COMMENTS ON THE DESCRIPTION OF SUB-BARRIER FUSION IN TERMS OF DISSIPATIVE TUNNELLING

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We discuss recently proposed methods to analyze sub-barrier fusion data. We show that these prescriptions can be derived from the assumption of a degenerate internal system which interacts with the relative motion. A simple numerical model illustrates the range of validity of this assumption and offers an improved prescription.

The interpretation of fusion reactions between light and medium A nuclei at energies near the Coulomb barrier is commonly and successfully carried out in terms of one-dimensional barrier penetration models [1]. Recently, the measurements of fusion cross sections were extended to sub-barrier energies [2–5]. While the tunnelling model fits very well the higher energy domain, the low energy data exceeds the model predictions by large factors ($\gtrsim 1000$ in some cases [5]). In order to overcome this apparent inconsistency the authors of ref. [5] modified the barrier penetration model by introducing two averaging procedures. The first accounts for the fluctuations in the barrier height due to the zero-point motion of surface modes [6], while the second averages over all orientations of the nuclear symmetry axis relative to the line which joins the centers of the colliding ions [2]. The fits obtained in this way reproduce the data extremely well, with parameters which agree nicely with the values expected from elementary considerations [6].

The impressive success of the averaging procedure calls for a closer scrutiny of its theoretical foundation and range of validity. To this end, we discuss in this note the modifications to the tunnelling model which are due to the fact that the relative motion of the colliding ions is coupled to the internal nuclear degrees of freedom. We show that the averaging procedure mentioned above is due to the coupling of the relative motion to low-frequency internal modes. We propose a criterion for the applicability of the averaging prescription and show also the modification necessary when the above mentioned criteria are not strictly fulfilled.

The potential barrier occurs at distances where the two ions hardly overlap. Therefore, the internal nuclear excitations during the tunnelling process are those corresponding to the excitations of the individual ions, as well as states which are obtained by exchange of a few nucleons. The formation of the neck and other degrees of freedom which characterize the combined system occur at smaller distances. Such degrees of freedom are not considered in the barrier penetration model since it is based on the assumption that whenever penetration occurs, the composite nucleus is formed.

The combined-system dynamics are described by the coupled-channels equations

$$\sum_{\beta} \{[(-\hbar^2/2M)\, d^2/dR^2 + V(R)]\, \delta_{\alpha\beta} + \langle\alpha|W(R)|\beta\rangle\} \phi_{\beta}(R) = (E - \epsilon_{\alpha})\phi_{\alpha}(R)\,, \qquad (1)$$

where $|\alpha\rangle$ stand for the internal states, ϵ_{α} are the corresponding energies, $V(R)$ is the effective elastic barrier and $\langle\alpha|W(R)|\beta\rangle$ is the coupling of the relative motion and the internal system. In writing eq. (1) for nucleon transfers, the no-recoil approximation has been used. Eq. (1) should be solved with boundary conditions which correspond to an incoming plane wave only in the ground-state channel ($\alpha = 0$). Beyond the barrier the ϕ_α are matched to waves which travel away from the barrier.

The fusion probability is taken as the *inclusive* penetrability P^{incl} to tunnel through the barrier irrespective of the final state of the internal system.

$$P^{\text{incl}} = \sum_\alpha \frac{k_\alpha}{k_0} |T_\alpha|^2 . \qquad (2)$$

Here, k_α is the wave number in the exit channel α and T_α is the corresponding tunnelling amplitude.

We shall present analytic solutions to eq. (1) in two limiting cases. Assume first that the internal spectrum is degenerate; that is, $\epsilon_\alpha = \epsilon_0 = 0$. If the interaction can be factorized $\langle\alpha| W(R) |\beta\rangle = f(R) W_{\alpha\beta}$, eq. (1) can be reduced to a set of uncoupled equations (the eigen-channels). This is done in terms of the matrix $U_{\alpha\beta}$ which diagonalizes $W_{\alpha\beta}$. Writing down the solutions which fulfill the boundary conditions and calculating the inclusive probability (2), one obtains

$$P^{\text{incl}} = \sum_\mu |U_{0\mu}|^2 P(\lambda_\mu) , \qquad (3)$$

where $P(\lambda_\mu)$ is the tunnelling probability in the μ eigen-channel, in which the effective potential is

$$V_\mu(R) = V(R) + \lambda_\mu f(R) , \qquad (4)$$

and λ_μ is the μ eigenvalue of $W_{\alpha\beta}$.

The result (3) indicates that the tunnelling probability is obtained by averaging the individual probabilities corresponding to the potentials $V_\mu(R)$, using a weight $|U_{0\mu}|^2$ which is the projection of the ground state on the eigen-channel states $|\mu\rangle$.

