The quantum graph as a limit of a network of physical wires

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Abstract. Quantum graphs are idealizations of networks of channels with thin yet finite section. To justify this physically motivated picture, we introduce and study a model system where the transition from a network of channels with finite width to the corresponding graph, consisting of infinitely thin wires, can be rigorously discussed and justified.

1. Introduction

Physical networks constrain particles or waves to propagate in restricted domains which are confined to the vicinity of a skeleton of one dimensional, interconnected wires which form a graph. They are realized in e.g., optical fibers where light waves do not leak out of the wires due to the difference in the dielectric constant of the fibers and the embedding space. In miniature electronic devices, as another example, the electron wave-functions are bounded transversally by electric fields. Naturally, in most applications, the transversal dimensions of the fibers or channels are comparable to the wave length of the propagating wave, but much shorter than the length of the wires. It is therefore quite tempting to model these systems by their infinitely thin analogues, were the transversal dynamics can be either neglected or incorporated effectively by a re-normalization of the physical parameters. The task in such a program is twofold – to justify the decoupling of the transversal and the longitudinal dynamics along a single wire, and to provide an adequate description of the junctions where the channels meet.

Various problems which arise in carrying out this program were addressed in the physical and mathematical literature. It is a task beyond the scope of the present note to review all the relevant literature. The interested reader is referred to the papers [**1, 3, 4, 5, 6**] and papers cited therein.

Transversal confinement can be introduced by potentials or by appropriate boundary conditions. The conditions under which the transversal and longitudinal dynamics can be decoupled along a single channel were discussed at length in e.g., [**2**] for potential induced confinement, and in [**1, 3**] when confinement is achieved by boundary conditions. Here, we shall not deal with this issue and assume from the

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start that the channel dynamics is governed by the separable classical hamiltonian,

$$\frac{1}{2M}p_x^2 + Q(x) + \frac{1}{2m}p_y^2 + V(y) \quad ; \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^d, \tag{1.1}$$

where x, (y) stand for the longitudinal (transversal) coordinates and p_x, p_y for their corresponding conjugate momenta. In (1.1) $Q(x)$ stands for a longitudinal potential whose properties will be specified later, see Theorem 2.4. In the present introductory discussion we shall put $Q(x) = 0$. Of all the possible choices for the transversal potentials, we consider only the harmonic potential, $V(y) = \frac{1}{2}Ky^2$. For the sake of simplicity we shall study the situation when $d = 1$. The results, however, can be easily extended to any number of transversal degrees of freedom. The differential operator corresponding to (1.1) is

$$H_{channel} = -\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right); \quad (x, y) \in \mathbb{R}^2$$

where convenient units were chosen for the coordinates, and ω stands for the oscillator quantum of energy.

Let $I = [-L, L]$ be a real interval and denote by $\mathcal{D}$ the infinite strip $I \times \mathbb{R}$. Consider the solutions $\Psi(x, y) \in L^2(\mathcal{D})$ of the Schrödinger equation

$$H_{channel}\Psi(x, y) = E\Psi(x, y) \quad ; \quad E > \frac{w}{2}.$$

$\Psi(x, y)$ can be expanded in terms of the complete set of normalized eigenfunctions of the harmonic oscillator $\chi_n(y)$,

$$\begin{aligned}\Psi(x, y) \quad = \quad & \sum_{n=0}^{N(E)} (a_n \cos k_n x \quad + \quad b_n \sin k_n x \quad) \chi_n(y) \\ & + \sum_{n > N(E)} (a_n \cosh \kappa_n x + b_n \sinh \kappa_n x \ ) \chi_n(y) \ .\end{aligned} \tag{1.2}$$

Here, a_n, b_n are arbitrary coefficients (which are only restricted by the requirement that $\Psi(x, y) \in L^2(\mathcal{D})$), and $N(E) = [(E - \frac{\omega}{2})/\omega]$, where $[\cdot]$ denotes the integer part. For $0 \le n \le N(E)$, $k_n = \big(E - (n + \frac{1}{2})\omega\big)^{\frac{1}{2}}$. For $n > N(E)$, $\kappa_n = \big((n + \frac{1}{2})\omega - E\big)^{\frac{1}{2}}$. The physical interpretation of (1.2) is that at the given energy E, there are $N(E) + 1$ "conducting modes" where waves propagate along the channel with a longitudinal wave number k_n. When $n > N(E)$ the modes are "evanescent", with exponential decay determined by κ_n. Note that in all the quantities defined above, E appears always in the combination $\varepsilon = E - \frac{\omega}{2}$. This is the energy excess above the vacuum energy of the oscillator.

