Statistical Mechanics Fall 2014 — Problem Set 5

due: Wednesday, January 14, 2015

5.1 Recursion relations of n-vector model (15 points)

Consider the Landau-Ginzburg Hamiltonian of the n-vector model

$$\beta\mathcal{H} = \int d^d r \left[\frac{t}{2}\vec{m}\cdot\vec{m} + \frac{1}{2}(\nabla m)^2 + u(\vec{m}\cdot\vec{m})^2\right], \tag{1}$$

$$with\ (\nabla m)^2 = \sum_{i=1}^{n}\sum_{\alpha=1}^{d}(\partial_\alpha m_i)^2, \tag{2}$$

where $\vec{m}$ is an n components vector. The recursion relations of this model to leading non-trivial order in u are

$$\frac{dt}{dl} = 2t + \frac{A_n K_d \Lambda^d}{t+\Lambda^2}u \tag{3}$$

$$\frac{du}{dl} = (4-d)u + \frac{B_n K_d \Lambda^d}{(t+\Lambda^2)^2}u^2 \tag{4}$$

$$\tag{5}$$

In class we have derived these recursion relations for the Ising model ($n = 1$), yielding $A_1 = 12$ and $B_1 = 36$. Demonstrate that for general n,

$$A_n = 4(n+2) \quad ; \quad B_n = 4(n+8). \tag{6}$$

Show how these factors are derived using the appropriate diagrams counting.

5.2 Anistotropic criticality (45 points)

In this question we shall consider the Landau-Ginzburg theory of an anisotropic materials, such as liquid crystals, which behave differently along distinct directions, which shall be denoted parallel and perpendicular. Let us assume that the d spatial dimensions are grouped into n parallel directions $x_{||}$ and $d-n$ perpendicular directions $x_\perp$. Consider a one-component field $m(x_{||}, x_\perp)$ subject to Landau-Ginzburg Hamiltonian, $\beta\mathcal{H} = \beta\mathcal{H}_0 + U$, with

$$\beta\mathcal{H}_0 = \int d^n x_{||} d^{d-n} x_\perp \left[\frac{K}{2}(\nabla_{||} m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm\right] \tag{7}$$

$$U = u\int d^n x_{||} d^{d-n} x_\perp m^4. \tag{8}$$

(Note that $\beta\mathcal{H}$ depends on the first gradient in the $x_{||}$ directions, and on the second gradient in the $x_\perp$ directions.) In this question we shall treat only the unperturbed Hamiltonian $\mathcal{H}_0$ (namely $u = 0$).

(a) Write $\beta\mathcal{H}_0$ in terms of the Fourier transforms $m(q_{||}, q_\perp)$.

(b) Construct a renormalization group transformation for $\beta\mathcal{H}_0$, by rescaling coordinates such that $q'_{||} = bq_{||}$ and $q'_\perp = cq_\perp$ and the field as $m'(q') = m(q)/z$. Note that parallel and perpendicular directions are scaled differently. Write down the recursion relations for K, L, t, and h in terms of b, c, and z (no need to evaluate complicated integrals).

(c) Choose $c(b)$ and $z(b)$ such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h for the rescalings of t and h.

(d) Write the relationship between the (singular parts of) free energies $f(t, h)$ and $f'(t', h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form $f(t, h) = t^{2-\alpha} g_f(h/t^\Delta)$, and identify the exponents α and Δ.

(e) How does the unperturbed zero-field susceptibility $\chi(t, h = 0)$ diverge as $t \to 0$?

(f) For $h = 0$, calculate the expectation value $\langle m(q)m(q')\rangle_0$, and the corresponding susceptibility $\chi_0(q) = \langle |m_q|^2\rangle_0$, where q stands for $(q_{||}, q_\perp)$.

5.3 Long range interactions (40 points)

Long range interactions between spins can be described by adding a term

$$\int d^d x \int d^d y J(|x - y|)\vec{m}(x) \cdot \vec{m}(y), \tag{9}$$

to the usual Landau-Ginzburg Hamiltonian.

(a) Show that for $J(r) \sim r^{-d-\sigma}$, the Hamiltonian can be written as

$$\beta\mathcal{H} = \int \frac{d^d q}{(2\pi)^d} \frac{t + K_2 q^2 + K_\sigma q^\sigma + ...}{2} \vec{m}(q) \cdot \vec{m}(-q) \tag{10}$$

$$+u \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \vec{m}(q_1) \cdot \vec{m}(q_2)\vec{m}(q_3) \cdot \vec{m}(-q_1 - q_2 - q_3). \tag{11}$$

(b) For $u = 0$, construct the recursion relations for (t, K_2, K_σ) and show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K'_\sigma = K_\sigma$, in which case K_2 is irrelevant. What is the fixed Hamiltonian now?

(d) For $\sigma < 2$, calculate the generalized Gaussian exponents ν, η, and γ from the recursion relations. Show that u is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.

(e) What is the critical behavior if $J(r) \sim \exp(-r/a)$? Explain.