

# Irreversible quantum graphs

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## Abstract

Irreversibility is introduced to quantum graphs by coupling the graphs to a bath of harmonic oscillators. The interaction which is linear in the harmonic oscillator amplitudes is localized at the vertices. It is shown that for sufficiently strong coupling, the spectrum of the system admits a new continuum mode which exists even if the graph is compact, and a *single* harmonic oscillator is coupled to it. This mechanism is shown to imply that the quantum dynamics is irreversible. Moreover, it demonstrates the surprising result that irreversibility can be introduced by a ‘bath’ which consists of a *single* harmonic oscillator.

## 1. Introduction

Quantum graphs emulate many properties of quantum chaotic systems, and their use is now widespread in the physics and mathematics literature [1–7]. In many potential applications the physical systems are irreversible, and it is desirable to include this feature in quantum graphs, and thus enlarge their range of applicability.

The standard way to introduce irreversibility to a quantum system is by coupling it to a ‘bath’ which consists of a set of ‘irrelevant’ degrees of freedom. The bath dynamics is governed, however, by a well defined quantum Hamiltonian [8–11]. The enlarged system, composed of the original system and the bath, is a proper quantum system, with Hamiltonian

$$H = H^{\text{system}} + H^{\text{bath}} + W. \quad (1)$$

$H^{\text{system}}$  and  $H^{\text{bath}}$  are operators in the ‘system’ and ‘bath’ Hilbert spaces, respectively, and  $W$  is the coupling. In the physics literature, irreversibility is generally associated with the bath having a continuous spectrum. The physical justification is based on the observation that the Poincaré recurrence time is infinite, or, stated differently, the mean probability of returning back to the initial state vanishes, which is a prerequisite for irreversibility. This argument can also be formulated mathematically. To simplify the presentation, assume that  $H^{\text{system}}$  has a discrete spectrum, while the bath spectrum is continuous, with a spectral measure  $d\mu(\mathcal{E})$  and

$\delta$  normalized eigenfunctions  $|\chi_\varepsilon\rangle$ . Denoting by  $\Psi_{\mathcal{P}}$  the projection of an eigenstate of  $H$  on the system subspace, The Schrödinger equation can be reduced to

$$E\Psi_{\mathcal{P}} = H^{\text{system}}\Psi_{\mathcal{P}} + \lim_{\eta \searrow 0} \int d\mu(\mathcal{E}) W_{\mathcal{P}}(\mathcal{E}) \frac{1}{E - \mathcal{E} + i\eta} W_{\mathcal{P}}^\dagger(\mathcal{E})\Psi_{\mathcal{P}}, \quad (2)$$

where  $W_{\mathcal{P}}(\mathcal{E})$  is the projection of  $W|\chi_\varepsilon\rangle$  on the system subspace. Using the identity

$$\lim_{\eta \searrow 0} \frac{1}{x + i\eta} = -i\pi \delta(x) + \mathcal{P}\left(\frac{1}{x}\right), \quad (3)$$

one traditionally associates irreversibility with the operator

$$\Gamma(E) = -i\pi \int d\mu(\mathcal{E}) \delta(E - \mathcal{E}) W_{\mathcal{P}}(\mathcal{E}) W_{\mathcal{P}}^\dagger(\mathcal{E}). \quad (4)$$

$\Gamma(E)$  is an anti-Hermitian operator, which introduces decay to the quantum time evolution. (This can be immediately shown when  $\Gamma(E)$  can be approximated by an energy-independent operator.) Two important points must be emphasized. First, the discussion presented above is valid only when the bath spectrum is continuous. Second,  $\Gamma(E)$  vanishes unless  $E$  is in the support of the spectral measure. Similar expressions appear, e.g., for the width of resonances in the Breit–Wigner theory [12].

Various approximations and models were used to extract the main features of quantum irreversibility as described by (4). Amongst the most popular is the harmonic bath model where the bath consists of a continuous set of harmonic oscillators which are linearly coupled to the system degrees of freedom [10]. In this case, some of the integrations can be carried out, and using first order perturbation theory, the decay width introduced by the bath can be computed explicitly.

The purpose of this work is to carry out the above programme for a system which consists of a graph that is coupled linearly to a harmonic bath. The system will be described in section 2 and the corresponding Schrödinger operator will be discussed. Because of the form of the coupling, the question of finding a self-adjoint extension is not simple. In the present work, a physically motivated form of the Schrödinger operator is constructed, and the rigorous proof of its validity is given in another paper in this issue by Solomyak [13].

