

## Hyperbolic Billiards on Surfaces of Constant Curvature

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**Abstract:** We establish sufficient conditions for the hyperbolicity of the billiard dynamics on surfaces of constant curvature. This extends known results for planar billiards. Using these conditions, we construct large classes of billiard tables with positive Lyapunov exponents on the sphere and on the hyperbolic plane.

### 1. Introduction

From the point of view of differential dynamics, billiards are the geodesic flows on manifolds with a boundary. Since the early beginnings of the study of classical and quantum chaos, billiards have been used as a paradigm. Billiards are one of the best understood classes of dynamical systems that demonstrate a broad variety of behaviors: from integrable to chaotic. In fact, several key properties of chaotic dynamics were first observed and demonstrated for billiards. Many popular models of statistical mechanics, e.g., the Lorenz gas, the hard sphere (Boltzmann–Sinai) gas, etc., can be reduced to billiards in special domains.

Among chaotic dynamical systems, the billiards with nonvanishing Lyapunov exponents are of special interest. For brevity we will often call them hyperbolic billiards. The Pesin theory of smooth nonuniformly hyperbolic systems [Pe], extended by A. Katok and J.-M. Strelcyn to systems with singularities [KS], implies that hyperbolic billiards have strong mixing properties: at most countable number of ergodic components, positive entropy, Bernoulli property, etc.

In the present paper we consider billiards on surfaces of constant curvature. For simplicity of exposition, we restrict the details of our analysis to the simply connected surfaces of constant curvature: the plane, the sphere and the hyperbolic plane. Employing a uniform method, we establish widely applicable conditions, sufficient for positivity of the Lyapunov exponent. The study of billiards on curved surfaces is partially motivated by recent technical advances in semiconductor fabrication techniques. They allow to manufacture solid state (mesoscopic) devices where electrons are confined to a curved

surface (e.g. sphere) [FLBP]. Many properties of these devices can be theoretically derived, using billiards as simplified models.

The billiard dynamics crucially depends on the curvature of the surface. On the plane, billiard trajectories separate only linearly with time, so that the motion between collisions with the boundary is neutral. Exponential separation of billiard trajectories can occur only if the reflections from the boundary introduce sufficient instability. On the hyperbolic plane, geodesics diverge exponentially, so that the main role of the boundary is to confine the mass point to the billiard table. Thus, the boundary can be neutral (i. e., with zero curvature), and the “stretching and folding” necessary for chaotic dynamics, will be provided by the metric. This phenomenon contrasts the billiard dynamics on the sphere, where any two geodesics intersect twice, at focal points. Thus, the boundary reflections have to compensate for the focusing effect of the sphere, in order to produce chaotic dynamics.

Up to now, the study of billiards on surfaces (and hyperbolic billiard dynamics in particular) has been by and large restricted to the Euclidean plane. See, however, [Ve] for a study of integrable billiards on surfaces of constant nonzero curvature. See also [Ta] for some results on chaotic billiards on the hyperbolic plane, and [Vet1,Vet2,KSS] for some results on hyperbolic billiards on a general Riemannian surface. There are many results in the literature concerning hyperbolic dynamics for planar billiards [Si,Bu1-Bu4,Wo2,Ma,Do]. In the present work we generalize Wojtkowski’s criterion of hyperbolicity [Wo2] to billiards on arbitrary surfaces of constant curvature.

We interpret Wojtkowski’s condition [Wo2] in terms of a special class of trajectories, which generalize two-periodic orbits. Let  $Q$  be a billiard table on a surface of constant curvature. The billiard map  $\phi : V \rightarrow V$  acts on the phase space  $V$ , which consists of pairs  $v = (m, \theta)$ . Here  $m$  is the position of the ball on the boundary  $\partial Q$  of  $Q$ , and  $\theta$  is the angle between the outgoing velocity and the tangent to  $\partial Q$  at  $m$ . The billiard map preserves a natural probability measure  $\mu$  on  $V$ . We denote the images of  $v$  after  $n$  iterations by  $(m_{n+1}, \theta_{n+1}) = \phi^n(v)$ . The trajectory  $\phi^n(v)$  is a *generalized two-periodic trajectory* (g.t.p.t.) if the following conditions are satisfied:

