Correlations in the Actions of Periodic Orbits Derived from Quantum Chaos

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We discuss two-point correlations of the actions of classical periodic orbits in chaotic systems. For systems where the semiclassical trace formula is exact and the spectral statistics follow random matrix theory, there exist nontrivial correlations between actions, which we express in a universal form. We illustrate this result with the analogous problem of the pair correlations between prime numbers. We also report on numerical studies of three chaotic systems where the semiclassical trace formula is only approximate, but nevertheless these unexpected action correlations are observed.

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Semiclassical trace formulas relate the quantum spectral density to classical periodic orbits [1]. In this Letter we use this link to express a *classical* two-point correlation function, involving the actions and stabilities of the periodic orbits of chaotic systems, in terms of a two-point statistic of the *quantum* energy spectrum. We first consider systems for which the trace formulas are exact. Assuming that the spectral fluctuations follow the predictions of random matrix theory (RMT) [2,3], we derive a universal expression for the classical correlation function. This expression represents in some cases a tendency towards action repulsion. Studying in exactly the same way the correlations between pairs of prime numbers, i.e., assuming that the zeros of the Riemann zeta function follow RMT, we get a correlation which is consistent with the Hardy-Littlewood conjecture of number theory [4,5]. Surprisingly, we find that such correlations also occur in other chaotic systems which we studied numerically, and for which the semiclassical approximation (SCA) is not expected to be valid in the limit of long times. For definiteness we consider Hamiltonian flows with 2 freedoms and maps with 1 degree of freedom.

Trace formulas express the spectral density by the Selberg-Gutzwiller sum [1],

$$d(E)=\bar{d}(E)+d_{\rm osc}(E)=\bar{d}(E)+\frac{1}{\hbar}\sum_{p,r}\frac{g_pT_p}{|\det(M_p^r-I)|^{1/2}}\exp\left[\frac{i}{\hbar}rS_p-\frac{i\pi}{2}r\nu_p\right], \tag{1}$$

where $\bar{d}(E)$ is the mean level density, and the summation is over primitive periodic orbits p and repetitions r (both positive and negative). S_p, T_p, g_p, M_p, and ν_p are the action, period, multiplicity, monodromy matrix, and Maslov index of the pth orbit, respectively. We label the preexponential factor in (1) by A_j, where j signifies the pair (p,r), and define $S_j=rS_p$, $\nu_j=r\nu_p$, and $T_j=rT_p$. Equation (1) is strictly valid only for a few systems, e.g., some billiards on Riemann surfaces of negative curvature [6], but it comprises the leading term in $\hbar$ for generic systems. We concentrate first on cases where (1) is exact.

The spectra form factor $K(\tau)$ is a useful statistic [3], which is defined as the Fourier transform of the spectral density autocorrelation function,

$$K(\tau)=\frac{1}{\bar{d}(E)}\int_{-\infty}^{\infty}d\epsilon\exp\left[\frac{i}{\hbar}\epsilon T\right]\int_{-\infty}^{\infty}\frac{dE'}{\Delta E}d_\chi\left[E'+\frac{\epsilon}{2}\right]d_\chi\left[E'-\frac{\epsilon}{2}\right]. \tag{2}$$

Here $\tau=T/h\bar{d}(E)$ is the normalized time, and the notation $d_\chi(E')=d_{\rm osc}(E')\chi(E'-E)$ has been introduced to limit the integrations to a spectral band of width ΔE around E which contains many levels but which is classically small, $\bar{d}^{-1}\ll\Delta E\ll E$. We choose [7] the weight function χ as $\chi(u)=\exp(-\pi u^2/2\Delta E^2)$, which transforms to $\tilde{\chi}(t)=(\sqrt{2}/\Delta T)\exp(-2\pi t^2/\Delta T^2)$, where $\Delta T=h/\Delta E$.

$K(\tau)$ can be expressed in terms of classical quantities by substituting (1) into (2). If we limit ΔE to the range $\Delta E\ll\sqrt{\hbar E/T}$ (or, equivalently, $\sqrt{\hbar T/E}\ll\Delta T\ll T$), we have $S_j(E')=S_j(E)+(E'-E)T_j$, which leads to a semiclassical approximation for $K(\tau)$:

$$K^{\rm sc}(\tau)=\frac{1}{\Delta E\bar{d}(E)}\left\{\sum_j A_j^2\tilde{\chi}^2(T-T_j)+\sum_{j\neq j'}A_jA_{j'}\cos\left[\frac{1}{\hbar}(S_j-S_{j'})-\frac{\pi}{2}(\nu_j-\nu_{j'})\right]\tilde{\chi}(T-T_j)\tilde{\chi}(T-T_{j'})\right\}. \tag{3}$$

In systems without systematic degeneracies the first (diagonal) term can be approximated, for times long enough for the

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classical behavior to become ergodic, by

$$K_D^{sc}(\tau)=(1/h\bar{d})\sum_j A_j^2\delta(T-T_j)\simeq g|\tau|\,,$$

where g is the mean multiplicity [3,8]. The second (nondiagonal) term $K_N^{sc}(\tau)$ is the main object of our investigation. We emphasize that as $K^{sc}(\tau)$ is not self-averaging, smoothing over some parameter is absolutely necessary [7].