Irreversibility is demonstrated by computing the vertex scattering matrices [1, 2], which are responsible for the transfer of current between the bonds of the graph (section 3). For sufficiently strong coupling, the vertex scattering matrices are sub-unitary, which means that the total probability current is not conserved. Rather, it decreases upon scattering at the vertex. In other words, the quantum mechanical evolution, restricted to the graph degrees of freedom, is not unitary. The surprise is that this happens even when a *single* harmonic oscillator is coupled at the vertex! This mechanism should not be confused with what happens naturally when an additional lead to infinity is coupled to the vertex, and its effect on the original scattering is to draw current. This mechanism which induces an ‘escape width’ in the graph dynamics is not what is considered here. Rather, non-unitarity is due to the coupling to a harmonic oscillator.

The rest of the paper attempts to explain this result in physical terms. To this end, a simple graph is presented and solved approximately in section 4. Using this model, one can demonstrate and explain how the coupling of a single harmonic oscillator to a compact graph can modify the spectrum in a major way. If the coupling is weak, the spectrum remains pure point. Beyond a critical strength, a continuous component appears in the spectrum. The appearance of the continuum occurs at the same value of the coupling constant at which the vertex scattering matrix becomes non-unitary. A rigorous treatment of this spectral problem is also provided in Solomyak’s paper [13].

**2. The Schrödinger operator on a graph and the coupling to a harmonic bath**

A graph  $\mathcal{G}$  consists of  $V$  vertices connected by  $B$  bonds according to the connectivity matrix  $C_{i,j}$  whose elements take the value 1 if the vertices  $i, j$  are connected, and vanish otherwise. We assign the natural metric to the bonds, and associate a length  $L_b$  to each. The position  $x$  of a point on the graph is determined by specifying on which bond  $b$  it is, and its distance  $x_b$  from the vertex with the *smaller* index,  $0 \leq x_b \leq L_b$ . Sometimes, it will be convenient to denote the bond connecting the vertices  $i$  and  $j$  by  $(i, j)$ . The number of bonds which emanate from a vertex is the ‘valency’  $v_i = \sum_{j=1}^V C_{i,j}$

In the absence of a harmonic bath, the Schrödinger operator can be defined in the following way: let  $x \in \mathcal{G}$  and let  $\Psi(x)$  be a real valued and continuous function on  $\mathcal{G}$ , such that  $\Psi(x) = \psi_b(x_b)$  for  $x \in b$ , and  $0 \leq x_b \leq L_b$ , where  $\psi_b(x_b)$  are twice differentiable in the interior of the bond. We restrict  $\Psi(x)$  to being square integrable, and require that it is uniquely defined at the vertices. That is, all the functions  $\psi_b$  which correspond to bonds connected at a vertex  $i$  attain the same value  $\phi_i$  at the common vertex. Given an arbitrary set of  $V$  non-negative constants  $\lambda_i$ , one constructs the quadratic form

$$L_0[\Psi] = \sum_{b=1}^B \int_0^{L_b} dx_b \left| \frac{d\psi_b(x_b)}{dx_b} \right|^2 + \sum_{i=1}^V \lambda_i \phi_i^2. \tag{5}$$

The Euler–Lagrange variational principle selects the stationary solutions of the quadratic form, as the solution of the Schrödinger equation on the bonds:

$$\left( -\frac{d^2}{dx_b^2} - E \right) \psi_b(x_b) = 0, \tag{6}$$

subject to the boundary conditions

$$\sum_{j=1}^V C_{i,j} \left. \frac{d}{dx_{(i,j)}} \psi_{(i,j)}(x_{(i,j)}) \right|_i - \lambda_i \phi_i = 0. \tag{7}$$

The summation is over all the bonds  $(i, j)$  which emanate from the vertex  $i$ , and the derivatives are computed at the vertex. The quadratic form (5) is positive definite, and, therefore, the boundary condition (7) provides a self-adjoint extension of the Schrödinger operator for any choice of the (non-negative) constants  $\lambda_i$ . This operator determines the quantum dynamics of the ‘system’. One can extend this operator to non-compact graphs, by adding bonds from the vertices to infinity. The corresponding scattering problem has been amply discussed in the literature [2, 4–7].

