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Spectral properties of systems with dynamical localization: II. Finite sample approach

T Dittrich† and U Smilansky

Department of Nuclear Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

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Abstract. We study correlations in the quasi-energy spectrum of the quantum kicked rotor restricted to a Hilbert space of *finite* dimension. The spectral correlations depend on the ratio γ of the localization length to the basis size. We derive semiclassical expressions for the two-point cluster function which interpolate between COE behaviour for $\gamma \to \infty$ and Poissonian (lack of correlations) for $\gamma \to 0$. We show how the diffusive nature of the classical dynamics finds its expression in the quantal spectral correlations.

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1. Introduction

This is the second of two articles dedicated to the study of the fluctuations in the spectrum of the one-step evolution operator for systems displaying dynamical localization. In particular, we shall concentrate on the quantum kicked rotor (QKR) as the paradigm of this class of systems. The spectrum of the evolution operator consists of unimodular numbers, whose phases (the quasi-energies) are the subject of our investigation. The QKR evolves in an infinite-dimensional Hilbert space (corresponding to the cylindrical classical phase space $-\infty < l < \infty$, $0 \le \theta < 2\pi$). Hence the quasi-energy (QE) spectrum consists of an infinite (denumerable or non-denumerable) number of points which densely cover the unit circle. To study the spectral fluctuations, one has to introduce a mechanism which enables focusing on a *finite* subset of points, and the choice of this subset should be both systematic and physically motivated.

In the previous article of this series [1] (to be referred to as I), we showed that the local spectrum approach provides a selection mechanism which satisfies the requirements mentioned above. This is done, however, at the expense of assigning a weight to each QE, thus treating the various QE values on unequal footing. In contrast to the local spectrum approach, we shall discuss here the QE spectrum from a 'democratic' point of view: all QEs are given equal weights in the definition of the spectral density. To this end we consider the dynamics of the rotor in a *truncated* space of eigenvectors of the unperturbed system H_0, and our objective is to study the dependence of the spectral correlations on the dimension of the truncated basis. A closely related question, the dependence of the degree of localization of the QE

† Permanent address: Institut für Physik, Universität Augsburg, Memminger Straße 6, D-8900 Augsburg, Federal Republic of Germany.

eigenstates in the finite sample on the basis size, has recently been addressed by Casati *et al* [2].

The truncation can be accomplished by various means. One can restrict the classical phase space to a torus. A quantization of this model is achieved by restricting the effective Planck's constant to rational values [3]. Another method consists in using broadened 'pulses' instead of infinitely sharp kicks for the driving force [4]. The broadened pulses do not contain high frequencies and therefore do not couple high-lying states. The effective size of the space is determined by the pulse width. A third way of truncation, which we use in the present work, is introduced as follows.

Let the time-dependent Hamiltonian be represented by the *Hermitian* matrix

$$\langle l \,|\, H^{(L,l_c)}(t) \,|\, l' \rangle = \tfrac{1}{2} l^2 \delta_{l-l'} + k \langle l \,|\cos\theta\,|\, l' \rangle \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \qquad |l - l_c|, \quad |l' - l_c| \le \hat{L}. \tag{1.1}$$

l_c denotes the centre of the angular momentum interval considered, and the dimension of the basis is

$$L = 2\hat{L} + 1. \tag{1.2}$$

We use the same system of units and notations as introduced in I. The Hamiltonian (1.1) controls the *unitary* evolution in the Hilbert space spanned by the L eigenvectors of the free rotor. If the dimension L exceeds by far the localization length (given, say, by the participation ratio ξ), this evolution approximates that of the unmodified quantum kicked rotor ($L \to \infty$). Therefore, this method of truncation is also the most natural one to be used in numerical simulations of the QKR.

The classical analogue of the quantum system described above is obtained by considering a finite phase-space cylinder whose length is L and whose midpoint on the l axis is l_c. The classical equations of motion are modified by imposing elastic reflection conditions at the boundaries $|l - l_c| = \hat{L}$, corresponding to local antisymmetry of the wavefunction in the quantal case. This means that, once a trajectory l_n, θ_n reaches one of these boundaries, the angular momentum increment is reversed by the replacement $\theta_n \to 2\pi - \theta_n$. Since this amounts to switching from a trajectory to its conjugate counterpart (see I, (2.4)), essential properties like periodicity and boundary conditions with respect to l are conserved under this operation.

The unitary one-cycle evolution operator is defined by

$$U^{(L,l_c)} = T \exp\left(-\mathrm{i} \int_{\epsilon}^{\tau+\epsilon} \mathrm{d}t \, H^{(L,l_c)}(t)\right) \tag{1.3}$$

where T denotes time ordering (the operations of truncation and exponentiation do not commute!). The eigenvalues $\exp(\mathrm{i}\omega_\alpha)$, $\alpha = 1, \ldots, L$, of $U^{(L,l_c)}$ are the object of our study.

We define a smoothed spectral density as

$$d^{(L,l_c)} = \sum_{\alpha=1}^{L} \delta^{(N)}(\omega - \omega_\alpha) \tag{1.4}$$

with

$$\delta^{(N)}(x) = \frac{1}{2\pi} \sum_{n=-\hat{N}}^{\hat{N}} e^{inx} \tag{1.5}$$

a smooth periodic function which approaches the periodic delta function for $N \to \infty$, $N = 2\hat{N} + 1$. Its width is of the order of magnitude $\pi/\hat{N}$ and $\delta^{(N)}(0) = N/2\pi$.

The choice of N is restricted by the following considerations: in order to resolve the L eigenphases ω_α to a degree better than their mean separation, the condition $L < N$ has to be satisfied. Since, on the other hand, we are going to apply semiclassical techniques to evaluate the spectral fluctuations, we must also limit N from above, in order to avoid convergence problems as they emerge in trace formulae if the underlying classical dynamics is chaotic [5]. For the time being, we assume a choice $N \gtrsim L$ and defer a more thorough clarification of this issue to a later stage of the discussion.

