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The domino billiard*

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We study one-dimensional chains of billiards, where it is possible to switch between order and disorder in two respects: regular versus chaotic scattering in the billiards, and periodic versus random length sequence of the connecting pipes. Classically, the dynamics is largely dominated by the type of the scatterers. Quantum mechanically, the connector length sequence is also decisive: there are extended states and a band spectrum if it is periodic, whereas if it is random, states are localized and the spectrum is discrete. In this case, the spectral statistics forms a transition between GOE (for short chains, compared to the localization length) and Poissonian (for very long chains) and can be related to the way a corresponding classical ensemble explores phase space.

1. The domino billiard

The search for the traces of classical chaos in quantum dynamics, in the last two decades, has shown that there exists a wealth of fascinating coherence phenomena along the way down from smooth and simple wave structures in the deep quantum regime, to the fractal roughness of chaotic dynamics in the classical limit. In this search, an important role has been played by prototype models which were not primarily designed as caricatures of some real microscopic system, but rather as free inventions, solely intended to form landmarks in the terra incognita between regular and chaotic, classical and quantal. Prominent among these, in turn, are so-called billiards: potential landscapes where the potential can only take on the values 0 or ∞, so that all the physics is implied by the shape of the boundary between the accessible and the inaccessible parts of space. For example, the "Bunimovich stadium" has not only, among similar billiards, helped to identify the fingerprints of chaos in quantal spectra [1–3], but has also served to demonstrate the traces of unstable periodic orbits in quantal eigenfunctions [4]. The three-hard-disc system [5, 6] is one of the most intensely studied models of chaotic scattering, both classical and quantal.

With the present work, we intend to study a type of billiard half-way between bounded and scattering systems: shaped after the model of one- (or more-) dimensional solids. The domino billiard consists of a chain of stadia, connected by pipes: Extended to two dimensions, it would be closely related to the Sinai billiard with cyclic boundary conditions, as well as to the Galton board. To be more specific, we use squares, with two adjacent sides open, as regular stadia (fig. 1(a)), and render them irregular scatterers by replacing the corner between the two open sides by a quarter circle of radius r, belly into the stadium (fig. 1(b)) [7]. The connectors are formed by waveguides, of the width w of the open sides.

In this class of systems, a transition from order

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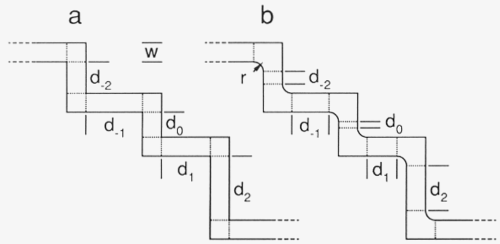

Fig. 1. Scheme of the domino billiard. Version (a), with a rectangular geometry, supports a pseudointegrable classical dynamics, version (b), with rounded corners, induces irregular scattering.

to disorder is possible in two respects: By increasing the ratio r/w from zero onwards, pseudointegrable dynamics is replaced by irregular dynamics. By choosing the sequence d_l, $l=0, \pm 1, \pm 2, \ldots$ of connector lengths from the various levels of numerical disorder ranging from periodic through quasiperiodic and pseudorandom to random, crystal- or amorphous-solid-like structures are formed. The simultaneous presence of these two order–disorder transitions offers a variety of features to be studied, both on the classical and the quantal level. The present contribution represents a report on work in progress, so we shall not provide conclusive answers but discuss preliminary results together with as yet unanswered questions.

2. The classical case

Classical trajectories in a billiard without magnetic field form straight lines in free space and specular reflections where they hit the boundary. Accordingly, the absolute value of the momentum, $|p|$, is a conserved quantity. For finite values of r/w, the defocusing effect of the curved part of the boundary leads to a fast mixing with respect to the angle of the trajectory. For $r/w = 0$, $|p_x|$ and $|p_y|$ are conserved separately (the axes are assumed parallel to the billiard edges),

