], Sect. 57.

Note the difference between the sufficient condition for convection onset, $Ra > Ra_{cr}$, formulated in terms of the control parameter Ra, which is global (a characteristics of the whole system), and a local necessary condition (1.9) found in Sect. 1.1.3.

2.3.

Consider the continuity of the fluxes of mass, normal momentum and energy:

$$\rho_1 w_1 = \rho_2 w_2 \, , \quad P_1 + \rho_1 w_1^2 = P_2 + \rho_2 w_2^2 \, , \tag{5.25}$$

$$W_1 + w_1^2/2 = \frac{\rho_2 w_2}{\rho_1 w_1}\big(W_2 + w_2^2/2\big) = W_2 + w_2^2/2 \ . \tag{5.26}$$

Excluding w_1, w_2 from (5.25),

$$w_1^2 = \frac{\rho_2}{\rho_1}\frac{P_2 - P_1}{\rho_2 - \rho_1} \ , \quad w_2^2 = \frac{\rho_1}{\rho_2}\frac{P_2 - P_1}{\rho_2 - \rho_1} \ , \tag{5.27}$$

and substituting it into the Bernoulli relation (5.26) we derive the relation called the shock adiabate:

$$W_1 - W_2 = \frac{1}{2}\big(P_1 - P_2\big)(V_1 + V_2) \ . \tag{5.28}$$

For given pre-shock values of P_1, V_1, it determines the relation between P_2 and V_2. Shock adiabate is determined by two parameters, P_1, V_1, as distinct from the constant-entropy (Poisson) adiabate PV^γ =const, which is determined by a single parameter, entropy. Of course, the after-shock parameters are completely determined if all the three pre-shock parameters, P_1, V_1, w_1, are given.

Substituting $W = \gamma P/\rho(\gamma-1)$ into (5.28) we obtain the shock adiabate for a polytropic gas in two equivalent forms:

$$\frac{\rho_2}{\rho_1} = \frac{\beta P_1 + P_2}{P_1 + \beta P_2}\,, \quad \frac{P_2}{P_1} = \frac{\rho_1 - \beta\rho_2}{\rho_2 - \beta\rho_1}\,, \quad \beta = \frac{\gamma-1}{\gamma+1}\,. \tag{5.29}$$

Since pressures must be positive, the density ratio ρ_2/ρ_1 must not exceed $1/\beta$ (4 and 6 for monatomic and diatomic gases respectively). If the pre-shock velocity w_1 is given, the dimensionless ratios ρ_2/ρ_1, P_2/P_1 and $\mathcal{M}_2 = w_2/c_2 = w_2\sqrt{\rho_2/\gamma P_2}$ can be expressed via the dimensionless Mach number $\mathcal{M}_1 = w_1/c_1 = w_1\sqrt{\rho_1/\gamma P_1}$ by combining (5.27,5.29):

$$\frac{\rho_1}{\rho_2} = \beta + \frac{2}{(\gamma+1)\mathcal{M}_1^2}\,, \quad \frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2}{\gamma+1} - \beta\,, \quad \mathcal{M}_2^2 = \frac{2+(\gamma-1)\mathcal{M}_1^2}{2\gamma\mathcal{M}_1^2 + 1 - \gamma}\,. \tag{5.30}$$

To have a subsonic flow after the shock, $\mathcal{M}_2 < 1$, one needs a supersonic flow before the shock, $\mathcal{M}_1 > 1$.

Thermodynamic inequality $\gamma > 1$ guarantees the regularity of all the above relations. The entropy is determined by the ratio P/ρ^γ, it is actually proportional to $\log(P/\rho^\gamma)$. Using (5.30) one can show that $s_2 - s_1 \propto \ln(P_2\rho_1^\gamma/P_1\rho_2^\gamma) > 0$ which corresponds to an irreversible conversion of the mechanical energy of the fluid motion into the thermal energy of the fluid.

See Sects. 85,89 of [10] for more details.

2.4.

Simple estimate. We use a single shock, which has the form $u = -v\tanh(vx/2\nu)$ in the reference frame with the zero mean velocity. We then simply get $\langle u^2u_x^2\rangle = 2v^5/15L$ so that

$$\epsilon_4 = 6\nu[\langle u^2u_x^2\rangle + \langle u^2\rangle\langle u_x^2\rangle] = 24v^5/5L\,.$$

Substituting $v^5/L = 5\epsilon_4/24$ into $S_5 = -32v^5x/L$ we get

$$S_5 = -20\epsilon_4 x/3 = -40\nu x[\langle u^2u_x^2\rangle + \langle u^2\rangle\langle u_x^2\rangle]\,. \tag{5.31}$$

Sketch of a theory. One can also derive the evolution equation for the structure function, analogous to (2.10) and (2.32). Consider

$$\partial_t S_4 = -(3/5)\partial_x S_5 - 24\nu[\langle u^2u_x^2\rangle + \langle u_1^2u_{2x}^2\rangle] + 48\nu\langle u_1u_2u_{1x}^2\rangle + 8\nu\langle u_1^3u_{2xx}\rangle\,.$$

Since the distance x_{12} is in the inertial interval then we can neglect $\langle u_1^3 u_{2xx}\rangle$ and $\langle u_1 u_2 u_{1x}^2\rangle$, and we can put $\langle u_1^2 u_{2x}^2\rangle \approx \langle u^2\rangle\langle u_x^2\rangle$. Assuming that

$$\partial_t S_4 \simeq S_4 u/L \ll \epsilon_4 \simeq u^5/L\,,$$

we neglect the lhs and obtain (5.31). Generally, one can derive

$$S_{2n+1} = -4\epsilon_n x \frac{2n+1}{2n-1}$$

2.5.

We write the equation of motion (1.29):

$$\frac{d}{dt}\rho_0 V(t) u = \rho V(t)\dot{v} + \frac{d}{dt}\rho V(t)\frac{v-u}{2}\ . \tag{5.32}$$

The solution is

$$u(t) = a\sin\omega t\frac{3\rho}{\rho+2\rho_0} + \frac{a}{\omega}(\cos\omega t - 1)\frac{2\rho}{\rho+2\rho_0}\frac{\alpha}{V(0)-\alpha t}\ . \tag{5.33}$$

It shows that the volume change causes the phase shift and amplitude increase in oscillations and a negative drift. The solution (5.33) looses validity when u increases to the point where $ku \simeq \omega$.

2.6.