General considerations imply that $K(\tau)\rightarrow 1$ for $\tau\gg 1$ [3]. Hence for (3) to be valid also for $\tau>1$, the various terms in $K_N^{sc}(\tau)$ should conspire to cancel the increasing $K_D^{sc}(\tau)$ so that $K^{sc}(\tau)$ approaches unity. However, as noted above we should expect this only if

$$\hbar\ll E\Delta T,\quad \hbar\ll V/\Delta T,\quad \hbar\ll E\Delta T^2/T\,, \tag{4}$$

where we have used the Thomas-Fermi estimate $\bar{d}(E)\approx V(E)/h^2$, with $V(E)$ the phase space volume of the energy shell. When combined, these inequalities provide an upper limit $\tau_{\max}=(ET^2/V)^{1/3}$ on the range of τ for which $K^{sc}(\tau)$ may be expected to approximate $K(\tau)$ [for systems which exhibit scaling the last inequality in (4) may be avoided and $\tau_{\max}$ relaxes to $(ET^2/V)^{1/2}$]. Note that ergodicity implies $T\gg\sqrt{V(E)/E}$. Corresponding behavior was confirmed for the octagon billiard on a Riemann surface and for the zeros of the ζ function.

We now specialize to systems for which all Maslov indices are even, and define the following *classical* function:

$$P(x;T)=\sum_{j\neq j'}A_jA_{j'}(-1)^{(\nu_j-\nu_{j'})/2}\delta(x-(S_j-S_{j'}))\times\Delta T\tilde{\chi}(T-T_j)\tilde{\chi}(T-T_{j'})\,. \tag{5}$$

$P(x;T)$ contains information about correlations between periodic orbits of energy E with periods within ΔT of T and action difference x. However, it involves their stability exponents (through the A_j's) and Maslov indices as well, and is thus a *weighted* action correlation function. Obviously $P(x;T)$ is easily related to $K_N^{sc}(\tau)$, and hence to $K(\tau)$. Specifically, keeping the classical parameters E, T, and ΔT fixed as $\hbar$ is varied, we find from (3) an explicit expression for it as a Fourier transform in h^{-1}:

$$\hat{P}(y)=\frac{V}{2\pi T^2}P\left(\frac{Vy}{2\pi T};T\right)\simeq\frac{1}{\pi}\int_0^\infty dz\,z[K(1/z)-g/z]\cos(yz)\,. \tag{6}$$

By introducing the dimensionless action difference $y=2\pi xT/V$ and normalizing, we have obtained a function which is independent of T, ΔT, and V (this scaling property is very useful in numerical investigations).

The conditions (4) limit the range of τ values for which $K^{sc}(\tau)$ gives a fair approximation of $K(\tau)$. This implies that the result (6) pertains only to the small-scale structure, $x\sim V/T$, and may be superimposed on a background with possible structure on scales $\gtrsim(EV^2/T)^{1/3}$. For systems in which all Maslov indices vanish, this background component is exponentially large for large T.

We now assume, supported by extensive numerical studies [2], that $K(\tau)$ can be taken from RMT. For systems which are invariant under time reversal, the Gaussian orthogonal ensemble (GOE) form factor must be used, with $g=2$ in K_D^{sc}, while for systems without time-reversal invariance, the Gaussian unitary ensemble (GUE) form factor is appropriate and $g=1$ [2]. Substituting the RMT form factors in (6), we find that

$$\hat{P}_{\rm GOE}(y)=-\frac{4}{\pi}\left(\frac{\sin(y/2)}{y}\right)^2+\frac{2}{y\pi}\{\cos y\,[\mathrm{si}(y)\cos y-\mathrm{Ci}(y)\sin y]+\mathrm{Ci}(2y)\sin 2y-\mathrm{si}(2y)\cos 2y\}\,, \tag{7a}$$

and

$$\hat{P}_{\rm GUE}(y)=-\frac{2}{\pi}\left(\frac{\sin(y/2)}{y}\right)^2. \tag{7b}$$

Note that $\hat{P}_{\rm GOE}$ has a logarithmic singularity at $y=0$, due to the quadratic behavior of $K(\tau)-g\tau$ near $\tau=0$. For the GUE case this difference vanishes in the interval $0\le\tau\le 1$ and the dip at $y=0$ is finite.

