Penumbra Diffraction in the Quantization of Dispersing Billiards

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Diffraction corrections to the semiclassical spectral density of dispersing (Sinai) billiards, due to orbits which are almost tangent to the concave part of the boundary, are studied here for the first time. We show that most periodic orbits needed for quantization must be corrected. For orbits which just miss tangency, the corrections are of the same magnitude as the semiclassical contributions themselves. For orbits which glance at an extreme forward direction, the new theory replaces the semiclassical term that approaches 0 at tangency with a finite one. These corrections are one of the most significant modifications of the trace formula considered so far.

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Dispersing (Sinai) billiards play a prominent role in the study of classical and quantum chaos [1,2]. They are defined by boundaries which consist of concave circular arcs and possibly straight segments. The best known example, which we refer to as the Sinai billiard, has a boundary Γ that consists of a square S and a concentric inscribed circle C. It was used as a standard paradigm to investigate some of the most important aspects of quantum chaos and of its applications in various fields [2].

The classical map of dispersing billiards is discontinuous when a trajectory is tangent to an arc. This singularity introduces diffraction effects when the billiards are semiclassically quantized. Because of the finite wavelength, the domain which is affected consists of a finite neighborhood of the tangent trajectories, and is called the *penumbra* (almost shadow) [3]. In this Letter we analyze the effect of diffraction in the penumbra on semiclassical quantization of dispersing billiards. We shall show that it introduces modifications to the standard semiclassical trace formula [4], which are of the same order as the leading terms, and thus are of prime importance. Moreover, diffraction corrections must be applied to most of the periodic orbits (PO's) which are relevant for semiclassical quantization.

Diffraction corrections in the penumbra are to be distinguished from the exponentially small contributions of creeping trajectories, which are described by the geometrical theory of diffraction [5], and which were previously included in the trace formula [6].

For clarity, we concentrate on the Sinai billiard. We analyze diffraction effects by using a variant of the boundary integral method (BIM). The eigenvalues of the billiard are those real values of k for which the equation

$$u(\mathbf{r}_s) = 2\int_S ds'\,\frac{\partial G_c}{\partial \hat{n}_s}(\mathbf{r}_s,\mathbf{r}_{s'})u(\mathbf{r}_{s'}) \qquad (1)$$

has a solution. In contrast to the standard BIM [7], we use the outgoing *circle's* Green function $G_c(\mathbf{r},\mathbf{r}')$, which is defined in the exterior of C. It satisfies $(\Delta + k^2)G_c(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$ and Dirichlet boundary conditions on C. The integration is carried out along the boundary *excluding* the circle, i.e., along S.

The contributions of PO's which bounce N times off S to the density of states $d(k)$ are derived from (1), by considering the Nth term in the multiple reflection expansion [8]

$$\mathrm{Im}\,\frac{2^N}{\pi N}\,\frac{d}{dk}\int_S ds_1\cdots ds_N\,\frac{\partial G_c}{\partial \hat{n}_1}(\mathbf{r}_1,\mathbf{r}_2)\cdots\frac{\partial G_c}{\partial \hat{n}_N}(\mathbf{r}_N,\mathbf{r}_1)\,. \qquad (2)$$

The saddle points of the integrand correspond to PO's, and their contributions are evaluated by the stationary phase approximation.

In the following, we consider the leading contributions to $G_c(\mathbf{r},\mathbf{r}')$. We shall briefly outline the derivation of the standard semiclassical and creeping results, and then derive the leading corrections in the penumbra. We start from the exact expression [5] $G_c(\mathbf{r},\mathbf{r}') = \sum_{m=-\infty}^{\infty} G_c^{(m)}(\mathbf{r},\mathbf{r}')$, where

$$G_c^{(m)}(\mathbf{r},\mathbf{r}') = \frac{i}{8}\int_{-\infty}^{\infty} dl\,\left[H_l^-(kr_<) + S_l(kR)H_l^+(kr_<)\right] \times H_l^+(kr_>)\exp(il\Delta\theta + 2\pi iml)\,. \qquad (3)$$

Here $r_>$ ($r_<$) is the larger (smaller) of r and r', $S_l(kR) = -H_l^-(kR)/H_l^+(kR)$ is the diagonal element of the circle scattering matrix, and the angle difference $\Delta\theta = |\theta - \theta'|$ is always taken so that $0 \le \Delta\theta \le \pi$. We assume that $r, r' \gtrsim R + R(kR)^{-1/3}$, i.e., the two points are not in the near vicinity of the circle. The analysis concentrates on $G_c^{(0)}(\mathbf{r},\mathbf{r}')$, which gives the dominant contribution.