Rough estimate can be obtained even without proper understanding the phenomenon. The effect must be independent of the phase of oscillations i.e. of the sign of A, therefore, the dimensionless parameter A^2 must be expressed via the dimensionless parameter $P_0/\rho gh$. When the ratio $P_0/\rho gh$ is small we expect the answer to be independent of it, i.e. the threshold to be of order unity. When $P_0/\rho gh \gg 1$ then the threshold must be large as well since large P_0 decreases any effect of bubble oscillations, so one may expect the threshold at $A^2 \simeq P_0/\rho gh$. One can make a simple interpolation between the limits

$$A^2 \simeq 1 + \frac{P_0}{\rho gh}\ . \tag{5.34}$$

Qualitative explanation of the effect invokes compressibility of the bubble (Bleich, 1956). Vertical oscillations of the vessel cause periodic variations of the gravity acceleration. Upward acceleration of the vessel causes downward gravity which provides for the buoyancy force directed up and vice versa for another half period. It is important that related variations of the buoyancy force do not average to zero since the volume of the bubble oscillates too because of oscillations of pressure due to column of liquid above. The volume is smaller when the vessel accelerates upward since the effective gravity and pressure are larger then. As a result, buoyancy force is lower when the vessel and the bubble accelerate up. The net result of symmetric up-down oscillations is thus downward force acting on the bubble. When that force exceeds the upward buoyancy force provided by the static gravity g, the bubble sinks.

Theory. Consider an ideal fluid where there is no drag. The equation of motion in the vessel reference frame is obtained from (1.29,5.32) by adding buoyancy and neglecting the mass of the air in the bubble:

$$\frac{d}{2dt}V(t)u = V(t)G(t)\,, \quad G(t) = g + \ddot{x}\ . \tag{5.35}$$

Here $V(t)$ is the time-dependent bubble volume. Denote z the bubble vertical displacement with respect to the vessel, so that $u = \dot{z}$, positive upward . Assume compressions and expansions of the bubble to be adiabatic, which requires the frequency to be larger than thermal diffusivity κ divided by the bubble size a. If, on the other hand, the vibration frequency is much smaller than the eigenfrequency (5.12) (sound velocity divided by the bubble radius) then one can relate the volume $V(t)$ to the pressure and the coordinate at the same instant of time:

$$PV^\gamma(t) = [P_0 + \rho G(h - z)]V^\gamma = (P_0 + \rho gh)V_0^\gamma\ .$$

Assuming small variations in z and $V = V_0 + \delta V \sin(\omega t)$ we get

$$\delta V = V_0 \frac{A\rho gh}{\gamma(P_0 + \rho gh)}\ . \tag{5.36}$$

The net change of the bubble momentum during the period can be obtained by integrating (5.35):

$$\int_0^{2\pi/\omega} V(t')G(t')\,dt' = \frac{2\pi V_0 g}{\omega}(1 - \delta V A/2V_0) + o\big(A^2\big)\ . \tag{5.37}$$

The threshold corresponds to zero momentum transfer, which requires $\delta V = 2V_0/A$. According to (5.36), that gives the following answer:

$$A^2 = 2\gamma\left(1 + \frac{P_0}{\rho gh}\right)\ . \tag{5.38}$$

At this value of A, the equation (5.35) has an oscillatory solution $z(t) \approx -(2Ag/\omega^2)\sin(\omega t)$ valid when $Ag/\omega^2 \ll h$. Another way to interpret (5.38) is to say that it gives the depth h where small oscillations are possible for a given amplitude of vibrations A. Moment reflection tells that these oscillations are unstable i.e. bubbles below h has their downward momentum transfer stronger and will sink while bubbles above rise.

Notice that the threshold value does not depend on the frequency and the bubble radius (under an implicit assumption $a \ll h$). However, neglecting viscous friction is justified only when the Reynolds number of the flow around the bubble is large: $a\dot{z}/\nu \simeq aAg/\omega\nu \gg 1$, where ν is the kinematic viscosity of the liquid. Different treatment is needed for small bubbles where inertia can be neglected comparing to viscous friction and (5.35) is replaced by

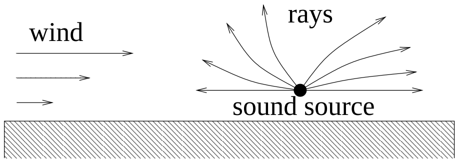
$$4\pi\nu a(t)\dot{z} = V(t)G(t) = 4\pi a^3(t)G/3 \ . \tag{5.39}$$

Here we used the expression (5.17) for the viscous friction of fluid sphere with the interchange water $\leftrightarrow$ air. Dividing by $a(t)$ and integrating over period we get the velocity change proportional to $1 - \delta aA/a = 1 - \delta VA/3V_0$. Another difference is that $a^2 \ll \kappa/\omega$ for small bubbles, so that heat exchange is fast and we must use isothermal rather than adiabatic equation of state i.e. put $\gamma = 1$ in (5.36). That gives the threshold which is again independent of the bubble size:

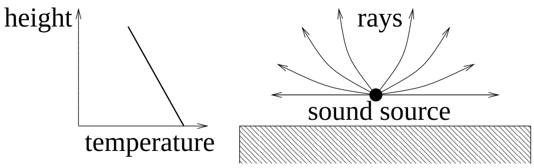
$$A^2 = 3\left(1 + \frac{P_0}{\rho g h}\right) \ . \tag{5.40}$$

2.7.

"The answer is blowing in the wind". The first that comes to mind is that for upwind propagation the distance is larger by the factor $1 + v/c$ where v is the wind speed. Viscous dissipation in the air decreases the intensity q with the distance r according to $q \propto \exp(-\nu\omega^2 r/c^3)$. Another factor is spreading of the acoustic energy flux over the half-sphere of an increasing radius: $q \propto r^{-2}$. Note however that near the ground (where most of our shouting happens), the wind speed is generally less than $30\,m/sec$ so that $v/c < 0.1$ and can be treated as a small factor. Therefore, the difference between upwind and downwind intensities would be of order v/c i.e. small. The real reason for the fast intensity drop upwind is the inhomogeneous wind speed profile near the ground. That bends sound rays which encounter stronger winds when they are further from the ground. As shown in the figure, that refraction makes the spreading of acoustic rays near the ground much faster upwind than downwind.