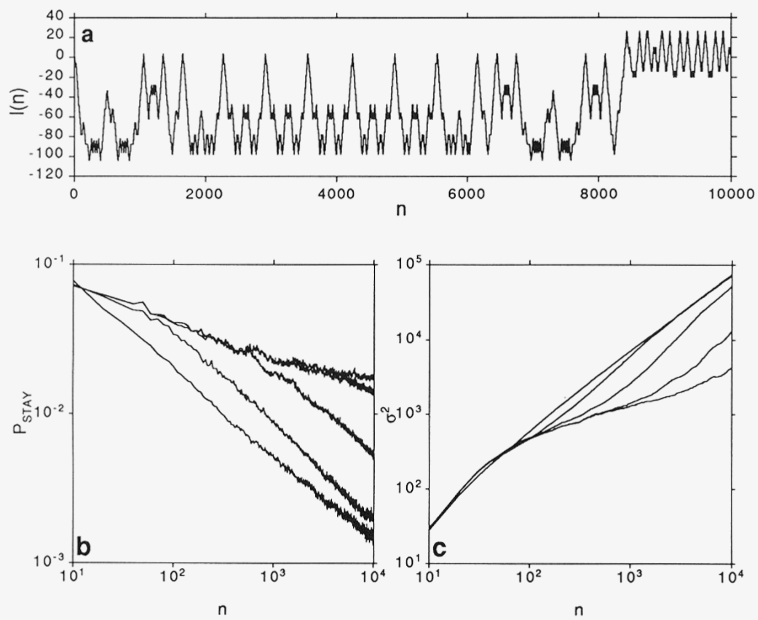

Fig. 2. Classical case. Coarse-grained typical trajectory (a), in terms of number l of the element of sojourn versus number n of passages through connectors, for the pseudointegrable case; time evolution of (b) probability to stay and of (c) variance of the distribution along the billiard, for various values of the relative radius of curvature r/w: 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1} (top to bottom in (b), bottom to top in (c)).

so that trajectories switch between four "sheets" of phase space, $(p_x, p_y) = (\pm|p_x|, \pm|p_y|)$. Therefore, this limiting case should be called pseudointegrable [8, 9].

The way a classical ensemble spreads along the domino billiard is largely determined by the nature of the scattering in the stadia. In the limit of large r/w, a trajectory can be described as a Markovian sequence of scattering events at the corners, and spreading is diffusive. In the pseudointegrable case, strong correlations persist over tens of thousands of successive scatterings (fig. 2(a)). The waiting time distribution in a single element of the chain then decays algebraically [10], and the spreading is either subdiffusive or superdiffusive (ballistic), according to the length sequence being random or periodic, respectively. For a finite value $r/w \lesssim 1$, there is a cross-over from sub- (or super-) diffusive behavior to ordinary diffusion, at a time $n_c \approx (r/w)^{-1}$ (figs. 2(b, c)).

3. The quantal case

Since, for a billiard in two dimensions, both Schrödinger's and Maxwell's equations reduce to the Helmholtz equation, the following results can be interpreted equally well in terms of, e.g., a quantum particle in a mesoscopic structure or in terms of microwaves in a cavity [7].

To solve the Helmholtz equation for an entire domino billiard, we adopt a three-step procedure: The solution for a single element, stadium

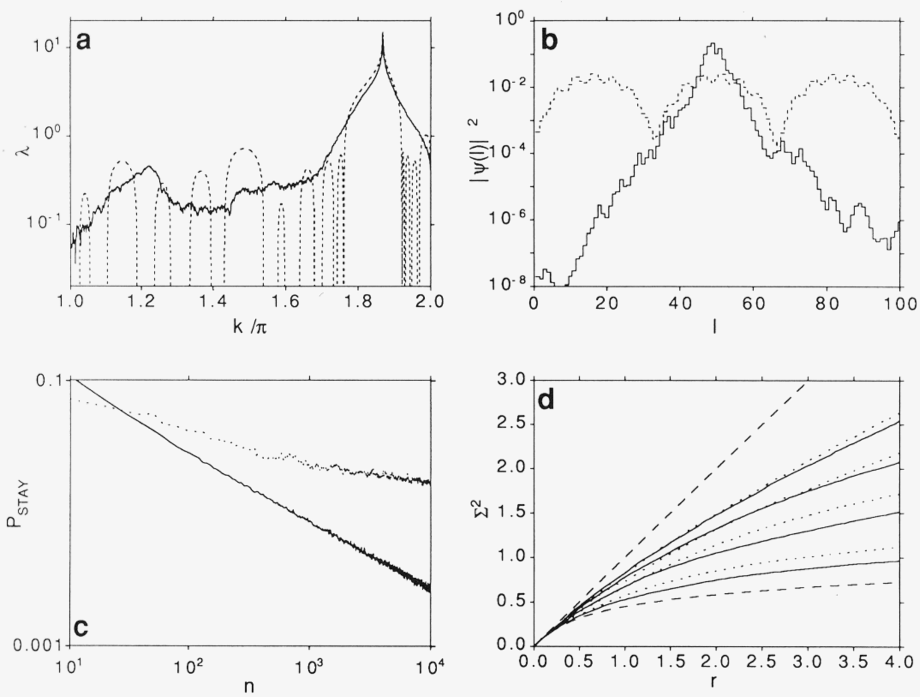

Fig. 3. Quantal case. (a) Exponent characterizing the growth of the trace of the successive product of transfer matrices along the chain vs. wave number; (b) typical eigenstates in a chaotic domino billiard ($r/w = 1$) consisting of 100 elements, both for a pseudorandom chain (full line) and a periodic chain (dashed line); (c) decay of the classical probability to stay for a chaotic billiard ($r/w = 1$, full line) and a pseudointegrable one ($r/w = 0$, dotted line); (d) variance of level separation distribution for various values of the chain length: 200, 100, 50, 20 (top to bottom) for the same cases as in (c), compared with Poissonian and GOE spectral statistics, respectively (dashed lines, top to bottom). All results are for a wave number range where only a single channel is open.

plus connector, can be achieved by a variety of numerical techniques [11–13]. It results in the transfer matrix for this element. The transfer matrix for the whole chain is given by the successive product of the single-element transfer matrices, and imposing reflecting boundary conditions for both ends selects eigenstates and eigenvalues.