The domain of $\mathbf{r}$ and $\mathbf{r}'$ in which the standard semiclassical result holds is called the *illuminated* region [see Fig. 1(a)]. In this region, the integral for $G_c^{(0)}(\mathbf{r},\mathbf{r}')$ is evaluated using the Debye approximation for the Hankel functions [9], and the integration is performed using the stationary phase approximation. There are two saddle points, which relate to the two classical trajectories from

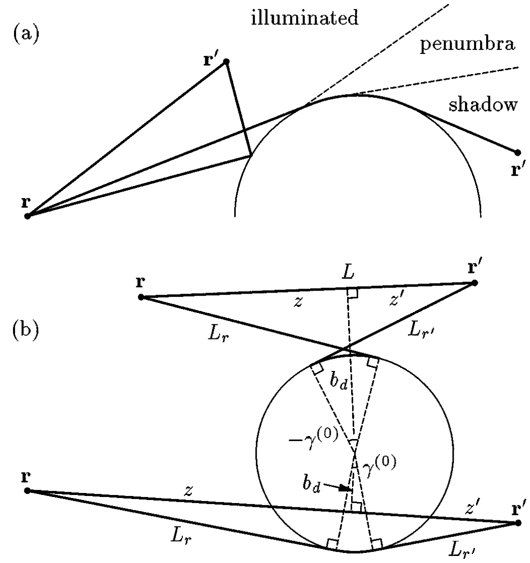

FIG. 1. (a) The three regions of $\mathbf{r}'$ for a fixed $\mathbf{r}$. The direct and reflected trajectories are shown for $\mathbf{r}'$ in the illuminated region, and the shortest creeping trajectory is shown in the shadow region. (b) The geometrical setup in the penumbra. In the upper part $\mathbf{r}$ and $\mathbf{r}'$ are in the classically illuminated region ($b_d > R$), and in the lower part they are in the classically shadowed region.

$\mathbf{r}$ to $\mathbf{r}'$: One is direct, and the other reflects once from the circle. Using the two contributions in (2), one recovers the standard semiclassical trace formula. The Debye approximation for $S_{l_r}(kR)$ fails if $kR - l_r \lesssim (kR)^{1/3}$, where l_r is the angular momentum of the reflected trajectory. This sets the limit of validity of the semiclassical approximation and defines the borderline between the illuminated region and the penumbra.

The term $G_c^{(0)}(\mathbf{r}, \mathbf{r}')$ gives the exponentially small contribution of a creeping trajectory in the *shadow* region [see Fig. 1(a)]. It is obtained by closing the contour of integration in (3) in the complex l plane, and summing over the poles of $S_l(kR)$ [5]. Using creeping contributions in (2) one recovers the results of [6]. This procedure fails if the creeping angle $\gamma^{(0)}$ is $\lesssim (kR)^{-1/3}$. The above condition defines the borderline between the shadow region and the penumbra.

In the penumbra, neither of the above approximations hold. Following Nussenzveig [10], we split $G_c^{(0)}(\mathbf{r}, \mathbf{r}')$ into a *direct* and a *glancing* part, which are, respectively, defined by

$$G_d(\mathbf{r}, \mathbf{r}') = \frac{i}{8} \int_{kR}^{\sigma_2 \infty} dl\, H_l^+(kr) H_l^+(kr') e^{il\Delta\theta}, \tag{4a}$$

$$G_g(\mathbf{r}, \mathbf{r}') = \frac{i}{8} \int_{\sigma_1 \infty}^{kR} dl\, S_l(kR) H_l^+(kr) H_l^+(kr') e^{il\Delta\theta} + \frac{i}{8} \int_{kR}^{\sigma_2 \infty} dl\, [S_l(kR) - 1] H_l^+(kr) H_l^+(kr') e^{il\Delta\theta}. \tag{4b}$$

The limits $\sigma_1 \infty$ and $\sigma_2 \infty$ are directed infinities in the complex plane, defined in [10].