1. The incidence angle and the curvature of the boundary  $\kappa_n$  at the bouncing points have period 2:  $\theta_{2n} = \theta_2, \theta_{2n+1} = \theta_1, \kappa_{2n} = \kappa_2, \kappa_{2n+1} = \kappa_1$ ;
2. The geodesic distance between consecutive bouncing points is constant:  $s = |m_n m_{n+1}|$  (see Fig. 1a).
3. If  $\theta_i = \pi/2$ , the g.t.p.t. is a two-periodic orbit, see Fig. 1b.

Along a g.t.p.t. the linearized map  $D_v\phi$  is two-periodic, and the stability of a g.t.p.t. is determined by  $D_v\phi^2$ . As we will see in Sect. 2, for each surface of constant curvature, the stability type of a g.t.p.t. is completely determined by the triple of parameters  $(d_1, d_2, s)$ , where  $2d_1$  (resp.  $2d_2$ ) is the signed length of the chord generated by the intersection of the line  $m_1 m_2$  with the osculating circle at  $m_1$  (resp.  $m_2$ ) (see Fig. 1a). We shall use the symbol  $T(d_1, d_2, s)$  for the g.t.p.t. with parameters  $(d_1, d_2, s)$ .

We will now discuss g.t.p.t.s for planar billiards in some detail. Here  $s$  is the euclidean distance between consecutive bouncing points, and  $d_i = r_i \sin \theta_i$ ,  $i = 1, 2$ , where  $r_i$  are the radii of curvature of the boundary  $\partial Q$  at the respective points. If the curvature of the boundary at the bouncing point is zero we take  $r_i = -\infty$  as the radius of curvature and  $d_i = -\infty$  respectively. By an elementary computation,  $T(d_1, d_2, s)$  is unstable if and

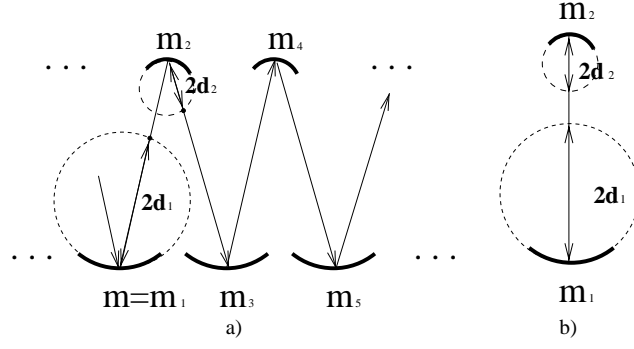


Fig. 1.

only if

$$s \in \begin{cases} [d_1, d_2] \cup [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [0, d_1 + d_2] \cup [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \quad (1.1)$$

Moreover, the trajectory is hyperbolic (i. e., strictly unstable) if  $s$  is in the interior of the corresponding interval, and the trajectory is parabolic if  $s$  is a boundary point (in the limiting case  $d_1 = d_2 = -\infty$  the trajectory is parabolic for any value of  $s$ ).

We introduce the notions of B-unstable and S-unstable g.t.p.t.s. The g.t.p.t.  $T(d_1, d_2, s)$  is B-unstable if in Eq. (1.1)  $s$  belongs to a “big interval”:

$$s \in \begin{cases} [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \quad (1.2)$$

On the contrary, if  $s$  belongs to a “small interval”, then  $T(d_1, d_2, s)$  is S-unstable:

$$s \in \begin{cases} [d_1, d_2] & \text{if } d_1, d_2 \geq 0 \\ [0, d_1 + d_2] & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \quad (1.3)$$

Note that a small interval shrinks to a point when  $|d_1| = |d_2|$ .