Consider now the internal degrees of freedom which describe the surface vibrations of a given multipolarity. As long as the interaction with the relative motion can be factorized as $W = f(R) g(\beta)$ [7] we can use the results obtained above in the following way. The operator $g(\beta)$ has eigenfunctions $\delta(\beta - \beta')$ with eigenvalues $g(\beta')$. The projection of the nuclear ground state $|0\rangle$ on this eigen-channel basis is $\phi_0(\beta) = (\pi^{1/2}\beta_0)^{-1/2} \exp[-\frac{1}{2}(\beta/\beta_0)^2]$ where $\beta_0 = (h/m\omega)^{1/2}$ is the RMS deviation of the zero point fluctuations in the ground state. Eq. (3) transforms into

$$P^{\text{incl}} = \int d\beta\, |\phi_0(\beta)|^2 P(\beta) , \qquad (5a)$$

and as in eq. (3) $P(\beta)$ is the penetrability through the potential $V_\beta(R) = V(R) + g(\beta) f(R)$.

The same arguments can be applied to the nuclear rotation modes. Assuming $W = f(R) Q_2 P_2(\cos\theta)$ [7] and neglecting effects which are due to angular momentum coupling we get

$$P^{\text{incl}} = \tfrac{1}{2} \int \sin\theta\, P(\theta)\, d\theta , \qquad (5b)$$

where now $P(\theta)$ is the tunnelling through the potential barrier $V(\theta) = V(R) + f(R) Q_2 P_2(\cos\theta)$.

Eqs. (5a), (5b) are the basis of the averaging method which was used in refs. [2,5,6]. This procedure is exact when the internal spectrum is degenerate. It provides a good approximation for those modes of excitation in which the mean excitation energy $\langle\epsilon_\alpha\rangle$ is sufficiently small that

$$\langle\epsilon_\alpha\rangle \ll \hbar/\tau , \qquad (6)$$

where τ is the "tunnelling time"

$$\tau = \int_{R_1}^{R_2} \frac{dR}{\{2M[V(R) - E]\}^{1/2}}$$

$$\cong 2\pi \{[d^2V(R)/dR^2]_{R=R_b}/M\}^{1/2} ,$$

$$V(R_1) = V(R_2) = E .$$

For most fusion reactions which involve medium A nuclei, condition (6) is well satisfied for excitations of the order of $\langle\epsilon_\alpha\rangle < 2$ MeV, so that the application of (5a)–(5b) for the cases studied in ref. [5] are well justified.

In practical calculations, one may use eq. (3) in order to discuss the effect of rotation and vibration modes. This is done by truncating the spectra to a finite number of states. The expression for the coupling matrix is straightforward and reliable results are obtained using a space of about 10 states.

The role of excitations with very high energy $\langle\epsilon_\alpha\rangle$ can also be analyzed. This corresponds to the adiabatic limit, and under these circumstances the penetrability should be calculated in terms of the modified potential $V(R) + \lambda_0(R)$ where $\lambda_0(R)$ is the lowest eigenvalue of $\langle\alpha| W(R) |\beta\rangle + \epsilon_\alpha \delta_{\alpha\beta}$ which, for $R \to \infty$, coincides with the ground state energy. $\lambda_0(R)$ is the well known "polarization" or "adiabatic" conservative potential which is due to the presence of high-frequency modes. It affects not only the tunnelling probability but also elastic scattering. Thus, when the potential $V(R)$ is derived by fitting the calculated elastic cross-sections to the data, $V(R)$ already includes the polarization contributions.

In summary, we have shown that out of all the internal degrees of freedom that might modify the tunnelling process, only those with low frequencies should be considered in an averaging procedure over the "zero point fluctuations". The integration over zero point fluctuations was demonstrated to be due to the summation over all final internal states in calculating the *inclusive* penetrability. High frequency modes introduce effective adiabatic potentials which renormalize the barrier in a well understood manner.

To investigate the effects of a non-degenerate spectrum, we devised the following simple model. In eq. (1) the effective elastic potential $V(R)$ was taken as a square barrier of height V, and the interaction term $\langle\alpha|W(R)|\beta\rangle$ was taken as $f(R)\,W_{\alpha\beta}$, with $f(R)$ a square barrier of height f overlapping $V(R)$, and W containing coupling only to the ground state: $W_{\alpha\beta} \neq 0$ only for $\alpha = 0$ or $\beta = 0$. The tunnelling penetrabilities can be calculated conveniently and reliably. No approximations are necessary, and the only possible source of error is numerical, which was kept very small.

In the degenerate case ($\epsilon_\alpha = \epsilon_0 = 0$), the eigenvalues are $\lambda_{1,2} = \pm(\Sigma_\beta W_{0\beta}^2)^{1/2}$, $\lambda_{\mu\geqslant 3} = 0$ and $|U_{01}|^2 = |U_{02}|^2 = \frac{1}{2}$, $U_{0\mu} = 0$ for $\mu \geqslant 3$. Eq. (3) is

$$P^{\text{incl}} = \tfrac{1}{2}(P_1 + P_2)\,, \tag{7}$$

where $P_{1,2}$ corresponds to tunnelling through the barrier $V \pm (\Sigma_\beta W_{0\beta}^2)^{1/2} f$.