The evaluation of the direct part is similar to that of the direct trajectory contribution in the illuminated region. The only difference is that here we take into account the boundary of the integral at kR. As a result, the semiclassical contribution of the direct trajectory is multiplied by the factor $[F(\infty) - F(\nu)]/\sqrt{2i}$, where $F(x) = C(x) + iS(x)$ is the Fresnel integral function [9], and $\nu = (R - b_d)\sqrt{kL}/\sqrt{\pi z z'}$. The impact parameter of the direct path is denoted by b_d, $z = \sqrt{r^2 - b_d^2}$, $z' = \sqrt{r'^2 - b_d^2}$, and $L = z + z'$ is the length of the direct trajectory [see Fig. 1(b)]. The Fresnel factor equals $\frac{1}{2}$ for exact tangency, and goes to 1 (0) when approaching the borderline of the illuminated (shadow) region.

The main contribution to the glancing part (4b) comes from the vicinity of $l = kR$. In the integrand, $S_l(kR)$ is replaced by its transition region approximation for $l \approx kR$ [9]. The rest of the integrand is evaluated at $l = kR$, where the Debye approximation is used for the Hankel functions. One obtains

$$G_g(\mathbf{r}, \mathbf{r}') = -\frac{C}{4\pi} \frac{(kR)^{1/3}}{k\sqrt{L_r L_{r'}}} e^{ik(L_r + L_{r'} + R\gamma^{(0)})}, \tag{5}$$

where $C \approx 0.996\,193\,019\,928\, e^{i\pi/3}$ was obtained by a numerical integration. The length of the line segment from $\mathbf{r}$ ($\mathbf{r}'$) to the point where it is tangent to the circle is L_r ($L_{r'}$), and $\gamma^{(0)}$ is the directed creeping angle [see Fig. 1(b)]. When $b_d < R$ the length $L_r + L_{r'} + R\gamma^{(0)}$ is equal to the length of a creeping path from $\mathbf{r}$ to $\mathbf{r}'$. When $b_d > R$ the angle $\gamma^{(0)}$ becomes negative.

The contribution of PO's to the density of states $d(k)$ is now found by substituting in (2) for each segment the appropriate approximation for $G_c(\mathbf{r}, \mathbf{r}')$. In all regions this approximation is of the form $Ae^{ikL(\mathbf{r}, \mathbf{r}')}$ (where A varies smoothly with $\mathbf{r}$ and $\mathbf{r}'$), which allows the use of the stationary phase approximation. For isolated PO's that traverse the penumbra once, with a direct segment, the standard semiclassical contribution [which is $O(k^0)$] is multiplied by the Fresnel factor. Thus the correction to the trace formula will be of the same order as the contribution itself. In the case of a single glancing traversal in the penumbra, the contribution is proportional to $k^{-1/6}$, and is not simply related to the standard semiclassical contribution which it replaces. The latter approaches 0 at tangency due to an extreme classical instability, while the former remains finite. The extension to PO's with few segments that traverse the penumbra is straightforward. Since $G_c(\mathbf{r}, \mathbf{r}')$ gives a contribution in the classically shadowed region of the penumbra, there will also be significant contributions of PO's with classically forbidden segments.

We shall present now numerical results which illustrate the importance of the penumbra corrections. This is well demonstrated in terms of the length spectrum [4]

$$D(x) = \int_0^\infty dk\, w(k) e^{ikx} d(k)\,, \tag{6}$$

where $w(k)$ is a weight function. Every PO contributes to $D(x)$ only in a small vicinity of its length. This allows us to isolate the contribution of a single PO (or of few orbits with a similar length) and to compare it to the theory. We consider the fourfold desymmetrized Sinai billiard (see Fig. 2), with side 1 and radius 0.5. The first 5667 exact eigenvalues in the range $0 \leq k \leq 300$ were obtained using the scattering method [11]. The classical PO's were calculated using the minimum and the unique coding principles [12]. In (6) $w(k)$ is a Gaussian centered around $k_0 = 150$ whose width is $\sigma = 40$.