In the model described above for a channel, the thin wire limit is achieved by $\omega \to \infty$ while keeping the excess energy ε constant. Thus, all the modes with $n \ge 1$ become evanescent, and the channel conducts in a single mode, exactly like a one dimensional wire. The remaining problem is to understand what happens at the junctions of a network of connected channels in this limit.

To describe a network of channels, one has to add a term to the hamiltonian (1.1) which allows transitions between the channels to occur in the vicinity of the junction points. This can be done by adding a potential function which is localized at the junction, and extends smoothly to the harmonic channels away from the junction. The spatial dimension of the junction potential should scale as $\omega^{-\frac{1}{2}}$, so that the limit of a thin network of wires is achieved by $\omega \to \infty$.

We shall adopt here an alternative way to introduce the junctions to the model. It is somewhat less intuitive and remote from the underlying physical picture, but it has the advantage of being more transparent mathematically, and allows for a simple and rigorous study of the limit of thin wires. We shall do it in two steps. First, we shall consider a junction with two emerging channels. Then we shall write down the equations which generalize the model to any number of channels.

Choose the junction position at the origin, and the two channels of length L each, along the positive and the negative x axis. The interaction between the channels is induced by

$$H_\eta = \alpha y \frac{1}{\sqrt{\pi\eta}} \exp(\frac{-x^2}{\eta})$$

where the real parameter α is an arbitrary coupling constant. When η is small, this form localizes the interaction to a close x vicinity of the junction. In previous papers [**7, 8**] the present authors discussed a formally identical problem obtained in the limit $\eta \to 0$. They have shown that in spite of the singular behavior of the resulting $\delta(x)$ potential, and the fact that H_η is not bounded from below, the operator $H_{channel} + H_{\eta\to 0}$ is self-adjoint, when considered on an appropriate domain. Moreover one can study its spectrum by replacing the singular potential by boundary conditions at $x = 0$, namely,

i. $\Psi(x, y)$ is continuous at $x = 0$.

ii. $\lim_{\epsilon\to 0} \left[\epsilon^{-1} \left(\Psi(\epsilon, y) - \Psi(-\epsilon, y)\right)\right] = \alpha y \Psi(0, y)$.

Substituting the general form of the wave function (1.2) in the above boundary conditions provide relations between the coefficients a_n, b_n on the two channels, which in turn, describe the coupling between the transversal and the longitudinal dynamics at the junction. The derivation of the boundary conditions is most naturally done by studying the quadratic form which corresponds to the hamiltonian $H_{channel} + H_{\eta\to 0}$:

$$\begin{aligned} \mathbf{h}_{\omega,\alpha}[\Psi] &= \int_{-L}^{L} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \left[|\partial_x \Psi(x, y)|^2 + \frac{\omega}{2} \left(|\partial_y \Psi(x, y)|^2 + y^2 |\Psi(x, y)|\right)^2\right] \\ &\quad + \alpha \int_{-\infty}^{\infty} \mathrm{d}y\; y|\Psi(0, y)|^2. \end{aligned} \tag{1.3}$$

Our next task is to generalize this scheme to the case of arbitrary compact graphs. We do this in the next section.

2. Rigorous results

Below Γ is an arbitrary compact graph, with the set $\mathcal{E}$ of edges and the set $\mathcal{V}$ of vertices. The length of an edge $\epsilon \in \mathcal{E}$ is denoted by $|e|$ and the degree (valency) of a vertex $v \in \mathcal{V}$ by $d(v)$. We write $e \sim v$ if the edge e is incident to the vertex v. If $f(x)$ is a function on Γ which is differentiable along each edge, we denote

$$[f'](v) = \sum_{e\sim v} f'_e(v).$$

Here f_e is the restriction of $f(x)$ to the edge e and all the derivatives are taken in the outward direction with respect to the vertex v.