The bath degrees of freedom consist of  $M$  harmonic oscillators with coordinates  $q_m$ , and frequencies  $\omega_m$ , subject to the Hamiltonian

$$h_m = \frac{1}{2} \left( -\frac{d^2}{dq_m^2} + \omega_m q_m^2 \right); \quad q_m \in \mathbb{R}. \tag{8}$$

Write  $\mathbf{q} = (q_1, \dots, q_M)$  and  $H_{\text{osc}} = \sum_{m=1}^M h_m$ .  $H_{\text{osc}}$  is the ‘bath’ Hamiltonian. The coupling of the bath to the system is introduced by extending the quadratic form (5) to include the graph and harmonic bath coordinates. Consider  $\Psi(x, \mathbf{q}) = \psi_b(x_b, \mathbf{q})$  for  $x \in b$ , where each of the  $\psi_b(x_b, \mathbf{q})$  is square integrable. For every value of  $\mathbf{q}$ ,  $\Psi(x, \mathbf{q})$  is uniquely defined at the vertices: at the vertex  $i$  it assumes the value  $\phi_i(\mathbf{q})$ . To each vertex, we assign a real valued function  $\Lambda_i(\mathbf{q})$ , which is bounded on any compact subset of  $\mathbb{R}^M$  and is allowed to diverge algebraically as  $|\mathbf{q}| \rightarrow \infty$ . (In the following, we shall refer to these functions as the ‘form factors’.) We

construct the quadratic form

$$L_{\text{osc}}[\Psi] = \sum_{b=1}^B \int d^M q \left\{ \int_0^{L_b} dx_b \left( \left| \frac{d\psi_b(x_b, \mathbf{q})}{dx_b} \right|^2 + \psi_b^*(x_b, \mathbf{q}) H_{\text{osc}} \psi_b(x_b, \mathbf{q}) \right) + \sum_{i=1}^V \Lambda_i(\mathbf{q}) \phi_i^2(\mathbf{q}) \right\}. \quad (9)$$

Again, the Euler–Lagrange variational principle selects the stationary solutions of the quadratic form, as the solution of the Schrödinger equation on the bonds:

$$\left( -\frac{d^2}{dx_b^2} + H_{\text{osc}} - E \right) \psi_b(x_b, \mathbf{q}) = 0, \quad (10)$$

subject to the boundary conditions

$$\sum_{j=1}^V C_{i,j} \frac{d}{dx_{(i,j)}} \psi_{(i,j)}(x_{(i,j)}, \mathbf{q}) \Big|_i - \Lambda_i(\mathbf{q}) \phi_i(\mathbf{q}) = 0. \quad (11)$$

The coupling between the ‘system’ and the harmonic ‘bath’ is thus mediated through the boundary conditions (11).

When the form factors  $\Lambda_v(\mathbf{q})$  are non-negative the quadratic form  $L_{\text{osc}}[\Psi]$  is positive definite, and the Schrödinger operator (10), subject to the boundary conditions (11), is self-adjoint. The standard *linear* coupling of harmonic baths is written as

$$\Lambda_i(\mathbf{q}) = \sum_{m=1}^M \Lambda_{im} q_m, \quad (12)$$

with  $\Lambda$  a  $V \times M$  matrix with  $\mathbf{q}$ -independent real coefficients. For this choice of a form factor,  $L_{\text{osc}}[\Psi]$  is not definite, and the proof that the system (10), (11) is self-adjoint requires a special treatment, which is provided in [13]. The intuitive motivation behind this coupling scheme is that for a vertex with  $v = 2$ , and a single harmonic oscillator, the boundary condition at the vertex is equivalent to an effective potential  $\Lambda q \delta(x)$ . This represents a well localized coupling of the particle on the graph to the harmonic oscillator.

### 3. The vertex scattering matrix

The boundary conditions at the vertices of graphs (without coupling to any harmonic oscillators) (7) impose the conservation of the probability current across the vertex. In other words, the boundary conditions at a vertex can be described by a unitary *vertex scattering matrix* which provides the linear relation between the amplitudes of the waves which impinge upon the vertex and the waves scattered from it [1]. The vertex scattering matrix which corresponds to the boundary conditions (7) can be written explicitly as

$$\sigma_{b,b'}^{(i)}(k) = -\delta_{b,b'} + \frac{1 + \exp(-2i \arctan(\lambda_i/v_i k))}{v_i} \quad (13)$$

where  $i$  denotes the vertex under consideration,  $b$  and  $b'$  denote the bonds which emanate from the vertex, and  $k = \sqrt{E}$  is the wavenumber under consideration. Note that this  $v_i \times v_i$  vertex matrix distinguishes only between reflection ( $b = b'$ ) and transmission between different bonds ( $b \neq b'$ ), which is independent of their specific identity. This is an expression of the fact that the boundary condition at the vertex is invariant under the interchange of the bonds.  $\sigma_{b,b'}^{(i)}(k)$  is symmetric, and its unitarity can be easily demonstrated. The single vertex, together with the bonds emanating from it, is sometimes referred to as a ‘star graph’.