Exactly tangent orbits.—They always exist at the edge of the one-parameter families of neutral PO's ("bouncing balls") that bounce only between the straight segments of the billiard. The simplest of these families is of length 2 [see Fig. 2(a)]. The semiclassical contribution consists of two terms [13]: The contribution of the family $[O(k^{1/2})]$ and the contribution of the limiting orbit at the edge of the square $[O(k^0)]$. Comparison with the exact length spectrum shows a significant difference. Application of our analysis in the penumbra gives an additional term $O(k^{-1/6})$, which is due to an isolated exactly tangent orbit, and is obtained from $G_g(\mathbf{r}, \mathbf{r}')$. This successfully corrects the semiclassical approximation. The semiclassical result for the double repetition of this family is similar to the single repetition, and again shows a significant deviation from the exact result [Fig. 2(b)]. Application of our analysis in this case gives three diffraction terms of orders $O(k^0)$, $O(k^{-1/6})$, and $O(k^{-1/3})$, which reduce the difference substantially.

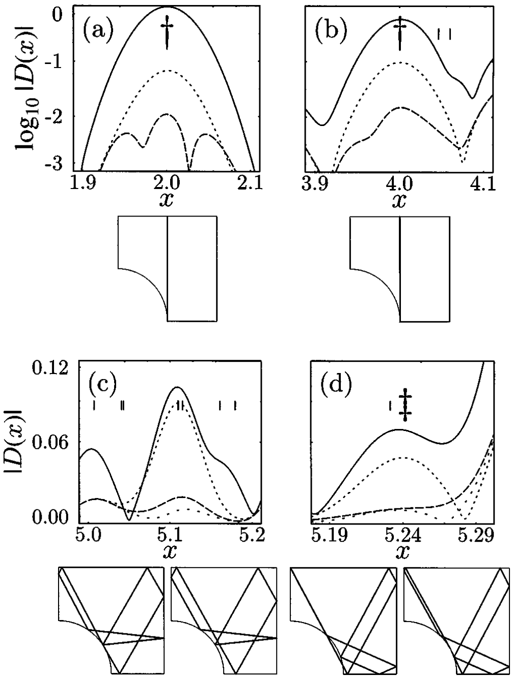

FIG. 2. Penumbra corrections of the length spectrum for four selected cases. Solid lines show the quantum (exact) length spectrum. All other lines show the *deviations from* various semiclassical approximations (SCA). The orbits considered are shown below each frame. Vertical bars indicate locations of unstable PO's, daggers indicate bouncing ball families. (a) Shortest tangent orbit ($x = 2$). Dotted line—SCA including bouncing ball and edge contributions according to [13]. Dashed line—glancing contribution also included. Note the logarithmic scale. (b) Double traversal of the orbit considered in (a). Dashed line includes three penumbra contributions (see text). (c) Pair of almost tangent PO's at $x \approx 5.10$. Dotted line—standard SCA, dashed line—direct term included, sparse dotes—glancing contribution also included. (d) Pair of classically forbidden PO's at $x \approx 5.24$, indicated by double dagger. Notation is as in (c).

Unstable isolated PO's traversing the penumbra.—We consider the two PO's of length ≈ 5.10 [see Fig. 2(c)]. The two orbits are geometrically similar, except that one (the direct) is almost tangent to the circle, and the other (the glancing) reflects in an extremely forward direction. The semiclassical amplitude of the direct orbit is multiplied by a Fresnel factor, which is $0.71 \exp(-0.23i)$ for $k = k_0 = 150$. This accounts for most of the difference. Including the corrected contribution of the glancing orbit further reduces the remaining difference by half.

Classically forbidden PO's which traverse the shaded part of the penumbra.—The two orbits shown in Fig. 2(d) of length ≈ 5.24 are classically forbidden. One cuts through the disk and the other creeps around it. (If we were to reduce the radius of the disk continuously, these orbits would coalesce at $R \approx 0.48$ and appear as a single tangent orbit.) Our theory successfully accounts for the large difference between the exact quantum length spectrum and its semiclassical counterpart at the relevant length. A naive application of the geometrical theory of diffraction [6] (including many modes) results in large deviations from the quantum mechanical length spectrum.