An alternative expression for the spectral density is [6]

$$d^{(L,l_c)}(\omega) = \frac{1}{2\pi} \sum_{n=-\hat{N}}^{\hat{N}} e^{in\omega} \operatorname{tr}\left((U^{(L,l_c)})^n\right). \tag{1.6}$$

The diagonal matrix elements of $(U^{(L,l_c)})^n$ are the probability *amplitudes* to stay at the initial state l after n periods of the driving force. Therefore this relation provides the link between the spectral distribution and the dynamics of the rotor.

We introduce the *pair distribution function* [7] by

$$R_2^{(L,l_c)}(\eta) = \int_0^{2\pi} d\Omega\, d^{(L,l_c)}((\Omega + \eta/2) \bmod 2\pi)\, d^{(L,l_c)}((\Omega - \eta/2) \bmod 2\pi) - L\,\delta(\eta) \tag{1.7}$$

where a transformation to the angles Ω, η on the unit torus as in (3.8) of I is assumed. This function is normalized to the number of pairs of non-identical eigenphases

$$\int_0^{2\pi} d\eta\, R_2^{(L,l_c)}(\eta) = L(L-1). \tag{1.8}$$

It can be written in the alternative form

$$\begin{aligned} R_2^{(L,l_c)}(\eta) &= \frac{1}{2\pi} \sum_{n=-\hat{N}}^{\hat{N}} e^{-in\eta} \left(|\operatorname{tr}\left((U^{(L,l_c)})^n\right)|^2 - L \right) \\ &= \frac{L}{2\pi} \left(L - 1 + 2 \sum_{n=1}^{\hat{N}} b^{(L,l_c)}(n) \cos(n\eta) \right) \end{aligned} \tag{1.9}$$

where the functions

$$b^{(L,l_c)}(n) = \frac{1}{L} |\operatorname{tr}\left((U^{(L,l_c)})^n\right)|^2 - 1 \tag{1.10}$$

are the Fourier coefficients of the spectral cluster function. In the limit $L \to \infty$ they correspond to Dyson's form factor $b_2(n)$ [7].

Finally, the normalized *two-point cluster function* $Y_2^{(L,l_c)}(r)$ [7], with a scaled argument $r = \eta L/2\pi$, is given by

$$Y_2^{(L,l_c)}(r) = \frac{1}{L}\left(1 - 2\sum_{n=1}^{\hat{N}} b^{(L,l_c)}(n)\cos\left(n\frac{2\pi}{L}r\right)\right). \tag{1.11}$$

So far the spectral functions where defined for a particular truncation of the Hilbert space which is determined by the values of L and l_c (see (1.1)). Since the spectral properties are statistically uniform along the angular momentum axis, it is advantageous to consider *mean values* of the spectral functions along the cylinder. This can be achieved by averaging over l_c. The averaging operation will be denoted by angular brackets in the following and indicated in the corresponding symbols by dropping the label l_c. Thus, the mean cluster function is

$$Y_2^{(L)}(r) = \frac{1}{L}\left(1 - 2\sum_{n=1}^{\hat{N}} b^{(L)}(n)\cos\left(n\frac{2\pi}{L}r\right)\right) \tag{1.12}$$

and

$$b^{(L)}(n) = \langle b^{(L,l_c)}(n)\rangle = \frac{1}{L_c}\sum_{l_c=-\hat{L}_c}^{\hat{L}_c} b^{(L,l_c)}(n). \tag{1.13}$$

The averaging interval is $-\hat{L}_c \le l_c \le \hat{L}_c$, $L_c = 2\hat{L}_c + 1$.

The calculation of the two-point correlation function and its Fourier coefficients will be discussed in the next section.

There are several scales of the angular momentum l which are of importance for the understanding of the dependence of the spectral fluctuations on the size of the system. Classical dynamics provides us with two characteristic scales: the first has to do with an intrinsic symmetry of the classical equations of motion in the unmodified phase space (see I, (2.5)). A shift of l by an integer multiple of $2\pi/\tau$ does not affect the dynamics (since the angles are defined mod 2π). Thus $L_u = 2\pi/\tau = (2D)^{1/2}2\pi/K$ determines the size of the 'unit cell' in the classical description. K and D are the classical stochasticity parameter and diffusion constant, respectively. Another classical scale arises from the fact that the classical evolution is diffusive along the l axis. Hence we can define $L_D = D^{1/2}$, which gives a measure of the broadening along the l axis of a phase-space distribution during one cycle of the driving force. Both L_D and L_u are proportional to $\sqrt{D}$. Localization provides a quantum mechanical scale through the mean localization length, for which the mean participation ratio ξ discussed in I is an adequate measure. Since $\xi \approx 0.3D$ (see I, the paragraph following (2.17)) and in the semiclassical limit $\xi \gg 1$, we have

$$L_u \lesssim L_D \ll \xi. \tag{1.14}$$

In the following we shall discuss the spectral fluctuations for systems with arbitrary length L and we shall see that their statistics depends crucially on the value of L relative to the scale established in (1.14). This ratio is given in terms of the parameter

$$\gamma = D/L \approx 3\xi/L. \tag{1.15}$$

The following timescales are relevant to the discussion of spectral correlations in the QKR: the shortest one is the time n_d which is inversely proportional to the classical Lyapunov exponent. It determines the time after which the diffusion approximation becomes valid. A cylinder of length L will be covered diffusively during a time $n_L \approx L^2/D$. It is convenient to define

$$n_L = \frac{L^2}{2\pi D}. \tag{1.16}$$

As long as $\gamma < 1$, we have $n_L > L > \xi$. For $\gamma > 1$ and increasing γ, n_L decreases till it reaches the value $n_L = 1$ at $\gamma \approx \xi^{1/2}$. At this point $L \approx L_\mathrm{u}$, in other words, the classical phase-space cylinder is truncated to the size of the unit cell, and the concept of diffusion along the l axis becomes inadequate. Quantum mechanics introduces another timescale n_qm which was discussed in I and will be a central issue here as well. This is the time after which the quantum evolution departs from the corresponding classical description due to the fact that quantum interference effects dominate the dynamics. We shall show that

$$n_\mathrm{qm} \approx \min(\xi, L). \tag{1.17}$$

2. The two-point correlation function

This section will be dedicated to the calculation of $Y_2^{(L)}(r)$ via its Fourier coefficients $b^{(L)}(n)$. Only a few exact results can be derived and they will be presented in the first part of this section. Some of the methods used in the first part will appear again in the second part, where we shall derive semiclassical expressions for the two-point cluster function.