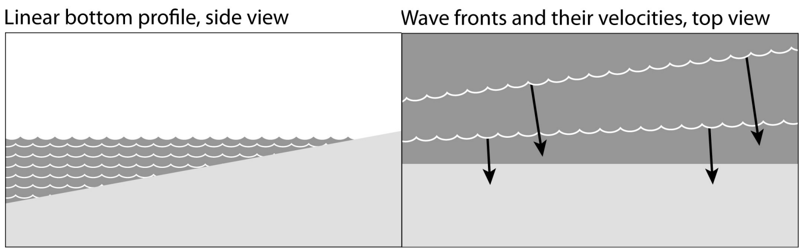


Similar upward curvature of sound rays (now isotropic in the horizontal plane) is caused by the temperature gradient in the atmosphere. For sound propagating at some angle to the ground, the part of the constant-phase surface which is higher moves slower which turns the surface up. The speed of sound is proportional to the square root of temperature in Kelvin degrees. When temperature drops by 6.5 degrees per kilometer (see Section 1.1.3) the speed of sound decreases by about $5\,m/sec$. Such refraction creates so-called "zone of silence" around the source on the ground; similar phenomenon is also known from optics where rays bend towards more optically dense medium.



3.1.

Wave velocity, $v_g = \sqrt{gh}$, grows with the depth. Depth usually decreases as one approaches the shore. Therefore, even if the wave comes from the deep at an angle, the parts of the wave fronts that are in deeper water move faster changing the orientation of the fronts (like in Exercise 2.7).



3.2.

The length of the wave packet is $L = N\lambda = 2\pi N/k$. The wave packet propagates with $v_{\rm group}$. The float will be disturbed by the wave packet during the time: $\tau = L/v_{\text{group}} = 2\pi N/kv_{\text{group}}$. For a quasi monochromatic wave packet $T = 2\pi/\omega(k)$, the number of "up and down" motions of the wave packet is $n = \tau/T = Nv_{\text{phase}}/v_{\text{group}}$. For gravity waves on deep water $\omega = \sqrt{gk}$ we have $n = 2N$. For gravity-capillary waves on deep water $\omega = \sqrt{(gk + \sigma k^3/\rho)}$ we have

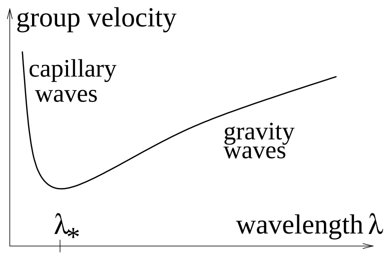
$$n = 2N\frac{(gk + \sigma k^3/\rho)}{k(gk + 3\sigma k^2/\rho)}$$

and for purely capillary waves $\omega = \sqrt{\sigma k^3/\rho}$, $n = 2N/3$.

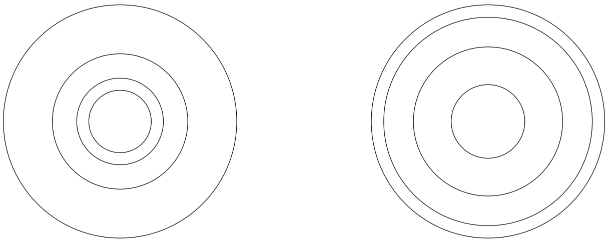
Remark. That was linear consideration (due to [20]). Nonlinear Stokes drift causes the float to move with the wave so that the relative speed is less: $v_{\text{group}} \to v_{\text{group}}[1 - (ak)^2]$.

3.3.

Qualitative analysis and simple estimate. The group velocity of surface waves depends on their wavelength non-monotonically as shown in the Figure:



The stone excites waves with the wavelengths exceeding its size. Indeed, a Fourier image of a bump with the width l is nonzero for wavenumbers less that $1/l$. Therefore, the wave pattern on a water surface depends on the relation between the size of the initial perturbation and λ_*. If we drop a large stone (with a size far exceeding λ_*) it will mostly generate gravity waves which are the longer the faster. As a result, the wavelength increases with the distance from the origin that is the larger circles are progressively sparser. On the contrary, small stones and raindrops can generate also capillary waves with shortest being fastest so that the distance between circles decreases with the radius. Since there is a minimal wave speed $v_* \simeq (g\alpha/\rho)^{1/4} \simeq 17\,\text{cm/s}$, then there are no waves inside the circle with the radius v_*t, this circle is a caustic. General consideration of caustics including stone-generated surface wave can be found in [11], Sect. 4.11.



Both gravity and capillary waves can be seen in Figure 5.2. For a given speed exceeding v_*, there are two distinct wavelengths that propagate together.

Figure 5.2 Long gravity waves and short capillary ripples propagating together.

Sketch of a theory for gravity waves is based on Sections 3.1.1 and 3.1.4. The perturbation can be considered as a force localized both in space and in time. Since we aim to describe time intervals far exceeding the time of stone sinking and wavelengths far exceeding its size we model this force by the product of delta functions $\delta(t)\delta(\mathbf{r})$. Adding such force to the equation of motion (3.4) we get

$$\frac{\partial v_z}{\partial t} = \frac{\partial^2 \zeta}{\partial t^2} = -g\frac{\partial \zeta}{\partial z} + \delta(t)\delta(\mathbf{r})$$

which gives $\zeta_k = i\exp(-i\sqrt{gk}\,t)/\sqrt{gk}$ and

$$\zeta(r,t) = \frac{i}{8\pi^2\sqrt{g}}\int \sqrt{k}dkd\theta\exp(ikr\cos\theta - i\sqrt{gk}\,t) \tag{5.41}$$

$$= \frac{i}{4\pi^2\sqrt{g}}\int_0^\infty dk\sqrt{k}J_0(kr)\exp(-i\sqrt{gk}\,t) \equiv \frac{i}{2\pi g^2t^3}\Phi\left(\frac{r}{gt^2}\right) ,$$

$$\Phi(y) = \int_0^\infty dzz^2J_0(yz^2)\exp(-iz) \ . \tag{5.42}$$

Here J_0 is the Bessel function. One can see that the crests accelerate with the gravity acceleration g. Before the leading crest, for $r \gg gt^2$, the surface is steady:

$$\zeta(r) \propto g^2t^{-3}(gt^2/r)^{3/2} \propto g^{-1/2}r^{-3/2} \ .$$

Behind the leading crest, for $r \ll gt^2$, the main contribution into (5.41) is given by $\theta = 0$, $r = \omega'(k)t = t\sqrt{g/k}/2$, that is by $k = gt^2/4r^2$, so that

$$\zeta(r,t) \propto \sin(gt^2/4r) \ .$$

The crests are at the following radial positions:

$$r_n = \frac{gt^2}{2\pi(4n+1)} \ .$$

For details, see http://www.tcm.phy.cam.ac.uk/˜ dek12/, Tale 8 by D. Khmelnitskii. General consideration that accounts for a finite stone size and capillary waves can be found in Sect. 17.09 of Jeffreys, H. and Swirles, B. (1966) *Methods of Mathematical Physics* (Cambridge Univ. Press, Cambridge).