We study the operator $\mathbf{H}_{\omega,\alpha}$ in the Hilbert space $L^2(\Gamma\times\mathbb{R})$, generated by the differential expression

$$\mathcal{H}_\omega\Psi = -\frac{\partial^2\Psi}{\partial x^2} + \frac{\omega}{2}\big(-\frac{\partial^2\Psi}{\partial y^2} + y^2\Psi\big), \qquad x\in\Gamma\setminus\mathcal{V},\ y\in\mathbb{R} \tag{2.1}$$

and the boundary (matching) conditions at each vertex v, which generalize the conditions *i, ii* in section 1:

i'. Ψ is continuous at each vertex $v\in\mathcal{V}$.

ii'. $[\Psi'_x](v,y) = \alpha y\Psi(v,y), \qquad \forall v\in\mathcal{V}.$

The above defined operator is quite similar to the operator $\mathbf{A}_\alpha$ studied in detail in the papers [**8, 9**]. The difference between these two versions of the operator is twofold: in [**8, 9**] we took $\omega=1$, and the matching condition *ii'* was imposed at only one chosen vertex $v=o$. This difference is not crucial, and the techniques developed in [**9**] extends to the present situation. In particular, the following result takes place, cf. theorems 1 – 3 in [**9**]. We present it without proof.

LEMMA 2.1. *For any real* α *the operator* (2.1) *under the conditions* i', ii' *admits exactly one self-adjoint realization which we denote by* $\mathbf{H}_{\omega,\alpha}$. *If*

$$|\alpha| < \alpha^*(\Gamma,\omega) = \frac{\omega d(\Gamma)}{\sqrt{2}}, \qquad d(\Gamma) := \min_{v\in\mathcal{V}} d(v), \tag{2.2}$$

then the operator $\mathbf{H}_{\omega,\alpha}$ *is bounded from below and its spectrum is discrete. The quadratic form domain of this operator is the same as for* $\alpha=0$.

If $|\alpha| > \alpha^*(\Gamma,\omega)$, *the operator is unbounded from below and its spectrum contains the absolutely continuous component* $\sigma_{a.c.}(\mathbf{H}_{\omega,\alpha})$ *which fills the whole of real axis. For* $|\alpha| = \alpha^* = \alpha^*(\Gamma,\omega)$ *the operator* $\mathbf{H}_{\omega,\alpha}$ *is bounded from below and* $\sigma_{a.c.}(\mathbf{H}_{\omega,\alpha^*}) = [0,\infty)$.

The operator $\mathbf{H}_{\omega,0}$ can be easily studied by separation of variables. It decomposes into the orthogonal sum, cf. §2 in [**8**]:

$$\mathbf{H}_{\omega,0} = \sum_{n\ge 0}^{\oplus}\big(-\Delta_\Gamma + \omega(n+1/2)\big) \tag{2.3}$$

where $-\Delta_\Gamma$ stands for the Neumann Laplacian on Γ. The equality (2.3) shows that the spectrum of $\mathbf{H}_{\omega,0}$ is completely determined by the spectrum of the Laplacian Δ_Γ.

The domain of the operator $\mathbf{H}_{\omega,\alpha}$ for $\alpha\neq 0$ admits an explicit description which is analogous to the description of the domain of $\mathbf{A}_\alpha$ in [**8, 9**]. We do not present it here, since we are interested in the limiting behavior of the operator as $\alpha = const$ and $\omega\to\infty$. For large values of ω the condition (2.2) is certainly satisfied, and the simplest way to study the operator uses quadratic forms. This approach makes it unnecessary to have the explicit description of the operator domain of $\mathbf{H}_{\omega,\alpha}$.

The quadratic form, which formally corresponds to the operator $\mathbf{H}_{\omega,\alpha}$, can be written as

$$\mathbf{h}_{\omega,\alpha}[\Psi] = \mathbf{h}_{\omega,0}[\Psi] + \alpha\mathbf{b}[\Psi] \tag{2.4}$$

where

$$\mathbf{h}_{\omega,0}[\Psi] = \int_{\Gamma\times\mathbb{R}} \Big(|\Psi'_x|^2 + \frac{\omega}{2}(|\Psi'_y|^2 + y^2|\Psi|^2\Big)\, dxdy \tag{2.5}$$

and

$$\mathbf{b}[\Psi] = \sum_{v\in\mathcal{V}} \mathbf{b}_v[\Psi], \qquad \mathbf{b}_v[\Psi] = \int_{\mathbb{R}} y|\Psi(v,y)|^2 dy.$$

This is a direct generalization of the quadratic form as in (1.3) to the case of an arbitrary graph. The quadratic form $\mathbf{h}_{\omega,0}$ is positive definite and closed on its natural domain $\mathfrak{d}$, defined by the condition $\mathbf{h}_{\omega,0}[\Psi] < \infty$.