2.1. Some exact results

(i) *Unperturbed motion ($k = 0$).*

In this case the Floquet operator $U^{(L,l_\mathrm{c})}$ is diagonal and has eigenvalues $\exp(-\mathrm{i}l^2\tau/2)$, with $|l - l_\mathrm{c}| \le \hat{L}$. The nature of the spectrum is determined by the number theoretic properties of the parameter $\tau/2\pi$. If this parameter is rational, i.e. $\tau/2\pi = p/q$, where p and q are relative primes, the QE spectrum (for $L \to \infty$) consists of the (at most) $2q$ values $(m\pi/q) \bmod 2\pi$, $m = 1, \ldots, 2q$, which are multiply degenerate [8]. We thus expect the function $Y_2^{(L)}$ to be composed of delta spikes representing this clustering of eigenphases. If, on the other hand, $\tau/2\pi$ is a typical irrational number, we expect that the eigenphases cover the unit circle at random (and densely as $L \to \infty$ [9]), so that $Y_2^{(L)} \equiv 0$.

To see how these expectations can be rigorously substantiated, consider the expression (1.10) for $b^{(L,l_\mathrm{c})}(n)$. In the present case,

$$\begin{aligned} b^{(L,l_\mathrm{c})}(n) &= \frac{1}{L}\left|\sum_{l=-\hat{L}}^{\hat{L}} \mathrm{e}^{-\mathrm{i}n(l+l_\mathrm{c})^2\tau/2}\right|^2 - 1 \\ &\approx \frac{1}{\hat{L}}\left|\sum_{l=1}^{\hat{L}} \mathrm{e}^{-\mathrm{i}nl^2\tau/2}\right|^2 - 1 + \mathrm{O}\left(\frac{1}{\hat{L}}\right). \end{aligned} \tag{2.1}$$

The sum appearing on the right-hand side of (2.1) has recently been investigated by Berry and Goldberg [10] (this work will be referred to as BG in the following). We shall adopt their techniques and notations to treat these sums, for which Berry and Goldberg coined the name 'curlicue sums'. As a matter of fact, the last expression in (2.1) follows from the approximate relation

$$\sum_{l=-\hat{L}}^{\hat{L}} \mathrm{e}^{-\mathrm{i}n(l+l_c)^2\tau/2} \approx \mathrm{e}^{-\mathrm{i}n(l_c)^2\tau/2} \sum_{l=-\hat{L}}^{\hat{L}} \mathrm{e}^{-\mathrm{i}nl^2\tau/2} \tag{2.2}$$

which can be derived by using the techniques of BG. Averaging (2.1) over l_c is trivial since l_c appears only in a phase factor. Hence,

$$b^{(L)}(n) \approx \frac{1}{\hat{L}} \left| S_L \left(n\frac{\tau}{2\pi} \right) \right| - 1. \tag{2.3}$$

Here S_L stands for the curlicue sum as defined in BG. For $\tau/2\pi$ a typical irrational number, it is shown in BG that, in the limit $L \to \infty$ and independent of n,

$$\left| S_L \left(n\frac{\tau}{2\pi} \right) \right| = \hat{L}^{\beta} \tag{2.4}$$

with $\beta = 1/2$ for almost all values of $\tau/2\pi$. This result corresponds to the simple picture where a curlicue is described as a random walk in the complex plane.

Substituting (2.4) with $\beta = 1/2$ in (2.3), we find that $b^{(L)}(n) \to 0$ for sufficiently large L, independently of the value of n. Thus, for all r,

$$\lim_{L\to\infty} Y_2^{(L)}(r) = 0 \tag{2.5}$$

which reflects the fact that two-point correlations do not exist in the QE spectrum. (A proof that the spectrum is Poissonian would, however, need to show in addition the absence of all higher-order correlations.)

For values of $\tau/2\pi$ 'close to a rational number', one gets $1/2 \le \beta \le 1$. Berry and Goldberg give a number-theoretical explication for the term 'close to rational' and show that the measure of the corresponding set is zero. We expect that it is closely related to the class of irrational values of $\tau/2\pi$, characterized by fast convergence of their continued fraction expansion, which lead to a singular continuous QE spectrum in the QKR, as has been proven by Casati and Guarneri [11]. In this case the expression for $Y_2^{(L)}(r)$ diverges with L in a complicated way.

Also for $\tau/2\pi = p/q$, the expected behaviour of $Y_2(r)$ follows from the behaviour of the corresponding curlicue sums. In the following paragraph we provide the (rather technical) proof.

If $\tau/2\pi = p/q$, $S_L(np/q)$ is periodic in n with a period $2q$. Hence

$$b^{(L)}(n) = b^{(L)}(n + 2mq) \qquad m = 0, 1, \ldots . \tag{2.6}$$

For a given n, $1 \le n \le 2q$, let $(p(n)/q(n)) \bmod 1$ be the form of np/q, reduced to relative primes. Then for $L \gg 1$, following BG,

$$b^{(L)}(n) \approx \begin{cases} \hat{L}c_e(n) & \text{for } p(n)q(n) \text{ even} \\ \dfrac{1}{\hat{L}}c_o(n) - 1 & \text{for } p(n)q(n) \text{ odd} \end{cases} \tag{2.7}$$

with $c_e(n)$ and $c_o(n)$ positive numbers of order unity, given explicitly in BG. Thus, to leading order in $1/L$,

$$Y_2^{(L)}(r) \approx - \sum_{n=0}^{[N/2q]} \sum_{m=1}^{2q} c_e(m) \cos\left((n+2mq)\frac{2\pi}{L}r\right). \tag{2.8}$$

Here, $[x]$ stands for the integer part of x, and the inner sum goes over those values of n for which the upper alternative in (2.7) holds. This result describes a sum of at most $2q$ smoothed delta functions, as expected.

The relation between the spectral features at issue and the properties of the curlicue sums will be used again in the subsequent discussion.