In contrast to the classical dynamics, the nature of spectrum and eigenstates is determined, apart from finer details, by the type of the sequence d_l: If it is periodic, the billiard resembles a crystal, the eigenstates are extended (Bloch) states (figs. 3(a, b), dashed lines), and the spectrum consists of continuous bands. If it is (pseudo)random, the billiard corresponds to an amorphous solid, the eigenstates are Anderson localized (figs. 3(a, b), full lines), and the spectrum is discrete [14]. In the latter case, the tools of spectral statistics and random matrix theory can be applied. The level statistics depends on the ratio γ of the localization length to the length of the chain: For large γ, the pseudointegrable or chaotic character of the classical dynamics implies the statistics of the Gaussian orthogonal ensemble (GOE) [8, 15, 16], just as for a single chaotic stadium. In the limit $\gamma \to 0$, the billiard houses a large number of localization neighborhoods, and the superposition of their statistically independent spectra renders the level distribution Poissonian. Between these extremes, the level statistics forms intermediates, not described by any one of the canonical ensembles, as shown in fig. 3(d) for the spectral variance [17, 18]. Instead, the spectral statistics can be derived from semiclassical arguments based on overall features of the classical spreading along the billiard, described, specifically, by the probability to stay (fig. 3(c)) [19–22].

4. Conclusion

There are four principal limiting cases of the domino billiard, given by the two transitions, pseudointegrable to chaotic scatterers and periodic to random connector length sequences. Among them, two are of particular interest: In the case of right-angle corners and aperiodic d_l, localization of the quantal eigenstates comes along with pseudointegrable classical dynamics. Thus, a subtle irregularity on larger scales in a geometry which is simple and ordered on small scales, proves sufficient to induce the randomization of phases necessary for Anderson localization to arise. The spectral statistics can be understood from the global way the corresponding classical ensemble explores phase space.

Conversely, in the case of curved corners and a periodic chain, classical deterministic diffusion corresponds to extended states and a band spectrum. Questions yet to be answered here concern, e.g., the fingerprints of classical chaos in a band structure, and the way Bloch states are able to emulate a diffusive dynamics in the semiclassical limit.

The versatility of the domino billiard and its surprisingly close resemblance to solids suggests a host of specific models yet to be studied, among them chains composed of sections of qualitatively different structure, and extensions to two dimensions.

References

[1] S.W. McDonald and A.N. Kaufman, Phys. Rev. Lett. 42 (1979) 1189.
[2] G. Casati, F. Valz-Gris and I. Guarneri, Lett. Nuovo Cimento 28 (1980) 279.
[3] O. Bohigas, M.J. Giannoni and C. Schmit, Phys. Rev. Lett. 52 (1984) 1.
[4] E.J. Heller, Phys. Rev. Lett. 53 (1984) 1515.
[5] B. Eckhardt, Europhys. Lett. 5 (1988) 107.
[6] P. Gaspard and S.A. Rice, J. Chem. Phys. 90 (1989) 2225.
[7] E. Doron, U. Smilansky and A. Frenkel, Phys. Rev. Lett. 65 (1990) 3072; Physica D 50 (1991) 367.
[8] P.J. Richens and M.V. Berry, Physica D 2 (1981) 495.
[9] B. Eckhardt, J. Ford and F. Vivaldi, Physica D 13 (1984) 339.
[10] W. Bauer and G.J. Bertsch, Phys. Rev. Lett. 65 (1990) 2213.
[11] R.L. Schult, D.G. Ravenhall and H.W. Wyld, Phys. Rev. B 39 (1989) 5476.
[12] Y. Avishai and Y.B. Band, Phys. Rev. Lett. 62 (1989) 2527.

[13] J. Martorell, S. Klarsfeld, D.W.L. Sprung and H. Wu, Solid State Commun. 78 (1991) 13.
[14] S. Fishman, D.R. Grempel and R.E. Prange, Phys. Rev. Lett. 49 (1982) 509; Phys. Rev. A 29 (1984) 1639.
[15] T. Cheon and T.D. Cohen, Phys. Rev. Lett. 62 (1989) 2769.
[16] D. Biswas and S.R. Jain, Phys. Rev. A 42 (1990) 3170.
[17] F.J. Dyson, J. Math. Phys. 3 (1962) 140, 1192.
[18] T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey and S.S.M. Wong, Rev. Mod. Phys. 53 (1981) 385.
[19] M.V. Berry, Proc. R. Soc. A 400 (1985) 229.
[20] T. Dittrich and U. Smilansky, Nonlinearity 4 (1991) 85.
[21] U. Smilansky, S. Tomsovic and O. Bohigas, preprint (1991).
[22] E. Doron, T. Dittrich and U. Smilansky, to be published.