We will outline a simple connection between the present approach and Wojtkowski’s method (for planar billiards). With any point  $v = (m_1, \theta_1) \in V$  of the phase space we associate a formal g.t.p.t.  $T(v)$ . Let  $\phi(v) = (m_2, \theta_2)$ . We set  $d_1 = d(v)$ ,  $d_2 = d(\phi(v))$  and  $s = |m_1 m_2|$ . The formal g.t.p.t.  $T(v)$  can be realized as an actual g.t.p.t.  $T(d_1, d_2, s)$  in an auxiliary billiard table  $Q_v$ , constructed from the boundary  $\partial Q$  around  $m_i$ , as shown in Fig. 2.

**Definition 1.** Let the notation be as above. A point  $v \in V$  of the billiard phase space is

- (a) B-hyperbolic (or strictly B-unstable) if the g.t.p.t.  $T(v)$  is strictly B-unstable;
- (b) B-parabolic if  $T(v)$  is B-unstable and parabolic (i. e.,  $s$  belongs to the boundary of the appropriate big interval in Eq. (1.2));
- (c) B-unstable if  $T(v)$  is B-unstable (i. e., B-parabolic or B-hyperbolic);

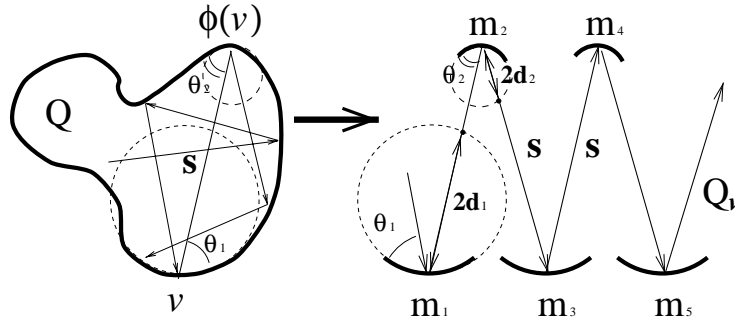


Fig. 2.

- (d) eventually strictly B-unstable if for some  $n \geq 0$  the point  $\phi^n(v)$  is strictly B-unstable, while  $\phi^i(v)$  are B-unstable for  $0 \leq i < n$ .

In our interpretation, Wojtkowski's hyperbolicity criterion [Wo2] is the condition that  $\mu$ -almost all points of the billiard phase space are eventually strictly B-unstable.

The concept of g.t.p.t.s and the associated structures make sense for billiards on any surface. In the body of the paper we will generalize the notions of the B-unstable and S-unstable g.t.p.t.s to arbitrary surfaces of constant curvature, thus extending Definition 1 to billiards on all of these surfaces. Now we formulate the main result of this work.

**Theorem 1 (Main Theorem).** *Let  $Q$  be a billiard table on a surface of constant curvature, and let  $\phi : V \rightarrow V$  be the billiard map. Let  $\mu$  be the canonical invariant measure on  $V$ . If  $\mu$  almost every point of  $V$  is eventually strictly B-unstable then the billiard in  $Q$  is hyperbolic.*

Later on in the paper we will derive geometric conditions on the billiard table that insure that the g.t.p.t.s are B-unstable. With these conditions, which depend on the curvature of the surface, Theorem 1 will become a geometric criterion for hyperbolicity of the billiard dynamics on surfaces of constant curvature. In particular, for planar billiards Theorem 1 yields Wojtkowski's criterion [Wo2].

Let  $\lambda(v) \geq 0$  be the Lyapunov exponent of the billiard in the table  $Q$ , which is defined for  $\mu$ -almost all  $v \in V$ . Recall that in our terminology the billiard in  $Q$  is hyperbolic if  $\lambda(v)$  is positive  $\mu$  almost everywhere. We denote by  $h(Q)$  the metric entropy (with respect to  $\mu$ ) of the billiard in  $Q$ . Following the approach of Wojtkowski's [Wo2], we will estimate from below the metric entropy of billiards satisfying the conditions of Theorem 1.

Let  $\phi_v$  be the map corresponding to the g.t.p.t.  $T(v)$ , and let  $\bar{\lambda}(v) = \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D\phi_v^n\| \geq 0$  be its Lyapunov exponent.