Describing the quantum evolution and dynamics in terms of waves travelling freely along the bonds and scattered at the vertices is a very useful concept in the theory of quantum graphs [1]. This is also true when harmonic oscillators are coupled, and we shall dedicate the rest of this section to the derivation of the vertex scattering matrices which correspond to the boundary conditions (11). We shall show that the coupling of the harmonic oscillators can have a profound effect—the vertex scattering matrix is not necessarily unitary for large values of the coupling  $\Lambda$ . Thus, the evolution on the graph is not conserving and hence irreversible!

To construct the vertex scattering matrix we consider a single vertex from which  $v_i$  bonds extend to infinity (this is a star). The presentation is simplified by coupling the vertex to a *single* harmonic oscillator ( $M = 1$ ), and choose the units such that  $\omega_1 = 1$ . The spectrum of the combined system of a harmonic oscillator and the star graph without coupling consists of overlapping continua with thresholds at  $E_n = n + 1/2$ . Further simplification is achieved by considering energies in the interval  $1/2 < E < E_1$ . In this energy range, and for  $x$  values away from the vertex, only the ground state ( $n = 0$ ) of the harmonic oscillator is populated, which corresponds to purely elastic scattering. We shall compute the vertex scattering matrix by considering an incoming wave incident on the bond  $b'$  only.

Writing

$$k^2 = E - \frac{1}{2}; \quad k_0 = k; \quad k_n = \sqrt{n - k^2} \quad \text{for } n \geq 1, \quad (14)$$

we expand the wavefunction in the complete orthonormal basis  $\chi_n(q)$  of eigenfunctions of the harmonic oscillator Hamiltonian  $h_1(q)$ . The bond wavefunctions are defined on the positive half-line, with the common vertex at  $x_b = 0$ . The expansion takes the form

$$\psi_b(q, x_b) = \frac{1}{\sqrt{k_0}} \eta_0^{(b)}(x_b) \chi_0(q) + \sum_{n=1}^{\infty} \frac{a_n^{(b)}}{\sqrt{k_n}} \eta_n^{(b)}(x_b) \chi_n(q). \quad (15)$$

The functions  $\eta_n^{(b)}(x_b)$  are given by

$$\eta_0^{(b)}(x_b) = \begin{cases} e^{-ik_0 x_b} + r e^{ik_0 x_b}, & b = b', \\ t e^{ik_0 x_b}, & b \neq b', \end{cases} \quad (16)$$

$$\eta_n^{(b)}(x_b) = e^{-k_n x_b} \quad \text{for } n \geq 1.$$

In this way,  $\Psi(x, q)$  corresponds asymptotically to an incoming wave together with scattered waves in the ground state channel. In all other (closed) channels, the wavefunctions are evanescent along the bonds. Note that in (16) we made use of the expected invariance of the scattering matrix under the interchange of bonds, so  $r$  stands for the reflection (diagonal) elements of the  $\sigma$  matrix, and  $t$  stands for the transmission (off-diagonal) elements of  $\sigma$ . Both parameters are independent of  $b$ .

The continuity of  $\Psi(x, q)$  at the vertex implies that the expansion coefficients  $a_n^{(b)}$  are the same for all  $b$ , and therefore they will be denoted in what follows by  $a_n$ . Moreover,

$$1 + r = t. \quad (17)$$

Substituting in (11) and using (17), we get

$$t = \frac{2}{v} + \frac{\lambda}{2i} \frac{1}{\sqrt{k_0 k_1}} a_1, \quad (18)$$

and

$$2a_1 + \lambda \left( a_2 \sqrt{\frac{2}{k_1 k_2}} \right) = -\frac{\lambda}{\sqrt{k_0 k_1}} t, \quad (19)$$

$$2a_n + \lambda \left( a_{n+1} \sqrt{\frac{n+1}{k_n k_{n+1}}} + a_{n-1} \sqrt{\frac{n}{k_n k_{n-1}}} \right) = 0 \quad \text{for } n \geq 2,$$

where  $\lambda \equiv \sqrt{2}\Lambda/v$ . Equation (19) is an infinite set of linear, inhomogeneous equations, which are defined in terms of the infinite Jacobi matrix  $J(k, \lambda)$  whose elements are explicitly given in (19). For large  $n$ ,

$$\sqrt{\frac{n}{k_n k_{n-1}}} = 1 + \mathcal{O}\left(\frac{1}{n}\right). \quad (20)$$