Our aim is to prove the following result. In its formulation, by $\lambda_n(\mathbf{T})$ we denote the eigenvalues of a non-negative self-adjoint operator $\mathbf{T}$ with discrete spectrum, arranged in a non-decreasing order and counted according to their multiplicities.

THEOREM 2.2. *For an arbitrary compact graph, any fixed $\alpha > 0$, any natural number n and $\omega \to \infty$, the n-th eigenvalue of the operator $\mathbf{H}_{\omega,\alpha} - \frac{\omega}{2}\mathbf{I}$ tends to the corresponding eigenvalue of the Neumann Laplacian on Γ:*

$$\lambda_n(\mathbf{H}_{\omega,\alpha}) - \omega/2 \to \lambda_n(-\Delta_\Gamma) \qquad \text{as } \omega \to \infty.$$

Note that the substraction of the term $\omega/2$ plays the same role as in the special case discussed in section 1.

PROOF. Let us expand functions $\Psi \in \mathfrak{d}$ in the Fourier series with respect to the orthonormal system of Hermite functions,

$$\Psi(x,y) = \sum_{n=0}^{\infty} u_n(x)\chi_n(y),$$

cf. (1.2). Then we get

$$\begin{aligned}\mathbf{h}_{\omega,0}[\Psi] &= \sum_{n\ge 0} \int_\Gamma \big(|u'_n|^2 + \omega(n+1/2)|u_n|^2\big)\, dx,\\ \mathbf{b}_v[\Psi] &= \sum_{n>0} \sqrt{2n}\,\mathrm{Re}(u_n(v)\overline{u_{n-1}(v)}).\end{aligned}$$

Further, we subtract from $\mathbf{h}_{\omega,\alpha}[\Psi]$ the term $\omega\|\Psi\|^2/2$ and add $\|\Psi\|^2$. Denote the resulting quadratic form by $\widetilde{\mathbf{h}}_{\omega,\alpha}[\Psi]$. In particular, for $\alpha = 0$

$$\widetilde{\mathbf{h}}_{\omega,0}[\Psi] = \sum_{n\ge 0} \int_\Gamma \big(|u'_n|^2 + (\omega n + 1)|u_n|^2\big)\, dx.$$

The quadratic form $\mathbf{b}_v[\Psi]$ can be estimated as follows, cf. [**9**], Eq. (9.6):

$$|\mathbf{b}_v[\Psi]| \le \sqrt{2}|u_0(v)||u_1(v)| + \sum_{n\ge 2} \sqrt{\frac{n}{2}}\,\big(|u_n(v)|^2 + |u_{n-1}(v)|^2\big)$$

and further, keeping in mind that $\sqrt{n/2} + \sqrt{(n+1)/2} < \sqrt{2n+1}$,

$$|\mathbf{b}_v[\Psi]| \le \frac{|u_0(v)|^2}{\omega^{1/4}} + (\frac{\omega^{1/4}}{2} + 1)|u_1(v)|^2 + \sum_{n\ge 2} \sqrt{2n+1}|u_n(v)|^2. \tag{2.6}$$

Now we need the following elementary estimate.

LEMMA 2.3. *For any function $w(x)$ from the Sobolev space $H^1(0,l)$, $l<\infty$ and any number $\gamma>0$ the inequality*

$$\gamma|w(0)|^2 \le \coth(\gamma l)\int_0^l (|w'|^2+\gamma^2|w|^2)dx \tag{2.7}$$

is satisfied.

PROOF. Consider the variational problem of minimizing the integral in the right-hand side of (2.7) over the set of functions $w\in H^1(0,l)$, such that $w(0)=1$. Any extremal function must satisfy the homogeneous Euler – Lagrange equation $-w''+\gamma^2 w=0$ and hence, can be written as

$$w(x)=A\sinh\gamma(l-x)+B\sinh\gamma x,$$

with some coefficients A,B. For any such function we have

$$\begin{aligned} J &:= \int_0^l (|w'|^2+\gamma^2|w|^2)dx \\ &= \gamma^2\int_0^l \left(|-A\cosh\gamma(l-x)+B\cosh\gamma x|^2+|A\sinh\gamma(l-x)+B\sinh\gamma x|^2\right)dx \end{aligned}$$

and, after the integration,

$$J=\gamma\sinh\gamma l\left((|A|^2+|B|^2)\cosh\gamma l-(A\overline{B}+\overline{A}B)\right).$$