**Theorem 2.** *Let  $Q$  be a billiard table satisfying the assumptions of the main theorem, and let the notation be as above. Then*

$$h(Q) \geq \int_V \bar{\lambda}(v) d\mu. \quad (1.4)$$

To explain the mysterious appearance of g.t.p.t.s, which bear the crux of our approach to hyperbolicity in billiard dynamics, we will outline a connection between them and the method of invariant cone fields of Wojtkowski [Wo1, Wo2]. Let  $\sigma : V \rightarrow V$  be the time-reversal involution:  $\sigma(m, \theta) = (m, \pi - \theta)$  and let  $\mathcal{W} = \{W(v) : v \in V\}$  be an invariant cone field defined in terms of a projective coordinate (each  $W(v)$  is an interval in  $\mathbf{R} \cup \infty$ ). We say that  $\mathcal{W}$  is *symmetric*, if  $W(v) = W(\sigma(v))$  for each  $v \in V$ . The invariant cone fields defined in [Wo2] are symmetric. It can be shown that the existence of a symmetric invariant cone field in  $V$  implies the instability of  $\mu$  almost all g.t.p.t.s  $T(v)$ ,  $v \in V$ . In the proof of Theorem 1 we will show that for our class of billiards the (quasi)converse holds. More precisely, if  $\mu$  almost all g.t.p.t.s  $T(v)$ ,  $v \in V$ , are B-unstable, then  $V$  has a symmetric invariant cone field. If, besides,  $\mu$  almost all g.t.p.t.s are eventually strictly B-unstable, then such cone field is eventually strictly invariant and the billiard dynamics is hyperbolic.

The plan of the paper is as follows. In Sect. 2 we provide the necessary preliminaries and study the geometric optics (i. e., the propagation and reflection of infinitesimal light beams) on surfaces of constant curvature. In Sect. 3 we apply these results to obtain explicit analogs of Eqs. (1.1–1.3). We derive linear instability conditions for g.t.p.t.s and show that they distinguish between B-unstable and S-unstable trajectories in a natural way. In Sect. 4, using invariant cone fields à la Wojtkowski, we prove the main theorem. We define our cone fields for billiards on all surfaces of constant curvature. Employing geometric optics, we show that under the assumptions of the main theorem these cone fields are invariant, and eventually strictly invariant. Also in Sect. 4 we prove Theorem 2. In Sect. 5 we derive hyperbolicity criteria for elementary billiard tables (the boundary consists of circular arcs). Then we apply the main theorem and its corollaries to construct several classes of billiard tables with hyperbolic dynamics on the sphere and on the hyperbolic plane. Finally, we formulate general principles for the design of billiard tables satisfying the conditions of Theorem 1. In particular, we obtain the counterparts of Wojtkowski's geometric inequality [Wo2] for surfaces of constant nonzero curvature. The calculations are involved, and we relegate them to the Appendix.

In a forthcoming publication [Gb] we will apply the methods developed here to investigate the dynamics of billiards in constant magnetic fields on arbitrary surfaces of constant curvature.

The results of Wojtkowski [Wo2] have been strengthened (for planar billiards) in [Bu3, Bu4], and [Do]. It turns out that the criteria of [Bu3, Bu4], and [Do] can be obtained using certain invariant cone fields, which are, in general, not symmetric. This suggests that our hyperbolicity criterion for billiards on surfaces of constant curvature can be considerably strengthened, by employing other invariant cone fields. In particular, we believe that the results of Bunimovich [Bu3, Bu4] and Donnay [Do] can be extended to billiards on surfaces of constant curvature.

## 2. Geometric Optics and Billiards on Surfaces of Constant Curvature

Let  $M$  be a simply connected surface of constant curvature, and let  $Q$  be a connected domain in  $M$ , with a piecewise smooth boundary  $\partial Q$ . For concreteness, we will assume that the curvature is either zero ( $M = \mathbf{R}^2$ ), or one ( $M = \mathbf{S}^2$ ), or minus one ( $M = \mathbf{H}^2$ ). In what follows,  $\partial Q$  is endowed with the positive orientation.