We shall denote by  $J_0(\lambda)$  the Jacobi matrix whose elements are defined by the equations with constant coefficients

$$2a_n + \lambda(a_{n+1} + a_{n-1}) = 0 \quad (21)$$

which approaches (19) for large  $n$ .

Denoting by  $G(z; k, \lambda)$  the resolvent

$$G(z; k, \lambda) = \frac{1}{J(k, \lambda) - z} \quad (22)$$

we get

$$a_1 = -\lim_{\epsilon \nearrow 0} G_{1,1}(i\epsilon; k, \lambda) \frac{\lambda}{\sqrt{k_0 k_1}} t. \quad (23)$$

Write

$$\gamma(k, \lambda) = \frac{\lambda^2}{2k_0 k_1} \lim_{\epsilon \nearrow 0} G_{1,1}(i\epsilon; k, \lambda). \quad (24)$$

Substituting (23) in (18), we find

$$t = \frac{2/v}{1 - i\gamma(k, \lambda)}; \quad r = t - 1 = \frac{(-1 + 2/v) + i\gamma(k, \lambda)}{1 - i\gamma(k, \lambda)}. \quad (25)$$

Note that when  $\lambda = 0$ , the reflection and transmission coefficients reduce to (13) (with  $\lambda_i = 0$ ). The flux in the leads is conserved if  $(v - 1)|t|^2 + |r|^2 = 1$ , which is satisfied only if  $\text{Im}(\gamma(k, \lambda)) = 0$ . We shall now show that this happens if and only if  $|\lambda| < 1$ . In other words, the dissipation of flux or, equivalently, quantum irreversibility occurs for  $|\lambda| > 1$ .

Following [14], we associate to the Jacobi matrix  $J$  the spectral measure  $\mu(x; k, \lambda)$ , defined by

$$G_{1,1}(z; k, \lambda) = \int \frac{d\mu(x; k, \lambda)}{x - z}. \quad (26)$$

Using the estimate (20), we find that the operator  $(J - J_0)$  is Hilbert–Schmidt. The main theorem in [14] guarantees that the absolutely continuous component of  $\mu(x; k, \lambda)$  is supported in  $\xi(k, \lambda) = [2(1 - |\lambda|), 2(1 + |\lambda|)]$ .

As long as  $|\lambda| < 1$ , the support  $\xi(k, \lambda)$  is confined to the positive half-line. Taking the limit  $z = i\epsilon \nearrow 0$  is straightforward, resulting in a real value for  $\gamma(k, \lambda)$ . Hence for  $|\lambda| < 1$  the flux in the  $x$  channel is conserved.

When  $|\lambda| > 1$ , the interval  $\xi(k, \lambda)$  includes the value 0. To compute (24) we use (3), and get

$$\text{Im} \gamma(k, \lambda) = -\frac{\pi \lambda^2}{2k_0 k_1} \varrho(0, \lambda) \neq 0 \quad (27)$$

where we write  $d\mu(x; k, \lambda) = \varrho(x, \lambda) dx$ . Thus, conservation of flux is violated, which is equivalent to the onset of irreversibility. This is the main result of the present work. It proves the fact that  $\text{Im} \gamma(k, \lambda) \neq 0$ , but it does not provide an explicit expression for  $\gamma(k, \lambda)$ .

However, once it is computed numerically or approximately (see the next paragraph) it can be used together with (25) to compute the vertex scattering matrix,

$$\sigma_{b,b'}(k, \lambda) = -\delta_{b,b'} + t(k, \lambda). \quad (28)$$

An approximate computation of the resolvent for energies in the vicinity of the elastic threshold ( $E \gtrsim \frac{1}{2}$ ) can be obtained by replacing the Jacobi matrix  $J(k, \lambda)$  by  $J_0(\lambda)$ . The matrix elements of  $J_0(\lambda)$  are independent of  $n$ , and an exact solution of the inhomogeneous equations for  $\lambda > 1$  reads

$$a_n = e^{in\alpha} \frac{t}{\sqrt{k_0 k_1}}; \quad \cos \alpha = -\frac{1}{\lambda}. \quad (29)$$

Thus,

$$\gamma(k, \lambda) \cong -\frac{\lambda}{k_0 k_1} e^{i\alpha}. \quad (30)$$

This is an explicit expression which can be directly used in (28).