3.4.

Hamiltonian is real therefore $A(k)$ is real, we denote A_1, A_2 respectively its symmetric and antisymmetric parts. Also, $B(k) = B(-k)$ and we can consider $B(k)$ real, absorbing its phase into $b(k)$. The same is true for the transformation coefficients u, v which also could be chosen real (as is also clear below). The canonicity of u-v transformation requires

$$u^2(k) - v^2(k) = 1\,, \quad u(k)v(-k) = u(-k)v(k)$$

and suggests the substitution

$$u(k) = \cosh[\zeta(k)]\,, \quad v(k) = \sinh[\zeta(k)] \ .$$

In this terms, the transformation equations take the form

$$\omega_k = A_2 + A_1/\cosh(2\zeta)\,, \quad A_1\sinh(2\zeta) = B[\cosh^2(\zeta) + \sinh^2(\zeta)] \ .$$

That gives

$$\omega_k = A_2(k) + \operatorname{sign} A_1(k)\sqrt{A_2^2(k) - B^2(k)}\ .$$

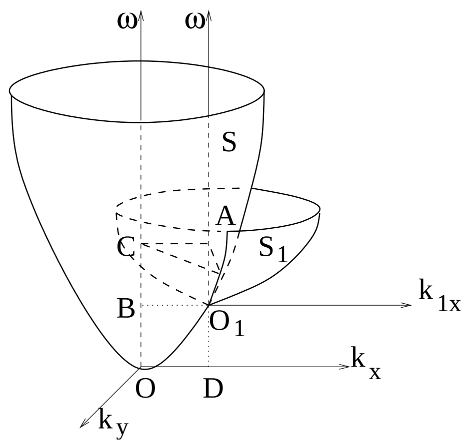
If $A_1(k)$ turns into zero when $B(k) \neq 0$, we have complex frequency which describes exponential growth of waves i.e. instability. The quadratic Hamiltonian can be reduced to the form

$$\mathcal{H}_2 = \int C(k)[a(k)a(-k) + a^*(k)a^*(-k)]\, d\mathbf{k}\,,$$

which describes the creation of the pair of quasi-particles from vacuum and the inverse process, see [21], Sec. 1.1.

3.5

For two-dimensional wave vectors, $\mathbf{k} = \{k_x, k_y\}$, the function $\omega(\mathbf{k})$ determines a surface in the three-dimensional space ω, k_x, k_y. In an isotropic case, $\omega(k)$ determines a surface of revolution. Consider two such surfaces: S determined by $\omega(k)$ and S_1 by $\omega(k_1)$. The possibility to find such $\mathbf{k}_1, \mathbf{k}$ that $\omega(\mathbf{k}_1 + \mathbf{k}) = \omega(k_1) + \omega(k)$ means that the second surface must be shifted up by $\omega(k)$ and right by $\mathbf{k}$ and that the two surfaces must intersect:



All three points with the coordinates $\{\omega, k\}$, $\{\omega_1, k_1\}$ and $\{\omega(\mathbf{k} + \mathbf{k}_1), \mathbf{k}+\mathbf{k}_1\}$ must lie on the intersection line O_1A. For example, $OD = k$, $OB = \omega$, $BC = \omega_1$. For intersection, $\omega(k)$ must define a convex surface. For power laws, $\omega(k) \propto k^\alpha$, that requires $\alpha > 1$. For $\alpha = 1$, only waves with collinear wave vectors can interact resonantly.

3.6.

The equation for the standing soliton analogous to (3.32) is now

$$\omega''\Delta A = 2T(A^3 - A_0^2 A)\ .$$

In polar coordinates, $\Delta = r^{-1}\partial_r r\partial_r + r^{-2}\partial_\phi^2$. It is thus clear that the dependence on the angle ϕ makes the condensate amplitude turning into zero on the axis: $A \propto r$ as $r \to 0$. Denote $f(r/r_0) \equiv A/A_0$. Considering $r \to \infty$ one concludes that $r_0^2 = \omega''/(2TA^2)$ i.e. the amplitude and the size are inversely related as in 1d. The resulting equation in the variable $\xi = r/r_0$ has the following form:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) + \left(1 - \frac{1}{\xi^2}\right)f - f^3 = 0\,, \tag{5.43}$$

It can be solved numerically. When $\xi \to 0$ then $f \propto \xi$ for $\xi \to 0$. When $\xi \to \infty$, $f = 1 + \delta f$ and in a linear approximation we get

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\delta f}{d\xi}\right) - \frac{1}{\xi^2} - 2\delta f - \frac{1}{\xi^2}\delta f = 0, \tag{5.44}$$

which has a solution $\delta f = -1/2\xi^2$. The solution is a vortex because there is a line in space, $r = 0$, where $|\psi|^2 = 0$. Going around the vortex line, the phase acquires 2π. That this is a vortex is also clear from the fact that there is a current J (and the velocity) around it:

$$J_\phi \propto i\left(\psi\frac{\partial\psi^*}{r\partial\phi} - \psi^*\frac{\partial\psi}{r\partial\phi}\right) = \frac{A^2}{r}\,,$$

so that the circulation is independent of the distance from the vortex. Note that the kinetic energy of a single vortex diverges logarithmically with the system size (or with the distance to another parallel vortex line with an opposite circulation): $\int v^2\, r dr \propto \int dr/r$. That has many consequences in different fields of physics, in particularly in two dimensions (like Berezinski-Kosterlitz-Thouless phase transition).

3.7

Qualitative answer. To integrate a general system of $2n$ first-order differential equations one needs to know $2n$ conserved quantities. For Hamiltonian system though, Liouville theorem tells us that n independent integrals of motion is enough to make the system integrable and its motion equivalent to n oscillators. For three modes, the number of degrees of freedom is exactly equal to the number of the general integrals of motion: Hamiltonian H, total number of waves $P = \sum |a_m|^2$ and the momentum $M = \sum m|a_m|^2$. That means that the system is integrable: in the six-dimensional space of a_0, a_1, a_{-1}, every trajectory is a 3-torus.