(ii) *The case $n = 1$, for $\gamma < 1$.*

For the QKR in a truncated basis, the form of the one-cycle propagator is simple and can be written explicitly as

$$\langle l \,|\, U^{(L,l_c)} \,|\, l' \rangle = e^{-il^2\tau/2} (-i)^{l-l'} J_{l,l'}^{(L)}(k) \tag{2.9}$$

where $(-i)^{l-l'} J_{l,l'}^{(L)}(k)$ is an element of the unitary matrix generated by exponentiation of the potential term $\langle l \,|\, k\cos(\theta) \,|\, l' \rangle$ of the truncated Hamiltonian (1.1). For $L \gg k$ (which is always the case for $\gamma < 1$ and $\xi \gg 1$) we may replace $J_{l,l'}^{(L)}$ by its counterpart for the QKR with infinite-dimensional Hilbert space,

$$J_{l,l'}^{(L)}(k) \approx J_{l-l'}(k) \tag{2.10}$$

with $J_l(x)$ the Bessel function of first kind. Again l_c appears only in a phase factor and the averaging is trivial. Hence,

$$b^{(L)}(1) = \frac{1}{L}(J_0(k))^2 \left| \sum_{l=-\hat{L}}^{\hat{L}} e^{il^2\tau/2} \right|^2 - 1. \tag{2.11}$$

The curlicue sums appear again and determine the large-L behaviour of $b^{(L)}(1)$ in the way familiar from the preceding discussion of the limit $k \to 0$. The most important feature of (2.11) is that $b^{(L)}(1)$ factorizes to a product: one factor contains information on the driving force alone, while the other depends exclusively, through interference of the phases generated by the free rotation, on the number-theoretical properties of τ. We shall show below that this factorization extends to the case of arbitrary n, as long as k is large enough.

For further reference we note that, if $\tau/2\pi$ is a typical irrational number, evaluation of the sum in (2.11) according to BG yields

$$b^{(L)}(1) = 2(J_0(k))^2 - 1. \tag{2.12}$$

(iii) *The limit of large n.*

For large times $n \gtrsim L$,

$$b^{(L)}(n) = \frac{1}{L}\left\langle \left| \sum_{\alpha=1}^{L} e^{in\omega_\alpha} \right|^2 \right\rangle - 1 \tag{2.13}$$

must fluctuate about the value zero as long as $2\pi/n$ is smaller than the minimum distance between any pair of phases and the ω_α are not rationally related to 2π. Under these conditions,

$$\lim_{n\to\infty} b^{(L)}(n) = 0. \tag{2.14}$$

We are not able to give an expression describing the approach to this asymptote. We shall argue, however, that this asymptote is reached already for $n > n_{\mathrm{qm}}$, the timescale defined in (1.17).

2.2. Semiclassical theory

In order to calculate the Fourier coefficients $b^{(L}(n)$ semiclassically, we recall that the diagonal elements of $(U^{(L,l_c)})^n$ in the unperturbed basis $\{|l\rangle\}$ represent the probability amplitude to find the system after n cycles of the driving force in the same state $|l\rangle$ where it was prepared initially. That is, in *the semiclassical approximation*,

$$\left((U^{(L,l_c)})^n\right)_{l,l} = \sum_s \sqrt{p_s(l,l;n)}\,\mathrm{e}^{\mathrm{i}\Phi_s(l,l;n)} \tag{2.15}$$

where s labels the classical trajectories which satisfy the boundary condition $l_0 = l_n = l$, $p_s(l,l;n)$ is the probability contributed by the trajectory s, and $\Phi_s(l,l;n)$ denotes the corresponding accumulated phase, determined by the action integral along s as well as the Maslov index. These trajectories are closed with respect to angular momentum, but not necessarily periodic. By performing the trace operation semiclassically, one obtains an expression analogous to Gutzwiller's trace formula (see, e.g., [6]),

$$\mathrm{tr}\left((U^{(L,l_c)})^n\right) = \frac{1}{2}\sum_{\substack{p\\ n_p=n}} \frac{1}{\sqrt{R_p}}\mathrm{e}^{\mathrm{i}\Phi_p}. \tag{2.16}$$

Here, the summation encompasses all those periodic points p of period $n_p = n$ of the classical map inside the cylinder $0 \le \theta < 2\pi$, $|l - l_c| \le \hat{L}$. R_p is the 'residue' [6] of the periodic orbit to which p belongs, and it is related to the stability matrix through

$$R_p = \frac{1}{4}\left|\det\left(I - \frac{\partial(\theta_n, l_n)}{\partial(\theta_0, l_0)}\right)\right| = \frac{1}{4}\left|2 - \mathrm{tr}\left(\frac{\partial(\theta_n, l_n)}{\partial(\theta_0, l_0)}\right)\right|. \tag{2.17}$$

Φ_p is the action along the periodic orbit, including the Maslov index. We get

$$b^{(L)}(n) = \frac{1}{4L}\left\langle 2n\sum_{\substack{p\\ n_p=n}}\frac{1}{R_p} + \sum_{\substack{p,p'\\ n_p=n_{p'}=n}} \frac{1}{\sqrt{R_pR_{p'}}}\mathrm{e}^{\mathrm{i}(\Phi_p-\Phi_{p'})}\right\rangle - 1 \tag{2.18}$$

where the factor $2n$ is due to constructive interference between contributions from pairs of points belonging to a conjugate pair of periodic orbits (cf (2.12) in I).

The relative importance of the two sums above depends on the time n. For not too large n the first term in the sum (the diagonal term) dominates. This is so since typical action differences are still relatively large, and the contribution of the double sum will vanish upon averaging over l_c. The diagonal term is the classical term. It gives the mean probability to stay at the initial state after n periods of the driving force.

For large n, the periodic orbits proliferate exponentially and action differences become very small. Therefore, the non-diagonal term in (2.18) cannot be neglected. Rather, the interfering contributions from all the trajectories in (2.18) must conspire to give the result (2.14), which was derived from more general considerations.