Action correlations for periodic orbits of discrete maps on a compact phase space can be derived in a similar manner. Consider the spectrum of the quantum one-step evolution operator U_L, where $L=1/h$ is the dimension of the Hilbert space. The spectral form factor is $K(\tau)=L^{-1}|\mathrm{Tr}U_L^n|^2$, where n is the number of applications of the map and $\tau=n/L$. For chaotic maps with $L\rightarrow\infty$, one can use RMT. The action correlation function is obtained by expanding $\mathrm{Tr}U_L^n$ semiclassically using periodic orbits. The results are the same as (3) and (5) with T replaced by n, $\tilde{\chi}(T-T_j)$ by δ_{nn_j}, and $\Delta E\bar{d}(E)$ by L.

We now turn to a mathematical model which captures all of the essential ingredients of this analysis and for which independent information about the analog of $P(x;T)$ is available—the nontrivial zeros of the Riemann zeta function [9]. According to Riemann's hypothesis, the zeros in question all lie on the line $s=\frac{1}{2}+iE$ with real E. The mean density of the zeros E_n along the E axis is $\bar{d}_R(E)\approx(1/2\pi)^{-1}\ln|E/2\pi|$. A form factor $K_R(\tau)$ can be defined exactly as in (2). Extensive numerical evidence [10] shows that $K_R(\tau)$ approaches the GUE form factor of $E\rightarrow\infty$.

For the zeta function, there is an exact formula for the form factor which is analogous to the periodic orbit expansion (3). It is obtained by identifying prime numbers p with primitive periodic orbits of period $T_p=\ln p$, action $S_p=E\ln p$, stability $|\det(M_p^r-I)|=p^{|r|}$, multiplicity $g_p=1$, and Maslov indices $\nu_p=0$. We also set $\hbar=1$. The diagonal contribution can be evaluated using the prime number theorem to give $K_D(\tau) \approx |\tau|$ for $E \to \infty$ [5]. One can define a prime number correlation function $P_R(x;T)$, in direct analogy to (5) and (6), and rewrite it as a Fourier transform (now with respect to E) of the difference between K_R and K_D, multiplied by $\bar{d}_R$. This may be evaluated [5] using RMT for $K_R(\tau)$, giving $P_R(x;T) \approx -1/2|x|$ for $|x|e^T \gg 1$. The only difference between this and the previous calculation comes from the difference in the functional form of the mean level densities. As before, the existence of a large, nonuniversal background component is not precluded, and indeed $P_R(x;T)$ behaves asymptotically as $\exp(T) - 1/(2|x|)$. Importantly, there exists independent information about the pairwise distribution of primes, the Hardy-Littlewood conjecture [4], which can be shown to directly imply this behavior (see also [5,11]), resulting in a pleasing consistency check.

We next turn to generic systems where it is expected that the SCA for $K(\tau)$ breaks down for long times [12]. We evaluated $\hat{P}(x;T)$ numerically, and found that the scaling property as well as the gross features predicted in (7a) and (7b) are reproduced by the data. In order to see the correlations we are looking for, one has to have an extensive database—hence the choice of systems.

(1) The hyperbola billiard.—This is a planar billiard bounded by the x and y axes and the hyperbola $y = 1/x$, with Dirichlet boundary conditions. Its area is unbounded, but its energy spectrum is purely discrete, with the mean level density given by V/h^2 with $V \sim \ln(E\hbar^{-2})$ weakly depending on $\hbar$. Correspondingly, the classical behavior is only approximately ergodic, with the relevant phase space volume increasing with T. Previous studies have shown that the SCA (1) is applicable down to the lowest energy eigenvalue [13,14]. The symmetry of the system was removed by adding a wall along the line $y = x$, and even and odd states were considered separately by taking appropriate phase factors for orbits hitting this wall. A list of the fist 101 265 periodic orbits with geometric lengths $l < 25$ was recently compiled using a three-letter code, and was used here to determine $K^{sc}(\tau)$. Even with this many periodic orbits, $K^{sc}(\tau)$ could be calculated only within a relatively limited range of τ. The results are shown in Fig. 1(a), averaged over even and odd states, and over different choices of T corresponding to orbits of length $12 < l < 25$. It is seen that $K^{sc}(\tau)$ follows the diagonal approximation K_D^{sc} for small τ, and then saturates at $K \simeq 1$ (the slope of K_D^{sc} is somewhat smaller than 2 because of the deviations from ergodic behavior). The results for $\hat{P}(y)$ are shown in Fig. 1(b), and compared with the RMT prediction [15]. Note that the hyperbola billiard is the only one of the three systems with nonzero Maslov indices.