Quantitative solution. The full system,

$$i\frac{da_0}{dt} = T\big(|a_0|^2 + 2|a_1|^2 + 2|a_{-1}|^2\big)a_0 + 2Ta_1a_{-1}a_0^* \ , \tag{5.45}$$

$$i\frac{da_1}{dt} = \beta a_1 + T\big(|a_1|^2 + 2|a_0|^2 + 2|a_{-1}|^2\big)a_1 + Ta_{-1}^*a_0^2 \ , \tag{5.46}$$

$$i\frac{da_{-1}}{dt} = \beta a_{-1} + T\big(|a_{-1}|^2 + 2|a_0|^2 + 2|a_1|^2\big)a_0 + Ta_1^*a_0^2 \ , \tag{5.47}$$

can be reduced explicitly to a single degree of freedom using the integrals of motion. Consider for simplicity $M = 0$ and put $a_0 = A_0\exp(i\theta)$, $a_1 = A_1\exp(i\theta_1)$ and $a_{-1} = A_1\exp(i\theta_{-1})$, where $A_1, \theta_1, \theta_{-1}$ are real. Introducing $\theta = 2\theta_0 - \theta_1 - \theta_{-1}$, the system can be written as follows:

$$\frac{dA_0}{dt} = -2\frac{dA_1}{dt} = -2TA_0A_1^2\sin\theta \ , \tag{5.48}$$

$$\frac{d\theta}{dt} = 2\beta + 2T(A_0^2 - A_1^2) + 2T(A_0^2 - 2A_1^2)\cos\theta \ . \tag{5.49}$$

This system conserves the Hamiltonian $H = 2\beta A_1^2 + TA_0^4/2 + 3TA_1^4 + 4TA_0^2A_1^2 + 2TA_1^2A_0^2\cos\theta$ and $P = A_0^2 + 2A_1^2$. Introducing $B = A_1^2$ one can write $H = (2\beta + TP/2)B + TB(P - 2B)(2\cos\theta + 3/2)$ and

$$\frac{dB}{dt} = 2TB(P - 2B)\sin\theta = -\frac{\partial H}{\partial\theta} \ , \tag{5.50}$$

$$\frac{d\theta}{dt} = 2\beta + 2T(P - 3B) + 2T(P - 4B)\cos\theta = \frac{\partial H}{\partial B} \ . \tag{5.51}$$

Expressing θ via H, B and substituting into (5.50), one can find the solution in terms of an elliptic integral. To understand the qualitative nature of the motion, it helps to draw the contours of constant H in coordinates $B, \cos\theta$. The evolution proceeds along those contours. The phase space is restricted by two straight lines $B = 0$ (which corresponds to $H = 0$) and $B = P/2$ (which corresponds to $A_0 = 0$ and $H = \beta P + TP^2/4$). It is instructive, in particular, to compare the phase portraits with and without the modulational instability. Indeed, exactly like in Sect. 3.3.2, we can consider $|a_0| \gg |a_1|, |a_{-1}|$ and $|a_0|^2 \approx P$, linearize (5.46,5.47) and for $a_1, a_{-1} \propto \exp(i\Omega t - iTPt)$ obtain $\Omega^2 = \beta(\beta + 2TP)$ in agreement with (3.5). When $\Omega^2 = \beta(\beta + 2TP) < 0$, there is an instability, while in the phase portrait of (5.50,5.51) there exist two fixed points at $B = 0$, $\cos\theta_0 = -1 - \beta/TP$, which are saddles connected by a separatrix. Even for trajectories that start infinitesimally close to $B = 0$, the separatrix makes it necessary to deviate to finite B, as seen from the right panel of Figure 5.3.

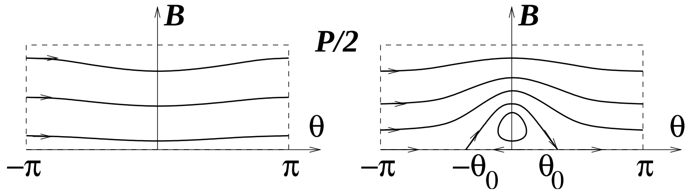
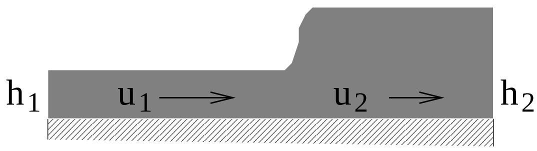


Figure 5.3 Sketch of the phase-space trajectories for large positive β (left, stable case) and $-2TP < \beta < 0$ (right, unstable case).

In the Hamiltonian integrable case, the separatrix is an unstable manifold for one fixed point and a stable manifold for another. When the system is perturbed (say, by small pumping and damping) and looses its Hamiltonian structure and integrability, the stable and unstable manifolds do not coincide anymore but have an infinite number of intersections. That separatrix splitting is responsible for creating a chaotic attractor.

3.8
As in any shock, mass and momentum are conserved but the mechanical energy is not.



Mass flux is ρuh so that the flux constancy gives $u_1h_1 = u_2h_2$. The momentum flux (1.17) includes the pressure which must be integrated over the vertical coordinate. Where the flow is uniform, the pressure can be taken hydrostatic, $P(z) = \rho g(h-z)$, which gives the momentum flux as follows:

$$\int_0^h \left[p(z) + \rho u^2\right] dz = \rho g h^2/2 + \rho h u^2 \ .$$

The momentum flux constancy gives $gh_1^2 + 2h_1u_1^2 = gh_2^2 + 2h_2u_2^2$. Substituting here $u_2 = u_1h_1/h_2$ we find $gh_1^2 - gh_2^2 = g(h_1-h_2)(h_1+h_2) = 2u_1^2h_1^2/h_2 - 2h_1u_1^2 = 2u_1^2h_1(h_1-h_2)/h_2$. That gives the height after the jump:

$$\frac{h_2}{h_1} = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2\frac{u_1^2}{gh_1}} \approx 1 + \frac{2\epsilon}{3} \ .$$

This is actually the first Rankine-Hugoniot relation (5.30) taken for $\gamma = 2$, $\mathcal{M}_1^2 = 1+\epsilon$.

