Statistical Mechanics 2011-12 — Problem Set 2

due: December 8, 2011

2.1 Ionization in the grand-canonical ensemble (20 points)

Consider M ions A^+, and n_e electrons. The ionization energy of the A atom is denoted by ϵ. For simplicity, the Coulomb interactions shall be ignored. All experiments considered below are done in a container of volume V at temperature T.

(a) Assume first that the A^+ ions are fixed in space. The electrons are now divided into two components - a classical gas of free electrons, and a 'gas' of bounded electrons. Calculate the fraction of electrons in the free gas by equating the chemical potentials of the two gases.

(b) Now consider the case where the A^+, e^- and A particles are all moving freely in the container. Compare your result here to those of (a).

2.2 Idea Bose gas in two dimensions (10 points)

Consider the relation that determines the chemical potential: the number of atoms equal to the sum of the Bose distributions over all momenta - see first paragraph in section 3.2.4 in the lecture notes. Is there a condensation at finite temperature in two dimensions ?

2.3 BEC in harmonic potentials (27 points)

Experimentally, Bose-Einstein condensates are usually created (here in the Weizmann institute and elsewhere) with particles trapped in an external potential (rather than free particles in a box). The trapping potential can often be considered harmonic to excellent approximation. In this question we will study the Bose-Einstein condensation under such a potential.

Consider a gas of N bosons in an anisotropic three-dimensional harmonic potential

$$V = \frac{1}{2}m(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2). \qquad (1)$$

The different frequencies w_i represent the fact that the trapping potential may be anisotropic.

(a) Calculate a general expression for the critical temperature of condensation. What would be the temperature for the experimental values of $N = 10^6$, $\omega_1 = 800Hz$, $\omega_2 = 600Hz$, $\omega_3 = 50Hz$ and the mass of Rubidium-87 atoms.

(b) Calculate the occupation number of the condensate as a function of temperature.

(c) Generalize the above result for an arbitrary dimension d. Is there a condenstation for $d = 2$? Does this result agree with the result of question 2.2 under the proper limit ?

2.4 Pauli paramagnetism (27 points)

In this question you will calculate the contribution of the spin of electrons to their magnetic susceptibility. Consider non-interacting electrons, each subject to a Hamiltonian

$$\mathcal{H}_1 = \frac{p^2}{2m} - \mu_0 \vec{\sigma} \cdot \vec{B}, \tag{2}$$

where $\mu_0 = e\hbar/2mc$, and the eigenvalues of $\vec{\sigma} \cdot \vec{B}$ are $\pm B$.

(a) The electrons pointing parallel and anti-parallel to the field may be considered as two separate gases that can exchange particles. Calculate the average number of particles $N_\pm$ in each of the gases as a function of the fugacity z and the temperature T, using the expression

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1} e^x + 1} \tag{3}$$

(you can read about the properties of $f_\nu(z)$ in most standard textbooks, e.g., Pathria, appendix E, p. 508). For a gas of N electrons, how can z be determined from N?

(b) Obtain the expression for the magnetization $M = \mu_0(N_+ - N_-)$, and expand the result for small B.

(c) Calculate the leading behavior of the zero field susceptibility $\chi(T, N) = \partial M/\partial B|_{B=0}$ at high and low temperatures, and sketch $\chi(T, N)/N$ as a function of T.

2.5 Intermediate statistics (16 points)

Consider a hypothetical system where each quantum state can be occupied by no more than p particles. Find the mean occupation number of the state with the energy ϵ when the chemical potential of the system is μ (system is considered within the grand-canonical ensemble). Check how the resulting formula goes into the Fermi or Bose distributions at the proper limits.