So far there is no proper theory which describes in detail the transition between the two time domains: the 'classical' and the 'quantal'. In the regime $\gamma < 1$ where the dynamics for short times does not feel the truncation of the phase space, we can assume that the interference effects become dominant in the same way as in the treatment of the local spectrum. We have shown in I that this happens at $n = n_{\mathrm{qm}} \approx \xi$. We shall assume that this same parameter marks the transition in the present context as well. For $\gamma \gg 1$ the classical dynamics is not diffusive even for the shortest times. The periodic orbits cover the bounded phase space in a uniform way, and one can apply the arguments put forward in [12] and [13] to show that in this domain $n_{\mathrm{qm}} \approx L$.

The further evaluation of (2.18) depends very much on whether $\gamma \sim \xi/L$ is smaller or larger than a critical value γ_c to be determined below. We shall therefore discuss the two cases separately.

(i) $\gamma < \gamma_c$.

This is the domain where the truncated classical dynamics is diffusive for some time, and the Floquet eigenstates are known to localize (for $\tau/2\pi$ sufficiently irrational). As was mentioned at the end of section 1, the classical map on the infinite cylinder is periodic in angular momentum with period $L_{\mathrm{u}} = 2\pi/\tau$, i.e. it is invariant under translations $l \to l + j2\pi/\tau$. Hence, for any periodic point in the unit cell $0 \le \theta < 2\pi$, $0 \le l < 2\pi/\tau$, there exist an infinite number of replicas generated by translating that point throughout the entire length of the classical phase space. It can readily be shown that the determinant R_p does not depend on j. On the other hand, we obtain for the action

$$\Phi_p(j) = \Phi_p(0) + \frac{(2\pi)^2}{\tau} jM_p + \frac{2\pi^2}{\tau} nj^2 \tag{2.19}$$

where

$$M_p = \frac{\tau}{2\pi} \sum_{i=1}^{n} l_i = (\theta_n - \theta_0)/2\pi \tag{2.20}$$

denotes the winding number of the periodic orbit to which the point belongs. Returning to the truncated phase space, we can use this symmetry to our advantage, as long as we limit the period of interest by requiring $n < n_L$.

We can now reorder the sum appearing in the trace formula (2.16) by collecting the contributions from the replicas of periodic points in the unit cell

$$\frac{1}{\sqrt{L}} \operatorname{tr}\left((U^{(L,l_c)})^n\right) = \frac{1}{2} \sum_{\substack{p \in u \\ n_p = n}} \left(\frac{1}{\sqrt{R_p}} e^{i\Phi_p(0)} \sqrt{\frac{\tau}{2\pi J}} \sum_{j=j_c-\hat{J}}^{j_c+\hat{J}} \exp\left(i\frac{(2\pi)^2}{\tau} (jM_p + nj^2/2) \right) \right) \tag{2.21}$$

where $\hat{J} = \hat{L}/L_{\mathrm{u}}$, $j_c = l_c/L_{\mathrm{u}}$, $J = 2\hat{J} + 1$, and the outer sum is limited exclusively to periodic points in the unit cell.

The summation in (2.21) now also includes trajectories which are reflected from the boundaries $|l - l_c| = \hat{L}$ and therefore do not belong to the set of trajectories with the translational symmetry mentioned above. Their contribution is relatively small for the following reason. The extension in the l direction of periodic orbits of period n is of the order of $\sqrt{nD}$ in the mean. Hence, the fraction of trajectories which scatter from the boundaries is of the order $\sqrt{nD}/L$. Since we are interested in length scales which are larger than ξ (which is of the same order of magnitude as D), we find that the above fraction will be small for times $n < \xi < n_L$. We have already shown above that this is exactly the time range where the semiclassical theory is applicable. The effect of the reflected periodic points will diminish as L increases.

The inner sum in (2.21) can be rewritten, using the Poisson sum formula and the saddle-point approximation as in BG, so as to simplify (2.21) to the final form

$$\frac{1}{\sqrt{L}}\mathrm{tr}\left((U^{(L)})^n\right) = \left(\frac{1}{2}\sum_{\substack{p\in u\\ n_p=n}} \frac{1}{\sqrt{R_p}}\mathrm{e}^{\mathrm{i}\Phi_p(0)}\right)\left(\sqrt{\frac{\tau}{2\pi n L}}\sum_{l=-n\hat{L}}^{n\hat{L}} \exp\left(-i\pi\frac{\tau}{2\pi n}l^2\right)\right). \tag{2.22}$$

Thus, we again achieved the factorization of $b^{(L)}(n)$ which we already encountered in the particular case $n = 1$: the first factor is independent of L and determined only by the periodic points of period n in the unit cell, through the actions and stability determinants associated with them. The second factor is entirely due to the translational symmetry of the phase space of the classical standard map. It can readily be checked that (2.22) indeed reduces to (2.12) for $n = 1$, if the Bessel functions $J_0(k)$ are replaced by their semiclassical counterparts $\sqrt{2/\pi k}\,\cos(k - \pi/4)$ [14].

For typical irrational values of $\tau/2\pi$ we finally get

$$b^{(L)}(n) = \frac{\tau}{4\pi}\left|\sum_{\substack{p\in u\\ n_p=n}} \frac{1}{\sqrt{R_p}}\mathrm{e}^{\mathrm{i}\Phi_p(0)}\right|^2 - 1. \tag{2.23}$$

As long as n is not too large one can neglect the interference between contributions from different periodic orbits. For $n \lesssim n_{\mathrm{qm}}$ (2.23) reduces to

$$b^{(L)}(n) = 2n\frac{\tau}{4\pi}\sum_{\substack{p\in u\\ n_p=n}} \frac{1}{R_p} - 1. \tag{2.24}$$

The origin of the factor $2n$ was discussed above (cf (2.18)).

In the appendix we extend the uniformity principle proposed by Hannay and Ozorio de Almeida [15] (henceforth referred to as HO) to the case of an unbounded classical phase space. We show that

$$\sum_{\substack{p\in u\\ n_p=n}} \frac{1}{R_p} = \sqrt{\frac{8\pi}{n\tau^2 D}} \qquad n_{\mathrm{d}} < n < n_L. \tag{2.25}$$

n_{d} and n_L were defined in section 1.