Since $w(0)=A\sinh\gamma l$, we get

$$\begin{aligned} \frac{\coth\gamma l}{\gamma}J-|w(0)|^2 &= (|A|^2+|B|^2)\cosh^2\gamma l-(A\overline{B}+\overline{A}B)\cosh\gamma l-|A|^2\sinh\gamma l \\ &= |A-B\cosh\gamma l|^2\ge 0, \end{aligned}$$

which completes the proof. □

Note that for $l=\infty$ the estimate (2.7) should be replaced by

$$\gamma|w(0)|^2 \le \int_0^\infty (|w'|^2+\gamma^2|w|^2)dx. \tag{2.8}$$

The following useful estimate is evidently implied by Lemma 2.3:

$$|w(v)|^2\le \frac{\coth(\gamma\varepsilon)}{\gamma d(v)}\int_{S(v)}\left(|w'|^2+\gamma^2|w|^2\right)dx, \qquad \forall v\in\mathcal{V}$$

where $\varepsilon=\min_{e\in\mathcal{E}}|e|$ and by $S(v)$ we denote the star neighborhood of the vertex v, i.e. the union of all edges incident to v. Now we get from (2.6) and (2.8):

$$\begin{aligned} d(v)|\mathbf{b}_v[U]| \le{}& \omega^{-1/4}\coth\varepsilon\int_{S(v)}\left(|u_0'|^2+|u_0|^2\right)dx \\ &+(\omega^{1/4}/2+1)\frac{\coth(\sqrt{\omega+1}\varepsilon)}{\sqrt{\omega+1}}\int_{S(v)}\left(|u_1'|^2+(\omega+1)|u_1|^2\right)dx \\ &+\sum_{n\ge2}\coth(\sqrt{n\omega+1}\varepsilon)\sqrt{\frac{2n+1}{n\omega+1}}\int_{S(v)}\left(|u_n'|^2+(n\omega+1)|u_n|^2\right)dx. \end{aligned}$$

All the factors involving the coth function do not exceed $\coth\varepsilon$, all the other factors in front of the integrals do not exceed $C\omega^{-1/4}$ and by adding up the obtained estimates over all $v \in \mathcal{V}$, we arrive at the inequality

$$|\mathbf{b}[\Psi]| \le C\omega^{-1/4}\widetilde{\mathbf{h}}_{\omega,0}[\Psi], \tag{2.9}$$

with some constant factor C depending only on $d(\Gamma)$ and ε.

It follows from (2.9) that

$$(1 - C\omega^{-1/4}\alpha)\widetilde{\mathbf{h}}_{\omega,0}[\Psi] \le \widetilde{\mathbf{h}}_{\omega,\alpha}[\Psi] \le (1 + C\omega^{-1/4}\alpha)\widetilde{\mathbf{h}}_{\omega,0}[\Psi]. \tag{2.10}$$

An important consequence of this inequality (actually, of its left part) is that for $C\omega^{-1/4}\alpha < 1$ the quadratic form $\widetilde{\mathbf{h}}_{\omega,\alpha}[\Psi]$ is positive definite and closed on the domain $\mathfrak{d}$. The corresponding self-adjoint operator is, of course,

$$\widetilde{\mathbf{H}}_{\omega,0} = \mathbf{H}_{\omega,0} - (\frac{\omega}{2} - 1)\mathbf{I}.$$

Actually, a similar but a bit more involved estimate allows one to show that the quadratic form $\mathbf{h}_{\omega,\alpha}[\Psi]$ is bounded below for all $\alpha < \alpha^*$. However, for α close to α^* the important two-sided estimate (2.10) is no more true.

By the variational principle,

$$(1 - C\omega^{-1/4}\alpha)\lambda_n(\widetilde{\mathbf{H}}_{\omega,0}) \le \lambda_n(\widetilde{\mathbf{H}}_{\omega,\alpha}) \le (1 + C\omega^{-1/4}\alpha)\lambda_n(\widetilde{\mathbf{H}}_{\omega,0}) \tag{2.11}$$

for all ω, α such that $C\omega^{-1/4}\alpha < 1$. By (2.3), the eigenvalues of the operator $\widetilde{\mathbf{H}}_{\omega,0}$ are equal to the numbers $\lambda_k(-\Delta_\Gamma) + \omega m + 1$ (where $k = 1, 2, \ldots; m = 0, 1, \ldots$) rearranged in a non-decreasing order. From here we conclude that for any N there exists ω_0 such that

$$\lambda_n(\widetilde{\mathbf{H}}_{\omega,0}) = \lambda_n(-\Delta_\Gamma) + 1$$

for all $n \le N$ and $\omega > \omega_0$. For such n the inequality (2.11) takes the form (below λ_n stands for $\lambda_n(-\Delta_\Gamma)$)

$$(1 - C\omega^{-1/4}\alpha)(\lambda_n + 1) \le \lambda_n(\mathbf{H}_{\omega,\alpha}) - \omega/2 + 1 \le (1 + C\omega^{-1/4}\alpha)(\lambda_n + 1).$$