Note that the valency  $v$  of the vertex enters only through the scaled strength parameter  $\lambda = \sqrt{2}\Lambda/v$ .

When we relax the condition  $E < E_1$  the same phenomenon will occur, albeit now involving the inelastic scattering channels, whose number is finite and determined by the total energy  $E$ . It gives no new insight, and therefore it will not be discussed further.

The fact that the spectrum of the Jacobi matrix is continuous played a crucial role in the previous arguments. It appears in a way which reminds one of the discussion of quantum irreversibility presented in the introduction.

Coming back to the observation that the ingoing current exceeds the total outgoing current, one naturally asks where one can find the rest of the probability density. This, and other relevant questions, will be clarified in the next section.

#### 4. A simple model

The star graph with  $v = 2$  is equivalent to the Schrödinger equation

$$\left[ -\frac{\partial^2}{\partial x^2} + \frac{1}{2} \left( -\frac{\partial^2}{\partial q^2} + q^2 \right) + \Lambda q \delta(x) \right] \Psi(x, q) = E \Psi(x, q), \quad (31)$$

with  $(x, q) \in \mathbb{R}^2$ . The boundary condition at the vertex is replaced by a term which can be interpreted as a potential, linear in  $q$  and well localized in  $x$ . The Schrödinger equation describes a particle in the  $(x, q)$  plane, moving under the action of the potential shown in figure 1, where the  $\delta$  function is replaced by a narrow Gaussian. In this figure, the height of the potential is indicated by the grey level, where black (white) marks the domains where the potential is most negative (positive). Consider now a particle moving in two dimensions and acted upon by this potential. Away from the domain of strong coupling, the potential is independent of  $x$  and quadratic in  $q$ , thus forming a valley along the  $x$  axis. Asymptotically, the particle moves with a constant speed in the  $x$  direction, while oscillating in the  $q$  direction. At the vicinity of  $x = 0$ , the potential changes abruptly. Here a positive ridge (assuming  $\lambda > 0$ ) along the positive  $q$  axis makes the  $q > 0$  domain inaccessible to the particle, while a down sloping valley pointing towards the negative  $q$  axis may attract the particle. As long as we use a finite Gaussian to simulate the  $\delta$  potential, the quadratic potential in the  $q$  direction will take over, and will form a barrier from which the particle will be reflected back, and eventually the particle will find itself moving along the  $x$  axis in either the positive or the negative directions. The limit where the Gaussian becomes a  $\delta$  function is delicate, and our computations show

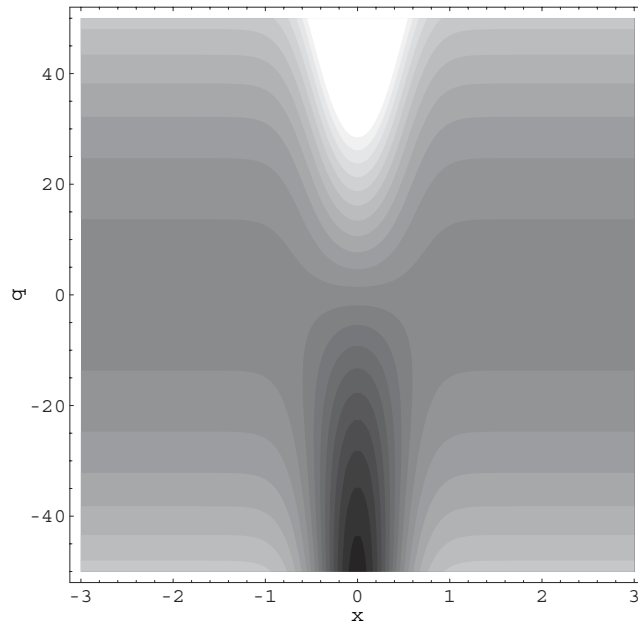


Figure 1. A grey level map of the potential in (31).

that it depends on the size of  $\lambda$ . Once  $|\lambda| > 1$ , the effective potential valley extends to infinity, and it is sufficient to draw flux in the  $q$  direction without ever reflecting it back. Going back to the original way of looking at the problem, we can say that, with a finite probability, the 'system' degree of freedom remains near the vertex at  $x = 0$  while the harmonic oscillator is excited in an outgoing stretching mode due to the strong interaction.