One of the observations made in I (and in a related context by Berry [12]) is that sums over periodic orbits of the type (2.23) display a characteristic dependence on the value of n (the period). As long as $n < n_{\mathrm{qm}}$ the semiclassical approximation (2.25)

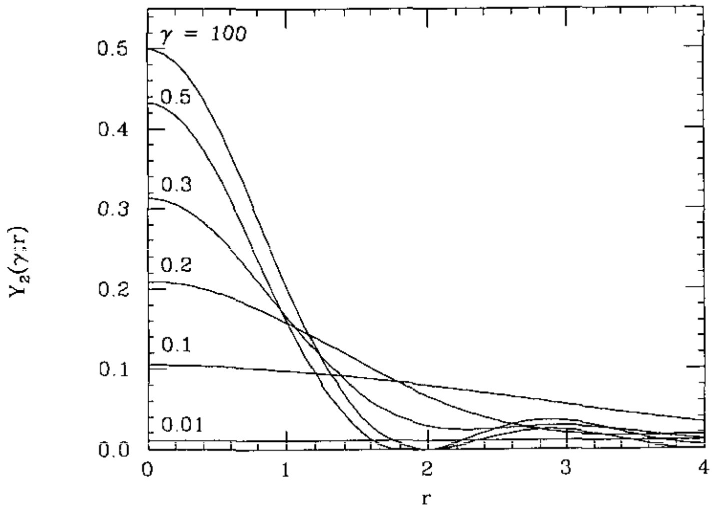

Figure 1. Two-level cluster function $Y_2(\gamma;r)$ for the quantum kicked rotor on a truncated Hilbert space ((2.28) and (2.31)), for various values of the effective localization length $\gamma = D/L$.

holds. A relatively sudden transition occurs beyond n_{qm} where the function attains its asymptotic value, which in the present case is null, as (2.14) demonstrates. We can summarize by writing

$$b^{(L)}(n) \approx \begin{cases} \sqrt{2n/\pi D} - 1 & n_{\mathrm{d}} < n < n_{\mathrm{qm}} \\ 0 & n > n_{\mathrm{qm}}. \end{cases} \tag{2.26}$$

In stating (2.26) we skipped the first few time steps $1 \leq n \leq n_{\mathrm{d}}$ where the dynamics is not yet generic. Lacking an expression which interpolates $b^{(L)}(n)$ between the two asymptotes in (2.26), we instead extrapolate both asymptotes to their intersection point $n = \pi D/2$. This amounts to a renormalization of n_{qm}, consistent with the required proportionality [1, 16] between the diffusion constant and the participation ratio ξ. By substituting (2.26) in the expression (1.12) for the two-point cluster function, we obtain

$$Y_2^{(L)}(r) = \frac{1}{L}\left(1 - 2\sum_{n=n_{\mathrm{d}}+1}^{n_{\mathrm{qm}}}\left(\sqrt{\frac{2n}{\pi D}} - 1\right)\cos\left(n\frac{2\pi}{L}r\right)\right). \tag{2.27}$$

If we go to the continuum limit $L, \xi \to \infty$, keeping their ratio γ constant and using it as a parameter instead of L, and if we set, within the same approximation, the lower limit in the sum in (2.27) to zero, we can rewrite this expression as

$$Y_2(\gamma; r) \approx \frac{1}{\pi}\int_0^{\pi^2\gamma} \mathrm{d}v \left(1 - \frac{1}{\pi}\sqrt{v/\gamma}\right)\cos(vr). \tag{2.28}$$

The integral in (2.28) can be evaluated numerically. The function $Y_2(\gamma; r)$ is shown in figure 1 for various values of γ. The fact that it is positive for small r and decays to zero for large r is an expression of the level repulsion which is consistent with the numerical results reported in [3]. $Y_2(\gamma; r)$ attains its maximum value $\gamma\pi/3$ at $r = 0$. As γ decreases, this maximum flattens out, which proves that the spectral correlations become Poissonian in the limit $\gamma \to 0$.

So far we assumed $\tau/2\pi$ to be a typical irrational number. For rational values, the spectrum is known to be continuous in the limit $L \to \infty$. Here, the expression for $Y_2(r)$ diverges in a similar way as for the limit $k \to 0$ discussed above.

$\gamma > \gamma_c$.

At the end of section 1 we showed that as γ increases beyond γ_c, the finite size of the accessible phase space limits the applicability of the free diffusion description to shorter and shorter times. At $\gamma \approx \xi^{1/2}$, classical diffusion becomes completely irrelevant. We shall now show how the change in the duration of the initial, diffusive regime of the dynamics is reflected in the spectral correlations.

We first consider the range $\xi^{1/2} > \gamma > 1$. This amounts to

$$\xi > L > n_L > 1 \qquad L > L_u. \tag{2.29}$$

In other words, there exists a range of time $n_L > n > n_d$ for which the classical dynamics can still be described in terms of the diffusion approximation. Hence, the semiclassical theory which yielded (2.26) also holds for the reduced range $n_L > n > n_d$.

In the range $n > n_L$ the discussion which led to (2.26) is no longer relevant and we should start again from (2.18). Now most of the classical periodic orbits with period $p > n_L$ are affected by the scattering from the boundaries of the truncated cylinder. Hence, we may apply the original version of the HO sum rule to estimate the diagonal contribution to (2.18), and neglect the contribution from the non-diagonal term for $n < n_{qm}$. Since in the present case for long times, the classical dynamics is that of a bounded system, we adopt Berry's result and set for the present case $n_{qm} \approx L$. Thus we get

$$b^{(L)}(n) \approx \begin{cases} \sqrt{2n/\pi D} - 1 & n_d < n < n_L \\ 2n/L - 1 & n_L < n < L \\ 0 & n > L \end{cases} \tag{2.30}$$

where again we skipped the non-generic regime $1 \leq n < n_d$ of the dynamics.

For $\gamma > \xi^{1/2}$, $n_L < 1$ and hence classical diffusion is suppressed from the very beginning. For this range the expression for $b^{(L)}(n)$ is obtained from (2.30) by skipping the first time domain. The resulting function is analogous to Berry's result, which in turn approximates rather well the asymptotic behaviour of the corresponding result obtained by Dyson [7] for the COE.