The flux of mechanical energy is as follows

$$\int_0^h \Big[p(z) + \rho u^2/2 + \rho g z\Big] u dz = \rho g u h^2 + \rho h u^3/2 \ .$$

Here the first term is the work done by the pressure forces, the second term is the flux of the kinetic energy and the third term is the flux of the potential energy. The difference in energy fluxes is then the energy dissipation rate:

$$\begin{aligned}
&\rho g u_1 h_1^2 + \rho h_1 u_1^3/2 - \rho g u_2 h_2^2 - \rho h_2 u_2^3/2 = \rho g u_1 h_1 (h_1 - h_2) \\
&+ \frac{\rho u_1}{4}[2h_2 u_2^2 + g(h_2^2 - h_1^2)] - \frac{\rho u_2}{4}[2h_1 u_1^2 + g(h_1^2 - h_2^2)] \\
&= \rho g u_1 h_1 (h_1 - h_2) + \frac{\rho(u_1+u_2)}{4} g (h_2^2 - h_1^2) \\
&= \rho g u_1 (h_2 - h_1)^3/4h_2 \approx 2\epsilon^3 \rho u_1^5/27g \ . \qquad (5.52)
\end{aligned}$$

The reader is advised to make observations in his/her kitchen sink to appreciate how complicated and turbulent the real hydraulic jump is. It is amazing then that the dissipation is completely determined by the pre-shock flow via the conservation laws of the mass and momentum.

Comparing (5.52) with the Rankine-Hugoniot energy-continuity relation (5.26) from Exercise 2.3, the difference is that for a gas we wrote the total energy which is, of course, conserved. Another difference is that the pressure is determined by height (analog of density) for a shallow fluid, so that two conservation laws of mass and momentum are sufficient in this case to determine the velocity and height after the shock.

3.9.

For a running wave of the form $u(x - vt)$ the equation

$$u_t + u u_x + \beta u_{xxx} - \mu u_{xx} = 0$$

takes the form

$$-vu + u^2/2 + \beta u_{xx} - \mu u_x = \text{const} \ .$$

Integrate it using the boundary conditions and introduce $\tau = x\sqrt{v/\beta}$ and $q(\tau) = u/v$:

$$\ddot{q} = -2\lambda \dot{q} + q - q^2/2 \quad 2\lambda \equiv \mu/\sqrt{\beta v} \ . \qquad (5.53)$$

It is a Newton equation for a particle in the potential $U = q^3/6 - q^2/2$

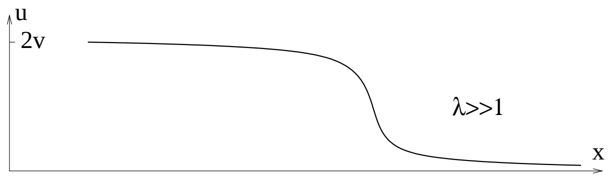
under the action of the friction force (we assume $\beta > 0$). The initial conditions in the distant past are $q(-\infty) = \dot{q}(-\infty) = \ddot{q}(-\infty) = 0$ so that the particle has zero energy: $E = \dot{q}/2 + q^3/6 - q^2/2$. We consider $q \geq 0$ since negative q goes to $q(\infty) = -\infty$, which gives non-physical u. Friction eventually bring particle to the minimum of the potential: $q \to 2$ as $\tau \to \infty$. Near the minimum we can use the harmonic approximation,

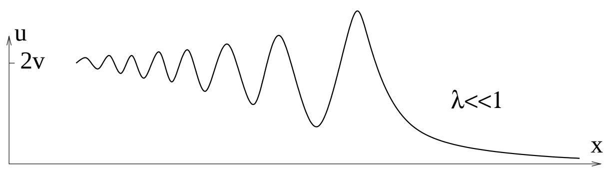
$$\ddot{q} \simeq -2\lambda\dot{q} - (q-2)\,, \tag{5.54}$$

which gives the law of decay: $q - 2 \propto \exp(rt)$ with $r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - 1}$. We see that for $\lambda < 1$ the asymptotic decay is accompanied by oscillations while for $\lambda \geq 1$ it is monotonic. For $\lambda \ll 1$, an initial evolution is described by the (soliton) solution without friction,

$$q = \frac{3}{\mathrm{ch}^2[(t-t_0)/2]}\,,$$

it brings the particle almost to $q \approx 3$, after which it goes back almost to $q \approx 0$, and then approach minimum oscillating. On the other hand, for very large friction, $\lambda \ll 1$, the particle goes to the minimum monotonically and the solution is close to the Burgers shock wave. It is clear that there exists an interval of $\lambda > 1$ where the solution (called collision-less shock) has a finite number of oscillations. It is an interesting question how large must be λ that no oscillations were present.





Note that adding small dispersion does not change the shock much while adding even small dissipation turns the soliton into the (collisionless) shock i.e. strongly changes the whole $x \to -\infty$ asymptotics; this is because friction is a symmetry-breaking perturbation as it breaks time-reversal symmetry.

3.10.

Simple estimate. Let us first use the virial theorem to estimate the dispersion relation for $v = 0$. This is done following (3.1,3.2). Both fluids are involved into the motion so that the kinetic energy per unit area can be estimated as $(\rho_1 + \rho_2)\omega^2 a^2 \lambda$, where a is the surface elevation. Under such elevation, the gravitational potential energy increases for the lower fluid and decreases for the upper fluid so that the net potential energy per unit area is $(\rho_2 - \rho_1)ga^2$. Potential energy of surface tension is the same $\alpha(a/\lambda)^2$. Virial theorem then generalizes (3.2) into

$$(\rho_1 + \rho_2)\omega^2 \simeq (\rho_2 - \rho_1)g\lambda^{-1} + \alpha\lambda^{-3} \ . \tag{5.55}$$

Complete solution. We need to combine the approach of Section 2.1 with that of Section 3.1. We introduce the velocity potentials ϕ_1 and ϕ_2 on both sides of the surface. Then the respective values of the pressure in the approximation linear with respect to the potentials ϕ_1, ϕ_2 and the elevation ζ are as follows:

$$P_1 = \rho_1 \left(g\zeta + \frac{\partial \phi_1}{\partial t} + v\frac{\partial \phi_1}{\partial x} \right) , \quad P_2 = \rho_2 \left(g\zeta + \frac{\partial \phi_2}{\partial t} \right) \ . \tag{5.56}$$

The pressure difference between the two sides is balanced by the surface tension as in (3.11):

$$\rho_2 \left(g\zeta + \frac{\partial \phi_2}{\partial t} \right) - \rho_1 \left(g\zeta + \frac{\partial \phi_1}{\partial t} + v\frac{\partial \phi_1}{\partial x} \right) = \alpha \frac{\partial^2 \zeta}{\partial x^2} \ . \tag{5.57}$$

We express the potential ϕ_1, ϕ_2 via the elevation ζ using the kinematic boundary conditions:

$$\frac{\partial \phi_2}{\partial z} = \frac{\partial \zeta}{\partial t} , \quad \frac{\partial \phi_1}{\partial z} = \frac{\partial \zeta}{\partial t} + v\frac{\partial \zeta}{\partial x} \ . \tag{5.58}$$