(2) The deformed cat map.—The cat map is a simple chaotic system whose classical and quantum properties were thoroughly investigated [16–18], but its classical actions are highly degenerate and the quantum spectral statistics do not follow RMT [18]. However, one can deform the cat map in a way that lifts the degeneracies while preserving its structural simplicity. This is done by modifying the Markov transition matrix [19], which is known from ergodic theory to uniquely define the mapping [20]. The time-reversal symmetry is also broken by the deformation. The SCA for $\mathrm{Tr}U_L^n$ can be written as $\mathrm{Tr}V_L^n$, where V_L is a semiclassical transfer operator, and the trace is taken over the classical phase space. V_L depends on the action which defines the mapping, and on the stability, and its form can be written down explicitly once the Markov transition matrix is specified [19,21]. A finite representation of V_L is obtained by coarse graining phase space using a Markov partition, whose level of refinement depends on the desired accuracy. This provides a very efficient method for calculating $K^{sc}(\tau)$. The results, averaged over $L \leq 50$ for $0 \leq \tau \leq 2$, are shown in Fig. 1(c). One sees clearly that the data follow the GUE curve within the statistical uncertainty. A systematic deviation does occur at larger values of τ, and its magnitude can be estimated from the deviation of the eigenvalues of V_L from the unit circle [19]. A direct evaluation of the action correlation function was not performed, because the individual orbits of the deformed cat map are not known. However, since $K^{sc}(\tau)$ and $\hat{P}(y)$ are related by (6), there is no doubt that the action correlations would follow (7b). We note that for both the hyperbola

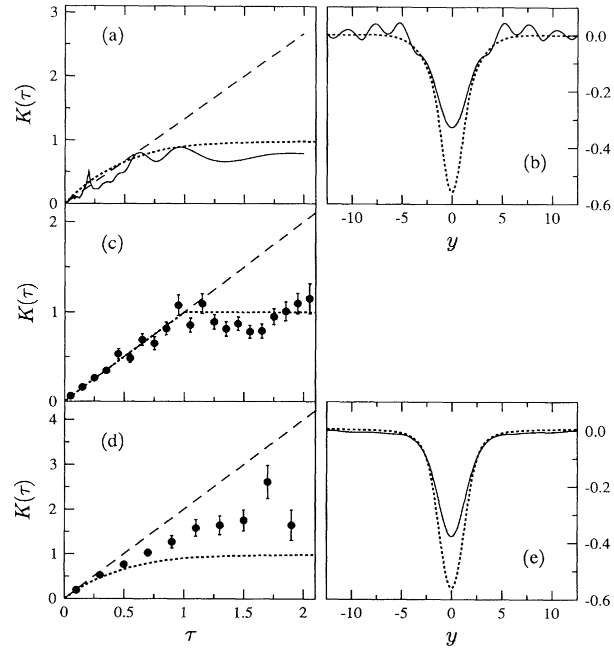

FIG. 1. $K^{sc}(\tau)$ and smoothed $\hat{P}(y)$ for (a) and (b) the hyperbola billiard; (c) the deformed cat map; and (d) and (e) the baker map. The results of the SCA (full lines and circles) are compared with the predictions of RMT and (7a) (dotted lines) and with the diagonal approximation for $K(\tau)$ (dashed lines). The results (d) and (e) for the baker map are scaled to take account of the additional unitary symmetry.

billiard and the deformed cat map it is essential to include the weights—replacing them by their mean values causes the correlations shown here to disappear.

(3) The baker map.—This is another simple and well-known chaotic system [22-24]. The classical dynamics is Bernoulli with an unrestricted binary code, and so the periodic orbits can be characterized by finite sequences, whose length n is the period of the orbit. Quantization of the baker map [22,23] yields evolution operators with RMT behavior [25]. We calculated $K^{sc}(\tau)$ and $\hat{P}(y)$ using two different approaches. In Figs. 1(d) and 1(e) we show the results of a straightforward calculation for orbit lengths in the range $18 \leq n \leq 29$ for $K^{sc}(\tau)$ and $18 \leq n \leq 26$ for $\hat{P}(y)$. τ, y, and $\hat{P}(y)$ were scaled by a factor of 2 to compensate for the unitary symmetry present in the baker map, and a smooth background component was removed from $\hat{P}(y)$. Here $\hat{P}(y)$ is a true action correlation function, since the Maslov indices vanish and the stabilities depend only on n. We also applied the semiclassical transfer operator technique, described above, and were able to reach L values of order 1000. These studies [26] show that the SCA error starts to be appreciable at times $n > n^*(L) \propto L^{1/2}$. In other words, $K^{sc}(\tau)$ diverges from the RMT prediction for τ of the order of $L^{-1/2}$ and higher. This fact would appear to seriously undermine our analysis, and so it is very interesting that the action correlations were still found to exist [26].