The generic part of the two-point cluster function for the present case can be derived easily. We obtain

$$Y_2(\gamma;r) \approx \frac{1}{\pi}\int_0^{\gamma^{-1}} \mathrm{d}v \left(1 - \frac{1}{\pi}\sqrt{v/\gamma}\right)\cos(vr) + \frac{1}{\pi}\int_{\gamma^{-1}}^{\pi} \mathrm{d}v \left(1 - \frac{1}{\pi}v\right)\cos(vr). \tag{2.31}$$

For $\gamma = 1/\pi$ the second integral in (2.31) vanishes and the first term coincides with the expression (2.28). Therefore, a consistent choice of the critical value γ_c is

$$\gamma_c = \frac{1}{\pi}. \tag{2.32}$$

For $\gamma \to \infty$ the first integral vanishes. The remaining second term gives

$$Y_2(\infty; r) = \frac{1}{(\pi r)^2}(1 - \cos \pi r). \tag{2.33}$$

We cannot expect to reproduce Dyson's result rigorously since we lack a semiclassical theory which interpolates between the three time domains in (2.30). Nevertheless, except for oscillations superposed on $Y_2(\infty; r)$ due to the singularity in the form factor $b^{(L)}(n)$ at $n = L$, there is rough quantitative agreement between (2.33) and the exact COE result [14]. Figure 1 shows the function $Y_2(\gamma; r)$ also for some values of $\gamma > \gamma_c$.

It is well known that the two-point cluster function is the input for the calculation of other spectral diagnostics such as the commonly used variance Σ_2 or the Δ_3 statistics. The former is related to $Y_2(r)$ by [17]

$$\Sigma_2(r) = r - 2\int_0^r (r - \rho) Y_2(\rho)\, \mathrm{d}\rho \tag{2.34}$$

and a further integration gives $\Delta_3(r)$. These functions will display the transition between the Poisson and COE behaviour as γ is varied, as does the two-point cluster function itself. The agreement between the approximation (2.33) for the COE limit and the exact result is much better in terms of the number variance, because the additional integration involved by (2.34) tends to damp out the oscillations in the cluster function. Figure 2 shows the function $\Sigma_2(\gamma; r)$ for various values of the parameter γ.

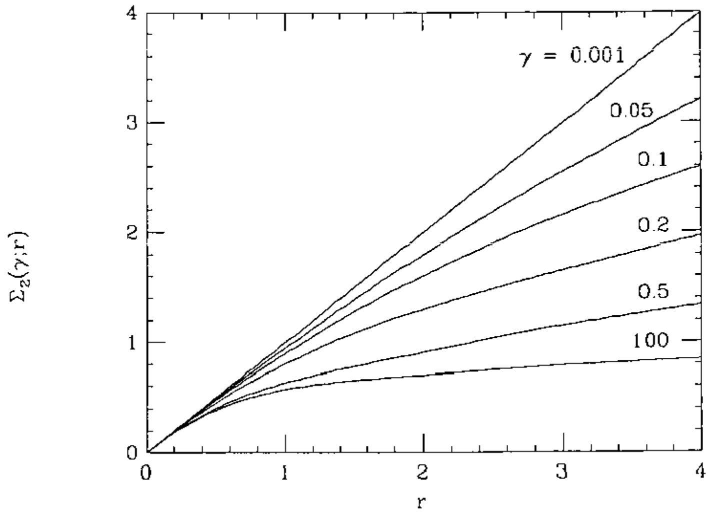

Figure 2. Number variance $\Sigma_2(\gamma; r)$ for the quantum kicked rotor on a truncated Hilbert space, for various values of the effective localization length $\gamma = D/L$.

The present theory cannot make any prediction on the statistics of nearest-neighbour spacings. This requires a more accurate and detailed information on the two- (and more) point correlation functions than the present crude semiclassical theory can provide. This is rather unfortunate, since the dependence of the nearest-neighbour spacing distributions as a function of γ were recently investigated numerically by

Izrailev [18]. Similar to our results, he finds a transition between the Poisson and COE limits which is controlled by the *single* parameter γ. His observations are based on detailed numerical evidence. Our semiclassical results substantiate these findings.

Before going to the summary, it remains to clarify two issues of slightly technical nature. The first one concerns our definition of the spectral density in terms of smoothed delta functions (see (1.4) and (1.5)). The semiclassical theory used n as the period of the periodic orbits in the trace formula (2.16) and its derivates. We have seen that for $n > n_{\mathrm{qm}}$ as defined above, there occurs a transition where the quantum interferences conspire to make $b^{(L)}(n)$ vanish. This is certainly the case for $n \gg L$. The surprising (and not yet proved) observation that the Fourier coefficients $b^{(L)}(n)$ vanish even earlier may be used to justify the truncation of the series (1.5) even at shorter times.

The second issue is the l_{c} averaging which we were so careful to introduce. It is necessary both in numerical investigations and to put our statements on firmer ground. In particular, the l_{c} averaging is responsible for the non-diagonal contributions in (2.13) and (2.18) vanishing for $n < n_{\mathrm{qm}}$.

3. Summary

In the present two papers we have investigated the quasi-energy spectrum of the kicked rotor in order to identify the fingerprints of dynamical localization in the spectral statistics. We took two different approaches to incorporate the specific spatial structure represented by localized Floquet eigenstates in the definitions of spectral correlation functions. In the case of the *local spectrum*, the contribution of each QE is endowed with an individual weight, equal to the overlap of the corresponding eigenstate with a reference state in the unperturbed basis. The *unbiased spectrum*, on the other hand, is defined by restricting the Hilbert space to a finite subset of the unperturbed basis, in analogy to a finite sample of a one-dimensional solid. Both approaches reveal features not encountered in any one of the canonical QE ensembles provided by random matrix theory.

The local spectrum of the QKR was discussed in I. Semiclassical arguments substantiated by numerical tests allowed us to investigate the QE spectrum in great detail and to show that dynamical localization implies spectral correlations which are very similar to those obtained for Anderson localization in disordered solids. Our semiclassical theory is essentially based on the important link between classical diffusion and quantum localization. This link is the basis of our scaling theory for the staying probability, and hence it substantiates the general validity of the functional form we found for the local two-point cluster function.