To determine the important parameters on which the inclusive penetrability depends, we made a large series of calculations comparing the degenerate approximation of the penetrability [eq. (7)] to the result calculated for the non-degenerate system. For each trial the internal spectrum ϵ_α and the coupling matrix elements $W_{0\beta}$ were chosen randomly, ϵ_α uniformly in the interval $0 \leqslant \epsilon_\alpha \leqslant \epsilon_{\max}$, and $W_{0\beta}$ in a gaussian distribution with second moment $\langle W_{0\beta} W_{0\gamma}\rangle = \bar{W}^2 \delta_{\beta\gamma}$. Typically, a space of 20–30 levels was used. Fig. 1 shows some representative results, using the following parameters: $U = 130$ MeV, $f = 1$ MeV, width of both U and $f = 3$ fm, $E = 110$ MeV, $\epsilon_{\max} = 5$ MeV, $\bar{W}^2 = 2$, dimension of space = 20 and reduced mass = 30 AMU. These represent a realistic barrier for the system Ar + Sm. Following the results of refs. [8,9], we took the barrier height

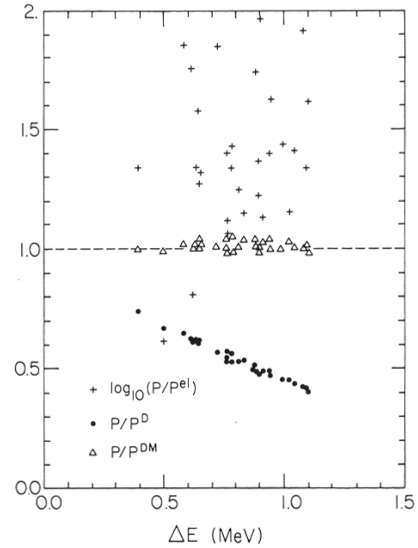

Fig. 1. Inclusive penetrabilities thruogh a square dissipative barrier. P^{el}, P^{D} and P^{DM} stand for three approximate ways to calculate the inclusive penetrability and P stands for the exact results.

to include polarization effects, and for each trial obtained a renormalized potential by subtracting from V the shift in ground state energy due to polarization. This is simply the lowest eigenvalue of the matrix $\epsilon_\alpha \delta_{\alpha\beta} + W_{\alpha\beta}$.

In fig. 1 each trial is identified by the energy loss

$$\Delta E = \sum_{\alpha=1}^{N} |T_\alpha|^2 \epsilon_\alpha \Bigg/ \sum_{\alpha=0}^{N} |T_\alpha|^2\,,$$

where T_α is the tunnelling amplitude as defined above. Three approximate penetrabilities are compared to the exact penetrability, all calculated with the renormalized barrier. (a) P^{el}: the penetrability through the renormalized elastic barrier. The large fluctuations apparent in $\log_{10}(P/P^{\text{el}})$ show that it has no strong systematic relationship with the exact penetrability. (b) P^{D}: the inclusive penetrability calculated for the corresponding degenerate hamiltonian [eq. (7)]. The ratio P/P^{D} shows a systematic dependence on E, while its scatter is reduced to a few percent. This behaviour shows that P^{D} reproduces the variations in P which depend on the particular choice of the $W_{0\beta}$ and which differ from trial to trial. More-over, it is clear that the correction to P^D depends almost entirely on the energy loss E and not on the details of the ϵ_α distribution. The behaviour of the ratio P/P^D versus energy leads us to define (c) P^{DM}, a modified penetrability calculated on the basis of eq. (7) for an effective energy $E - \Delta E$. One can clearly see from fig. 1 that this prescription reproduces the exact P to within ±5%. We checked the validity of the prescription P^{DM} in various cases and found that it invariably provides a better approximation than P^D. The quality of the approximation deteriorates when ϵ_{max} is increased or when the bombarding energy gets too close to the elastic barrier. Still, only in rare cases did we observe deviations which exceed the 50% range.

We propose to use the prescription P^{DM} for more realistic barriers and interactions. In order to determine ΔE, one needs the population of all channels after tunnelling. These can be approximated by the population of the corresponding states in the degenerate system, obtainable from the tunnelling amplitudes $\tilde{T}_\mu$ of the eigen-channels:

$$|T_\alpha|^2 \simeq \left| \sum_\mu U^{-1}_{\alpha\mu} \tilde{T}_\mu \right|^2 . \qquad (8)$$

This approximation was checked numerically and was found to give satisfactory results for cases with moderate values for ϵ_{max}. When applied to the system $^{40}Ar + ^{154}Sm$, considering only the rotational internal degree of freedom, we found P^{DM} to differ from P^D by only 10%. In the vibrational case the correction amounts to about 30%.

In summary, we have shown that the successful and intuitively appealing averaging procedure to account for rotations and zero-point vibrations can be derived from the single assumption of a degenerate spectrum of the internal hamiltonian. Numerical tests within a schematic model indicate that this approximation overestimates the exact answers. This can be rectified by taking into account the energy lost of the internal system.

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