For $\zeta(x,t) \propto \exp(ikx - i\Omega t)$ we obtain from (5.57,5.58) the dispersion relation

$$\Omega^2 - 2\frac{\rho_1}{\rho_1 + \rho_2} vk\Omega + \frac{\rho_1}{\rho_1 + \rho_2} v^2 k^2 - c_0^2 k^2 = 0 \ , \tag{5.59}$$

$$c_0^2 = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\frac{g}{k} + \frac{\alpha k}{\rho_1 + \rho_2} \ . \tag{5.60}$$

Here $k > 0$ and c_0 is the phase velocity of the gravity-capillary waves in the case $v = 0$. In this case, we see that when $\rho_1 > \rho_2$ the phase velocity and the frequency are imaginary for sufficiently long waves with $k^2 < (\rho_1 - \rho_2)g/\alpha$. That signals so-called Rayleigh-Taylor instability, which is responsible, in particular, for water spilling out of an overturned glass. In this case, inverted gravity causes the instability while surface tension stabilizes it.

Consider now a bottom-heavy configuration, $\rho_1 < \rho_2$, which is stable for $v = 0$ since $c_0^2 > 0$. For sufficiently high v, we have Kelvin-Helmholtz instability when one can find such k that the determinant of the quadratic equation (5.59) is negative:

$$v^2 > \frac{(\rho_1+\rho_2)^2}{\rho_1\rho_2}\min_k c_0^2(k) = \frac{\rho_1+\rho_2}{\rho_1\rho_2}\sqrt{4(\rho_2-\rho_1)g\alpha}\ . \tag{5.61}$$

Again, like in the Landau criterium for generating excitations in superfluid, the criterium for generating surface waves is the possibility that the flow moves faster than the minimal phase velocity of the waves. In this case, both gravity and surface tension are needed to provide for a nonzero minimal velocity; in other words, their interplay makes the flow stable for lower speeds. Application of (5.61) to wind upon water where $\rho_1/\rho_2 \simeq 10^{-3}$ gives the threshold value

$$v_{th} \approx 30\left(\frac{4g\alpha}{\rho_2}\right)^{1/4} \approx 7\, m/sec\ .$$

This is unrealistically high. By blowing air over the cup of tea, it is easy to observe that one can ruffle the surface with much smaller air velocity. Realistic theory of wave generation by wind not only requires an account of viscosity but also of the fact that the wind is practically always turbulent; strong coupling between water waves and air vortices makes such theory non-trivial.

Notes

Contents

1 Translated by A. Shafarenko

Chapter 1

1 The Deborah number was introduced by M. Reiner. All real solids contain dislocations which make them flow. Whether perfect crystals can flow under an infinitesimal shear is a delicate question, which is the subject of ongoing research.
2 To go with a flow, using Lagrangian description, may be more difficult yet it is often more rewarding than staying on a shore. Like sport and some other activities, fluid mechanics is better doing (Lagrangian) than watching (Eulerian), according to J-F. Pinton.
3 Temperature decays with height only in the troposphere that is until about -50° at 10-12 km, then it grows in the stratosphere until about 0° at 50 km
4 Convection excited by a human body at room temperature is always turbulent, as can be seen in a movie in [9], Section 605.
5 More details on the stability of rotating fluids can be found in Sect. 9.4 of [1] and Sect. 66 of [5] for details.
6 Actually, the Laplace equation was first derived by Euler for the velocity potential.
7 Conformal transformations stretch uniformly in all directions at every point but the magnitude of stretching generally depends on a point. As a result, conformal maps preserve angles but not the distances. These properties had been first made useful in naval cartography (Mercator, 1569) well before the invention of the complex analysis. Indeed, to discover a new continent it is preferable to know the direction rather than the distance ahead.
8 Second-order linear differential operator $\sum a_i \partial_i^2$ is called elliptic if all a_i are of the same sign, hyperbolic if their signs are different and parabolic if at least one coefficient is zero. The names come from the fact that a real quadratic curve $ax^2 + 2bxy + cy^2 = 0$ is a hyperbola, an ellipse or a parabola depending on whether $ac - b^2$ is negative, positive or zero. For

hyperbolic equations, one can introduce characteristics where solution stays constant; if different characteristics cross then a singularity may appear inside the domain. Solutions of elliptic equations are smooth, their stationary points are saddles rather than maxima or minima. See also Sects. 2.3.2 and 2.3.5.

9 Detailed discussion of minima and maxima of irrotational flows is in [3], p. 385

10 Presentation in Sect. 11 of [10] is misleading in not distinguishing between momentum and quasi-momentum.

11 That one can use the conservation of momentum inside an elongated cylindrical surface around the solid body follows from the consideration of the momentum flux through this surface. The contribution of the pressure, $\pi \int_0^{\mathcal{R}} [p(L,r) - p(-L,r)]dr^2 = \pi\rho \int_0^{\mathcal{R}} [\dot\phi(-L,r) - \dot\phi(L,r)]dr^2 = \pi\rho\dot u[1 - (1 - \mathcal{R}/L)^{-1/2}]$ vanishes in the limit $L/\mathcal{R} \to \infty$. The pressure contribution does not vanish for other surfaces, see Sect. 7.1 of [15].

12 Further reading on induced mass and quasi-momentum: [12] and Sects. 2.4-2.6. of [15].

13 The argument that the momentum transfer requires the resistance force to be proportional to the velocity squared goes back to Newton.

14 The general statement on a zero resistance force acting on a body steadily moving in an ideal fluid sometimes is called D'Alembert paradox, even though D'Alembert established it only for a body with a central symmetry.

15 No-slip can be seen in a movie in [9], Section 605. The no-slip condition is a useful idealization in many but not in all cases. Depending on the structure of a liquid and a solid and the shape of the boundary, slip can occur which can change flow pattern and reduce drag. Rich physics, and also numerical and experimental methods used in studying this phenomenon are described in Sect. 19 of [18].

16 One can see liquid jets with different Reynolds numbers in Sect. 199 of [9].

17 Movies of propulsion at low Reynolds numbers can be found in Sect. 237 of [9].

18 Photographs of boundary layer separation can be found in [19] and movies in [9], Sects. 638-675.

19 Another familiar example of a secondary circulation due to pressure mismatch is the flow that carries the tea leaves to the center of a teacup when the tea is rotated, see e.g. Sect. 7.13 of [6].

20 More details on jets can be found in Sects. 11,12,21 of D.J. Tritton, *Physical Fluid Dynamics* (Oxford Science Publications, 1988).