In summary, we have found numerical evidence for (weighted) correlations between the actions of classical periodic orbits, which, furthermore, appear to follow the predictions of our analysis. It would be interesting to extend the scope of the numerical study, e.g., to include systems with odd Maslov indices, for which the expression for the implied correlations is more involved, or to systems with focusing elements. The real challenge, though, is to find out whether these action correlations can be explained on a completely classical level.

The result (7b) was also obtained by Dr. J. H. Hannay (personal communication), with whom we have since enjoyed several helpful discussions on this subject.

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[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990); T. Paul and A. Uribe, C. R. Acad. Sci. Paris **313**, 217 (1991).

[2] O. Bohigas, in *Chaos and Quantum Physics*, Proceedings of the 1989 Les Houches Summer School, edited by M. J. Giannoni *et al.* (Elsevier, Amsterdam, 1991), p. 547.

[3] M. V. Berry, Proc. R. Soc. London A **400**, 229 (1985).

[4] G. M. Hardy and J. E. Littlewood, Acta Math. **44**, 1 (1923).

[5] J. P. Keating, in "The Riemann Zeta Function and Quantum Chaology," Proceedings of the International School of Physics "Enrico Fermi," Course CXIX (to be published); (to be published).

[6] See, e.g., E. B. Bogomolny, B. Georgeot, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **69**, 1477 (1992).

[7] N. Argaman, Y. Imry, and U. Smilansky, Phys. Rev. B **47**, 4440 (1993). Note that our use of a Gaussian form of $\chi(\epsilon)$ is equivalent to using d_{osc} instead of d_χ in (2), and using Gaussian averaging over both E' and T', as is done in this reference (see Ref. [28] therein). The fact that $K^{sc}(\tau)$ is the absolute square of a single sum over j is essential for manipulating a large number of orbits.

[8] J. H. Hannay and A. M. Ozorio de Almeida, J. Phys. A **17**, 3429 (1984).

[9] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function* (Clarendon, Oxford, 1986), 2nd ed.

[10] A. M. Odlyzko, "The 10^{20}-th Zero of the Riemann Zeta Function and 70 Million of Its Neighbours" (to be published).

[11] J. P. Keating, in *Quantum Chaos*, edited by M. A. Cerdeira *et al.* (World Scientific, Singapore, 1991), p. 280.

[12] See, e.g., M. A. Sepúlveda, S. Tomsovic, and E. J. Heller, Phys. Rev. Lett. **69**, 402 (1992).

[13] M. Sieber and F. Steiner, Phys. Rev. Lett. **67**, 1941 (1991).

[14] M. Sieber, Ph.D. thesis, University of Hamburg [DESY Report No. 91-030, 1991 (to be published)].

[15] R. Aurich and M. Sieber (to be published).

[16] V. I. Arnol'd and A. Aves, *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968).

[17] J. H. Hannay and M. V. Berry, Physica (Amsterdam) **1D**, 267 (1980).

[18] J. P. Keating, Nonlinearity **4**, 309 (1991).

[19] A. Yu. Kitaev (to be published).

[20] Ya. G. Sinai, *Introduction to Ergodic Theory* (Princeton Univ. Press, Princeton, NJ, 1977).

[21] E. B. Bogomolny, Nonlinearity **5**, 805 (1992).

[22] N. L. Balazs and A. Voros, Ann. Phys. (N.Y.) **190**, 1 (1989).

[23] M. Saraceno, Ann. Phys. (N.Y.) **199**, 37 (1990).

[24] A. M. Ozorio de Almeida and M. Saraceno, Ann. Phys. (N.Y.) **210**, 1 (1991).

[25] We find RMT behavior for generic values of $\hbar$, although special values of $\hbar$ give different results; see P. W. O'Connor and S. Tomsovic, Ann. Phys. (N.Y.) **207**, 218 (1991).

[26] F.-M. Dittes, E. Doron, and U. Smilansky (to be published).