# Statistical Mechanics 2014/2015 Problem Set 3

**Submission date: 17.12.14**

## 3.1 Spin 1 Ising model

Consider the mean-field Hamiltonian of a spin 1 Ising model

$$\mathcal{H} = -\frac{J}{2N}\sum_{i,j=1}^{N} S_i S_j - H\sum_{i=1}^{N} S_i, \tag{1}$$

where $S_i = 0, \pm 1$.

(a) Calculate the partition function and the free energy using mean-field approximation. Find the equation for the magnetization per spin $m \equiv \langle S_i \rangle$ and the critical temperature $T_c$ for the ferromagnetic phase transition when $H = 0$.

(b) Find the critical exponents $\beta$ and $\gamma$. Reminder: these exponents are defined by

$$|m| \propto (-t)^{\beta} \quad \text{for} \quad T < T_c \tag{2}$$

and

$$\chi_0 \propto t^{-\gamma} \quad \text{for} \quad T > T_c. \tag{3}$$

Here, $t \equiv (T - T_c)/T_c \ll 1$ and $\chi_0 = \frac{\partial m}{\partial H}|_{H\to 0}$.

## 3.2 Ising ferromagnet and anti-ferromagnet in the mean field approximation(26 points)

Consider the Hamiltonian

$$\mathcal{H} = J\sum_{\langle i,j\rangle} S_i S_j - H\sum_{i} S_i\ , \tag{4}$$

where $S_i = \pm 1$ and $\langle i, j\rangle$ indicates summation over nearest neighbors. Assume that the number of nearest neighbors per site is $\gamma$. Consider the case $J > 0$, where at zero field spins prefer to be anti-parallel.

(a) Develop the mean field theory of this system, by dividing it into two sub-lattices of nonzero net magnetization in the ground state. Write the mean field Hamiltonian and the self consistent equations for the magnetization of the two sub-lattices directly.

(b) Find the value of the magnetization for each of the sub-lattices in the paramagnetic phase, for small magnetic field $H$, to leading order in $H$. Explain why the order parameter in this system is the difference in magnetization between the two sub-lattices.

(c) Show that the transition temperature to the anti-ferromagnetic phase, at zero magnetic field, is equal to the transition temperature $T_c$ that was a found in class for a ferromagnet . Could you show this using symmetry considerations?

(d) Find the transition temperature to the anti-ferromagnetic phase as a function of magnetic field for small $H$, to leading order in $H$.

You might find useful the following identity

$$\tanh(x) - \tanh(y) = (1 - \tanh(x)\tanh(y))\tanh(x-y) \tag{5}$$

(e) Near the transition temperature and at small magnetic field, calculate the magnetization of each sub-lattice in the anti-ferromagnetic phase to leading order in $H$ and in $T - T_c$.

(f) How does the zero field magnetic susceptibility, $\chi = \frac{1}{2}(\chi_1 + \chi_2)$, behave as one approaches $T_c(H = 0)$ from above? from below? Compare it to the behavior of the susceptibility for the ferromagnet, equation (129) in the notes-can you explain the difference between the two?

## 3.3 Landau theory for anisotropic XY model

Consider the anisotropic XY model in a magnetic field in the x-direction

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + \sum_i D\left(S_{i,x}^2 - S_{i,y}^2\right) - HS_{i,x}, \tag{6}$$

where $\vec{S}_i = (S_{i,x}, S_{i,y})$ and $J, \Delta > 0$.

(a) Identify the appropriate order parameter (or order parameters) for this model, and write down the Landau free energy.

(b) At $H = 0$, identify the order parameter which orders (becomes non-zero) below the transition.

(c) Calculate the $H - T$ phase diagram from the Landau free energy, and show that it exhibits a critical line. What is the order parameter associated with this line? Identify the phases which exist in the $H - T$ plane and give an expression for the critical line separating them in terms of the parameters of Landau free energy.