For the unbiased spectrum, we find that the spectral correlations depend on the parameter γ, the ratio of the localization length to the dimension of the truncated Hilbert space. The spectral fluctuations show a smooth transition between COE behaviour for $\gamma \to \infty$ and Poissonian statistics for $\gamma \to 0$, the limit leading back to the unmodified QKR. This transition is described analytically. Resembling our results for the local spectrum, the decisive dynamical element which determines the character of the spectral correlations turns out to be the initial diffusive regime of the dynamics and the way it leads to the coverage of the accessible phase space. It should be emphasized that, for any finite value of γ, the cluster function deviates in an essential way from the predictions of any of the standard random matrix ensembles.

The expressions for the generic part of the two-point cluster function in the two regimes approach the forms corresponding to COE and Poissonian statistics in the limits $\gamma \to \infty$ and $\gamma \to 0$, respectively. The transitions to these two asymptotes are of a quite different nature: the limit $\gamma \to 0$ is characterized by the gradually increasing independence of distant localization neighbourhoods in the sample. In the domain $\gamma \gg \gamma_c$, where the critical value γ_c is of order unity, we see the last remaining traces of the influence of the initial diffusion fade as the system becomes smaller, till the point where diffusion no longer plays any role. This transition is reminiscent of the metal–insulator transition in disordered systems [19]: the range $\gamma < \gamma_c$ corresponds to the insulating regime, where the localization length is much smaller than the sample size, whereas $\gamma > \gamma_c$ corresponds to the metallic domain. In the present context, the transition occurs at $\gamma \approx \gamma_c \approx 1$.

An interesting observation emerges from our study of correlations in the unbiased spectrum: We have shown that the generic part of the Fourier coefficients of the two-point cluster function is characterized by two asymptotes. The long-time asymptote is of quantum origin. It sets a natural cut-off to the time correlations and hence to the QE correlations. The short-time asymptote is determined entirely by the classical dynamics. The only information required (through the Hannay and Ozorio de Almeida sum rule [15] and its extension) concerns the way phase space is filled by classical trajectories. As long as the accessible phase space is finite and covered uniformly, one obtains a behaviour which is similar to the one predicted by the canonical random matrix ensembles. This is intuitively clear: the main assumption underlying Dyson's random matrix theory is that there are no spatial preferences in the ensemble, and the only relevant parameter is the mean level spacing which depends only on the phase-space volume. The classical analogues of these ensembles are characterized by a finite phase space covered ergodically by classical orbits. The analogy fails if phase space is infinite and covered diffusively. This type of classical dynamics has no counterpart in the systems described by the standard random matrix ensembles—hence an agreement in the statistics of the spectral fluctuations cannot be expected, either.

The above considerations find some support in recent results of Bohigas *et al* [20], who show that a phase-space barrier which slows down the classical coverage of phase space induces deviations from the COE prediction in the spectral statistics of the corresponding quantum system. This result, together with our observations, might necessitate a revision of our thinking about the connection between classical chaos and spectral statistics.

Insofar as the kicked rotor is a paradigm for periodically perturbed systems with chaotic classical dynamics, the results achieved by means of the two approaches studied here should apply to systems with dynamical localization in general. Thus, it can be conjectured that classically chaotic systems which display deterministic diffusion, and whose quantum dynamics is bounded only effectively by quantum localization but not by boundary conditions, form a spectral universality class of their own.

Appendix

Hannay and Ozorio de Almeida [15] (HO) proposed a very powerful sum rule which is the cornerstone for Berry's proof [12] of the universal behaviour of the spectral two-point correlation function for Hamiltonian quantum systems chaotic in their classical limit. HO expression is valid only for systems defined on a *bounded* phase space. In the present appendix we shall generalize this sum rule to the case of an *unbounded, cylindrical* phase space, as it is used to describe the dynamics of the kicked rotor.

Consider a phase-space point $\boldsymbol{r}_0 = (l_0, \theta_0)$ in the unit cell, i.e. $-\pi/\tau \le l_0 < \pi/\tau$ and $0 \le \theta_0 < 2\pi$, and its iterates $\boldsymbol{r}_i(\boldsymbol{r}_0)$, $i = 1, \ldots, n-1$. Due to the mixing property of the dynamics, the evolution after a characteristic time n_{d} can be described, on average, by a distribution homogeneous in θ and diffusive in l. Thus the iterates of $\boldsymbol{r}_0$ are distributed around their initial point according to

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta(\boldsymbol{r}-\boldsymbol{r}_i(\boldsymbol{r}_0)) = \frac{1}{2\pi}\frac{\exp[-(l-l_0)^2/2nD]}{\sqrt{2\pi nD}} \tag{A.1}$$

if we allow for a small but finite width of the delta functions.

Since (A.1) is valid for arbitrary $\boldsymbol{r}$ and any typical choice of $\boldsymbol{r}_0$, take in particular $\boldsymbol{r} = \boldsymbol{r}_0$ and integrate over the unit cell: in this way, one obtains

$$\frac{1}{n}\sum_{\substack{p\in u\\ n_p\le n}}\frac{1}{R_p} = 4\int_{-\pi/\tau}^{\pi/\tau}\mathrm{d}l\int_0^{2\pi}\mathrm{d}\theta\,\frac{1}{2\pi}\frac{1}{\sqrt{2\pi nD}} = \sqrt{\frac{32\pi}{n\tau^2 D}}. \tag{A.2}$$

where the summation goes over all the periodic points of period $n_p \le n$ in u, and

$$R_p = \frac{1}{4}\det\left(I - \frac{\partial(l_{n_p}, \theta_{n_p})}{\partial(l_1, \theta_1)}\right). \tag{A.3}$$

In the present context, a more useful expression is

$$\sum_{\substack{p\in u\\ n_p=n}}\frac{1}{R_p} = \sqrt{\frac{8\pi}{n\tau^2 D}}. \tag{A.4}$$

An intermediate situation between the regime of validity of the original HO sum rule and the present result prevails in systems which are discussed in section 2 of this paper, where we consider a *truncated* cylindrical phase space of length L. For $n \lesssim n_L$ the result (A.4) remains valid.

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