Letting here $\omega \to \infty$, we get the desired result for $n \le N$, and hence for all $n \in \mathbb{N}$. □

The result admits a generalization to the case when the Laplacian is replaced by a differential operator of a rather general class, on graphs which are not necessarily compact.

Let Γ be an arbitrary (not necessarily compact) metric graph, whose sets of edges and of vertices are finite. Some of the edges are allowed to be of infinite length, and Γ is compact if and only if all the lengths are finite. Let $a(x), Q(x)$ be measurable real-valued functions on Γ, such that $a(x)$ is bounded and bounded away from zero, i.e. $a(x) \ge \delta > 0$ and $Q(x) \ge 0$ almost everywhere. Consider the quadratic form

$$\mathbf{a}[f] = \int_\Gamma \left(a(x)|f'(x)|^2 + Q(x)|f(x)|^2\right) dx, \tag{2.12}$$

defined on the domain $\mathfrak{d}(\mathbf{a}) = \{f \in H^1(\Gamma) : \mathbf{a}[f] < \infty$. Under the above assumptions, the quadratic form (2.12) is non-negative and closed, and we denote by $\mathbf{A}$ the corresponding self-adjoint operator in $L^2(\Gamma)$. For $a(x)$ smooth, the operator acts as

$$\mathbf{A}f = -(af'_x)_x + Qf.$$

For α small the hamiltonian $\mathbf{H}_{\mathbf{A},\omega,\alpha}$ in the space $L^2(\Gamma\times\mathbb{R})$ can be introduced via the quadratic form similar to (2.4) but with the first term in the integrand in (2.5) replaced by $\mathbf{a}[\Psi(\cdot,y)]$. Note that the hamiltonian as in (1.1) is a particular case of the general hamiltonians $\mathbf{H}_{\mathbf{A},\omega,\alpha}$.

THEOREM 2.4. *Let* Γ, $a(x)$ *and* $Q(x)$ *be as assumed above, and suppose that the operator* $\mathbf{A}$ *has discrete spectrum. Then*

$$\lambda_n(\mathbf{H}_{\mathbf{A},\omega,\alpha})-\omega/2\to\lambda_n(\mathbf{A})\qquad \text{as } \omega\to\infty.$$

The proof is essentially the same as that of Theorem 2.2. Indeed, an analogue of the orthogonal decomposition (2.3) is valid for the operator $\mathbf{H}_{\mathbf{A},\omega,0}$, and the estimate for $\mathbf{b}_v[U]$, which was our main technical tool in the proof, does not involve the potential. Note also that due to the inequality (2.8) the possible presence of infinite edges does not obstruct the estimate. An analogue of the inequality (2.9) is

$$|\mathbf{b}[\Psi]|\le C\delta^{-1}\omega^{-1/4}\widetilde{\mathbf{h}}_{\mathbf{A},\omega,0}[\Psi].$$

It follows that an inequality similar to (2.10) remains valid, whence the theorem.

Remark. To a certain extent, Theorem 2.2 resembles the well known results on the behavior of the spectrum of the Neumann Laplacian $\Delta_{-\Omega(\varepsilon)}$ in thin domains $\Omega(\varepsilon)$, shrinking to a planar metric graph as $\varepsilon\to 0$; concerning these results, see the papers [**3, 4, 5**] and the references therein. Our result is much simpler, partly due to the fact that we view the edges of the graph as line segments, rather than as planar curves. We also note that in order to obtain a meaningful result, we had to consider the numbers $\lambda_n(\mathbf{A}_{\omega,\alpha})-\omega/2$ rather than just $\lambda_n(\mathbf{A}_{\omega,\alpha})$. Similar normalization is known to be necessary in many problems on eigenvalue behavior of the Laplacian in thin domain, see e.g. [**1, 3**].

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