21 Shedding of eddies and resulting effects can be seen in movies in Sects. 210,216,722,725 of [9].

22 Elementary discussion and a simple analytic model of the vortex street can be found in Sect. 5.7 of [1], including an amusing story told by von Kármán about the doctoral candidate (in Prandtl's laboratory) who tried in vain to polish the cylinder to make the flow non-oscillating. Kármán vortex street is responsible for many acoustic phenomena like the roar of propeller or sound caused by a wind rushing past a tree.

23 Words and Figs 1.15,1.16, don't do justice to the remarkable transformations of the flow with the change of the Reynolds number, full set of photographs can be found in [19] and movies in [9], Sects. 196,216,659. See also Galleries at http://www.efluids.com/

24 One can check that for $Re < 10^5$ a stick encounters more drag when moving through a still fluid than when kept still in a moving fluid (in the latter case the flow is usually turbulent before the stick so that the boundary layer is turbulent as well). Generations of scientists, starting from Leonardo Da Vinci, believed that the drag must be the same (despite experience telling otherwise) because of Galilean invariance, which, of course, is applicable only to an infinite uniform flow, not to real streams.
25 One can generalize the method of complex potential from Sect. 1.2.4 for describing flows with circulation, which involves logarithmic terms. A detailed yet still compact presentation is in Sect. 6.5 of [3].
26 Newton argued that a rotating ball curves because the side that moves faster meets more resistance. Since he considered the resistance force proportional to the velocity squared that is to the pressure, this gives the same estimate (1.55) for the Magnus force.
27 Lively book on the interface between biology and fluid mechanics is S. Vogel, *Life in moving fluids* (Princeton Univ. Press, 1981).
28 It is instructive to think about similarities and differences in the ways that vorticity penetrating the bulk makes life interesting in ideal fluids and superconductors. An evident difference is that vorticity is continuous in a classical fluid while vortices are quantized in quantum fluids
29 Further reading on flow past a body, drag and lift: Sect. 6.4 of [3] and Sect. 38 of [10].

Chapter 2

1 Description of numerous instabilities can be found in [5] and in Chapters 8 of [6, 16].
2 Stability analysis for pipe and plane shear flows with the account of viscosity can be found in Sect. 28 of [10] and Sect. 9 of [1].
3 For a brief introduction into the theory of dynamical chaos see e.g. Sects. 30-32 of [10], full exposure can be found in E. Ott, *Chaos in dynamical systems* (Cambridge Univ. Press, 1992). See also Exercise 3.7.
4 Compact lucid presentation of the phenomenology of turbulence can be found in Sreenivasan's Chapter 7 of [16]. Detailed discussion of flux in turbulence and further references can be found in [4, 8].
5 While deterministic Lagrangian description of individual trajectories is inapplicable in turbulence, statistical description is possible and can be found in [4, 7].
6 It is presumed that the temporal average is equivalent to the spatial average, property called ergodicity.
7 Detailed derivation of the Kármán-Howarth relation and Kolmogorov's 4/5-law can be found in Sect. 34 of [10] or Sect. 6.2 of [8].
8 We also understand the breakdown of scale invariance for the statistics of passive fields carried by random flows, see [7].
9 Momentum and quasi-momentum of a phonon are discussed in Sect. 4.2 of [14]. For fluids, wave propagation is always accompanied by a (Stokes) drift quadratic in wave amplitude.
10 More detailed derivation of the velocity of Riemann wave can be found in Sect. 101 of [10].
11 Burgers equation describes also directed polymers with t being the coordinate along polymer and many other systems.

12 On experimental uses of the Döppler effect see [18].
13 Our presentation of a compressible flow past a body follows Sect. 3.7 of [1], more details on supersonic aerodynamics can be found in Chapter 6 of [16].
14 Passing through the shock, potential flow generally acquires vorticity except when all the streamlines cross the shock at the same angle as is the case in the linear approximation, see [10], Sects. 112-114.

Chapter 3

1 Derivation of the viscous decay rate of gravity waves via stresses calculated from an ideal flow can be found in Sect. 25 of [10].
2 For details on Stokes derivation of viscous dissipation of gravity surface waves see Sect. 3.5 of [11].
3 Galileo was the first to determine experimentally the air density. He then related the old-known practical fact that no suction pump can lift water higher than 10 meters to the atmospheric pressure. His pupils, Torricelli and Viviani, tested this idea with a heavier liquid, such as mercury, thus creating the first barometer (in 1643). See e.g. H.S. Lipson, *The Great Experiments in Physics* (Oliver and Boyd, Edinburgh 1968).
4 On the standing wave pattern, an enjoyable reading is Sect. 3.9 of [11] which contains a poem by Robert Frost with a rare combination of correct physics and beautiful metaphysics.
5 The details on Kelvin ship-wave pattern can be found in Sect. 3.10 and on caustics in Sect. 4.11 of [11]. Group velocity is stationary on caustics whose consideration thus requires further expansion of $\omega(k)$ up to cubic terms, with the Airy integral playing the same role as the Gaussian integral (3.15) at a general point.
6 The procedure for excluding non-resonant terms is a part of the theory of Poincare normal forms for Hamiltonians near fixed points and closed trajectories [2, 21].
7 More expanded exposition of two-cascade turbulence can be found in [4], detailed presentation is in [21].
8 Negative temperature can be seen in the simplest case of two groups of modes: for fixed $E = \omega_1 N_1 + \omega_2 N_2$, $N = N_1 + N_2$, the entropy $S = \ln N_1 + \ln N_2 \propto \ln(E - \omega_1 N) + \ln(\omega_2 N - E)$ gives the temperature $T^{-1} = \partial S/\partial E \propto (\omega_1 + \omega_2)N - 2E$, which is negative for sufficiently high E. For a condensate, $\omega_1 = 0$ so $T < 0$ when $2E = 2\omega_2 N_2 > \omega_2 N$ i.e. less than half of the particles are in the condensate. Such arguments go back to Onsager who considered 2d incompressible flows in finite domains where negative-temperature states correspond to clustering of vorticity into large coherent vortices.
9 Inverse cascades and persistent large-scale flow patterns exist also in rotating fluids and magnetized plasma.
10 A photo of a bore can be seen in [19].
11 Elementary discussion of hydraulic jumps can be found in Sect. 3.9 of [1], Sect. 2.12 of [11] or Sect. 2.16 of [6].
12 Detailed presentation of the Inverse Scattering Transform can be found e.g. in Ablowitz, M. and Segur, H. *Solitons and the inverse scattering transform.*

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