Another point of view can be obtained by studying the stationary wavefunction  $\Psi(q, x)$  (15) at  $x = 0$ . We consider its projection on the space of functions spanned by the oscillator states with  $n \geq 1$ , which correspond to evanescent modes in the  $x$  direction:

$$\Psi(q, 0) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{k_n}} \chi_n(q). \quad (32)$$

We use the approximate expression (29) for  $a_n$ , and replace the harmonic oscillator wavefunctions by their WKB approximation:

$$\chi_n(q) \approx \sqrt{\frac{2}{\pi}} \frac{\cos S(n)}{(2n+1-q^2)^{1/4}},$$

$$S(n) = \left(n + \frac{1}{2}\right) \left( \arcsin \frac{q}{\sqrt{2n+1}} + \frac{q}{\sqrt{2n+1}} \left(1 - \frac{q^2}{2n+1}\right)^{1/2} - \frac{\pi}{2} \right). \quad (33)$$

Substituting in (32), and using the saddle point approximation to perform the sum, we obtain

$$\Psi(x=0, q) \approx C \frac{\exp(-i\frac{q^2}{2} \tan \alpha)}{\sqrt{2\pi i \cos \alpha}} \quad (34)$$

where  $C$  is independent of  $q$ . This function is *not* square normalizable.

The simple computation above provides some heuristic evidence in support of the suggestion that the interacting system possesses a new continuum channel, where the particle can be trapped indefinitely. This issue is discussed in great detail and with rigour in Solomyak's

paper [13]. In the following we shall present another variant of the simple model, where the onset of a continuum at  $|\lambda| > 1$  will be elucidated from yet another perspective.

Consider (31) subject to the boundary conditions

$$\Psi(x = -L_1, q) = \Psi(x = L_2, q) = 0. \tag{35}$$

This Schrödinger operator describes a particle which is confined to the interval  $[-L_1, L_2]$  and a harmonic oscillator, which are coupled by an interaction which is very well localized at  $x = 0$ , and is linear in the oscillator amplitude. The spectrum of the uncoupled ( $\Lambda = 0$ ) problem is pure point:

$$E_{n,m} = \left(n + \frac{1}{2}\right) + \left(\frac{\pi}{L_1 + L_2}\right)^2 m^2; \quad n = 0, 1, \dots; \quad m = 1, 2, \dots \tag{36}$$

We shall turn now to the case  $\Lambda \neq 0$ . We consider in particular the domain of energies  $E < 1/2$ , just under the lowest energy of the unperturbed system. Again we expand  $\Psi(x, q)$  in the complete orthonormal basis  $\chi_n(q)$ , and use the notation  $k_n = +\sqrt{1/2 + n - E}$  for  $n = 0, 1, \dots$

$$\Psi(q, x) = \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{k_n}} \eta_n(x) \chi_n(q), \tag{37}$$

where  $\eta_n(x)$  satisfy the boundary conditions (35), are continuous at the origin, and normalized by  $\eta_n(0) = 1$ . Explicitly,

$$\eta_n(x) = \begin{cases} \frac{\sinh k_n(L_1 + x)}{\sinh k_n L_1}, & x \leq 0, \\ \frac{\sinh k_n(L_2 - x)}{\sinh k_n L_2}, & x \geq 0. \end{cases}$$

These functions are evanescent away from the origin  $x = 0$ . Substituting (37) in (31), we get a set of equations from which the coefficients  $a_n$  are to be determined:

$$\gamma_n a_n + \lambda \left( a_{n+1} \sqrt{\frac{n+1}{k_n k_{n+1}}} + a_{n-1} \sqrt{\frac{n}{k_n k_{n-1}}} \right) = 0, \tag{38}$$

where  $\lambda = \Lambda/\sqrt{2}$  and

$$\gamma_n = \coth k_n L_1 + \coth k_n L_2. \tag{39}$$

Equation (38) is a second order recursion relation with the boundary condition  $a_{-1} = 0$ . The coefficients depend on the energy  $E$ , and the spectrum of the Schrödinger operator is determined as the values of  $E$  for which the series  $a_n(E)/\sqrt{k_n}$  is square summable. Before we proceed to analyse this further, we should point out that in the limit  $n > E$ ,

$$\sqrt{\frac{n}{k_n k_{n-1}}} = 1 + \mathcal{O}\left(\frac{1}{n}\right); \quad \gamma_n = 1 + \mathcal{O}\left(\frac{1}{n}\right). \tag{40}$$

Thus, (38) reduces to the recursion relation with constant coefficients (21). This is of crucial importance to our discussion.