The above examples illustrated the success of our theory to account for the significant penumbra corrections for particular PO's. In the sequel, we provide arguments showing that most of the PO's that are relevant for quantization are affected (cf. [14]). One should bear in mind that the borderlines of the penumbra are k dependent. Any orbit (which is not exactly tangent) is excluded from the penumbra for a sufficiently high wave number. Hence the standard semiclassical contribution of any particular PO is valid in the semiclassical limit $k \to \infty$. Moreover, the fraction of phase space occupied by the penumbra is of order $(kL)^{-2/3}$, where L is a typical length of the billiard, and this fraction vanishes as $k \to \infty$. Thus one may naively conclude that the global effect of diffraction is negligible, which is not true because of the following reasons: For an orbit with n segments, define $\Delta = \min_j |l_j - kR|$, where l_j is the angular momentum of the jth segment with respect to the circle's center. The orbit traverses the penumbra at least once if $\Delta \lesssim (kR)^{1/3}$. Each segment of the orbit has an *a priori* probability $p \approx (kL)^{-2/3}$ to traverse the penumbra. Assuming statistical independence of the segments, and homogeneous coverage of phase space by long PO's, the probability that the orbit will avoid the penumbra is $(1 - p)^n \approx \exp[-n(kL)^{-2/3}]$. In order to achieve an energy resolution of the order of the mean level spacing, one has to include in the semiclassical theory all the PO's whose lengths are smaller than the Heisenberg length $L_H \approx kL^2$. An orbit of this length has $n_H \approx kL$ segments. Thus for all orbits with $n_H^{2/3} \lesssim n \lesssim n_H$ the probability that the orbit does not traverse the penumbra is very small. The exponential proliferation of PO's implies that most of the relevant orbits are in this range. Thus simple ergodic considerations give the surprising result that the semiclassical approximation fails for the *majority* of the relevant PO's in the semiclassical limit $k \to \infty$. The majority of orbits whose length is shorter than $\approx n_H^{2/3}$ avoid the penumbra. Their contribution to the spectral density is well accounted by the standard semiclassical theory. As $k \to \infty$, the number of these orbits grows, but they comprise a smaller fraction of the set of relevant PO's. We note that the above considerations are valid for general dispersing billiards.

In Fig. 3 we present the distribution of Δ for all the PO's in the quarter Sinai billiard for lengths between 7 and 10. The distribution is sharply peaked at the lower values, indicating that indeed most PO's include an almost tangent chord.

Diffraction effects in dispersing billiards were discussed for the 2- and 3-disk scattering problem [6,15]. However, in the geometrical setup used in order to avoid geometrical shadowing, penumbra effects are absent.

The results introduced in this Letter clearly show that penumbra corrections may play a very important role in the semiclassical quantization of dispersing billiards. There are two complementary factors which make the diffraction corrections in the penumbra significant: First, the corrections of amplitudes of individual orbits is of the same order as the semiclassical contribution itself. This should be contrasted with other types of corrections, which are of higher order (e.g., [16,17]) or exponentially small [6]. Second, the corrections should be applied to most of the relevant PO's. This is a direct consequence of the chaotic classical dynamics. The effect of penumbra corrections on an individual energy level is still to be investigated.

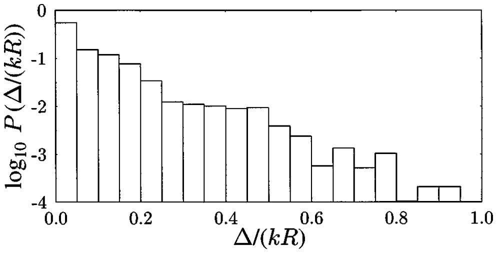

FIG. 3. The coarse-grained distribution of $\Delta/(kR)$, defined in the text. Only angular momenta l_j of reflections from the circle are considered. The data are based on 19 375 PO's which total to 107 029 reflections from the circle.

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