We shall now provide the heuristic arguments which show that for  $|\lambda| < 1$  the spectrum of the Schrödinger operator (31) retains its pure point character. However, for  $|\lambda| > 1$  it has a purely continuous component. Using the initial conditions  $a_{-1} = 0, a_0 = 1$ , one can apply the exact recurrence relation (38) and obtain  $a_N(E)$  for arbitrary large  $N$ . Chose  $N$  such that for  $n \geq N$  one introduces only a small error if one replaces (38) by (21). The solution of (21) can be written explicitly:

$$a_n^{\text{approx}} = A \xi^n + B \xi^{-n}; \quad \xi = \frac{1}{\lambda} (-1 + \sqrt{1 - \lambda^2}), \tag{41}$$

where  $A, B$  are arbitrary constants, to be determined by the initial conditions.

For  $|\lambda| < 1$ ,  $|\xi| < 1$ . Thus, to get a converging solution,  $B$  must vanish. The matching of the approximate series at  $n = N$  requires

$$\xi = \frac{a_{N+1}(E)}{a_N(E)}, \quad (42)$$

which is the spectral secular equation. Limiting our attention to the spectral interval  $E < 1/2$ , and with  $L_1, L_2$  sufficiently large, we can choose  $N = 0$ , and the secular equation reads

$$\xi = \frac{a_1(E)}{a_0(E)} = -\frac{2}{\lambda} \left( \left( \frac{1}{2} - E \right) \left( \frac{3}{2} - E \right) \right)^{1/4}. \quad (43)$$

Thus,

$$E = 1 - \frac{1}{2} \sqrt{1 + \frac{1}{4} (1 - \sqrt{1 - \lambda^2})^4} < \frac{1}{2}. \quad (44)$$

The approximate solution predicts the existence of a bound state of the system below the ground state of the unperturbed harmonic oscillator. This is rigorously proved in [13].

For  $|\lambda| > 1$ ,  $\xi = \exp(-i \arctan(\sqrt{\lambda^2 - 1})) = e^{-i\zeta}$ . Now, both terms in (41) contribute, and the secular equation reads

$$\frac{a_{N+1}(E)}{a_N(E)} = \frac{Ae^{-i(N+1)\zeta} + Be^{i(N+1)\zeta}}{Ae^{-iN\zeta} + Be^{iN\zeta}}. \quad (45)$$

In contrast with the previous case, for every  $E < \frac{1}{2}$ , the matching conditions are satisfied when

$$\frac{A}{B} = -e^{2i(N+1)\zeta} \frac{1 - (a_{N+1}(E)/a_N(E))e^{-i\zeta}}{1 - (a_{N+1}(E)/a_N(E))e^{i\zeta}}. \quad (46)$$

Hence, the secular equation can be solved for every  $E < \frac{1}{2}$ , and the spectrum is continuous. Moreover, the  $a_n(E)$  do not vanish in the limit of large  $n$ . Substituting the  $a_n$  in (37), the resulting  $\Psi(x, q)$  is bounded but not square integrable, which is typical for eigenfunctions in the continuous spectrum. We emphasize again that the onset of irreversibility coincides with the appearance of a continuum. This result, which was illustrated above for the simple model, is general, and is entirely due to the typical form of the boundary conditions at the vertices.

A last comment is in order. Often, when facing an infinite set of equations of the type studied here, it is tempting to truncate them at a physically or computationally motivated point. In the present context, one would naturally attempt to truncate the space of closed channels, and perform the computations within the subspace of opened channels, exclusively. This would *always* result in a unitary description of the vertex scattering matrix, and the possibility of flux dissipation would be overlooked.

To summarize—the main lesson from this work is that genuine quantum flux dissipation and irreversibility can be induced by a single harmonic oscillator—the transmitted and the reflected fluxes *do not* add up to the incoming flux. This result is of general interest because it shows that in a strongly interacting system, continuum channels, which are absent in the uncoupled system and bath, can be opened by the interaction, and induce dissipation. At such instances, the perturbative approach to dissipation might not reflect the true nature of the problem.

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