The billiard in  $Q$  is the dynamical system arising from the geodesic motion of a point mass inside  $Q$ , with specular reflections at the boundary. The standard cross-section,  $V \subset TQ$ , of the billiard flow consists of unit tangent vectors, with origin points on  $\partial Q$ ,

pointing inside  $Q$ . The first return associated with this cross-section is the billiard map,  $\phi : V \rightarrow V$ . We will use the standard coordinates  $(l, \theta)$  on  $V$ , where  $l$  is the arclength parameter on  $\partial Q$  and  $0 \leq \theta \leq \pi$  is the angle between the vector and  $\partial Q$ . We call  $V$  the phase space of the billiard map, associated with the billiard table  $Q$ . The invariant measure  $\mu = (2|\partial Q|)^{-1} \sin \theta dl d\theta$  is a probability measure,  $\mu(V) = 1$ .

We will study the natural action of the differential of  $\phi$  on the projectivization  $B$  of the tangent manifold of  $V$ . Abstractly,  $B$  consists of straight lines (as opposed to vectors) in the tangent planes to points of  $V$ . We will describe this space using the language of geometric optics. An oriented curve  $\gamma \subset M$ , of class  $C^2$ , defines a “light beam”, i. e., the family of geodesic rays orthogonal to  $\gamma$ . The geodesics which intersect  $\gamma$  infinitesimally close to a point,  $m \in \gamma$ , form an “infinitesimal beam”, which is completely determined by the normal unit vector  $v \in T_m M$  to  $\gamma$ , and by the geodesic curvature  $\chi$  of  $\gamma$  at  $m$ . We denote the infinitesimal beam by  $b(v, \chi)$ . Our convention for the sign of the curvature is opposite to the one used in [Si,Bu1–Bu4].

Infinitesimal beams yield a geometric representation of the projectivized tangent manifold to the unit tangent bundle of  $M$ . In particular, they give us a geometric realization of the space  $B$ . We will describe the differential of the billiard map in this realization. Let  $p : B \rightarrow V$  be the natural projection. Since  $\dim V = 2$ , each fiber  $p^{-1}(v) \equiv B_v \subset B$  is abstractly isomorphic to the projective line, and we take  $\chi \in \mathbf{R} \cup \infty$  as projective coordinate on  $B_v$  (this representation of  $B$  was discussed for the planar case by e. g., [Wo2]). In this coordinatization,  $B_v = \{b(v, \chi) : \chi \in \mathbf{R} \cup \infty\}$ .

Let  $X \subset TM$  be the set of unit tangent vectors with origin points in  $\partial Q$ , and let  $Y = \{b(v, \chi) : v \in X, \chi \in \mathbf{R} \cup \infty\}$  be the set of corresponding infinitesimal beams. Let  $\rho_m : T_m M \rightarrow T_m M$  be the linear reflection about the tangent line to  $\partial Q$ . As  $m$  runs through  $\partial Q$ , the reflections  $\rho_m$  yield a selfmapping  $\rho : X \rightarrow X$  whose differential acts on  $Y$ .

Let  $\Phi^s$  denote the geodesic flow of  $M$ . Let  $G(v)$  be the oriented geodesic defined by a unit tangent vector. For  $v \in V$  let  $s(v)$  be the distance along  $G(v)$  between the origin point of  $v$ , and the next intersection point of  $G(v)$  with  $\partial Q$ . Then  $\Phi^{s(v)}(v) \in X$ , and  $\rho \circ \Phi^{s(v)}(v) \in V$ . Let  $\Phi : V \rightarrow X$  be the mapping  $v \mapsto \Phi^{s(v)}(v)$ .

We will use the same letters,  $\phi$ ,  $\rho$ , and  $\Phi$ , for the (projectivized) differentials of these mappings. Since the billiard map is the composition:

$$\phi = \rho \circ \Phi, \quad (2.1)$$

it remains to compute the action of  $\Phi$  and  $\rho$  on infinitesimal beams.

Let  $b(v_-, \chi_-) \in Y$  be an infinitesimal beam, and let  $m \in \partial Q$  be the origin point of  $v_-$ . Set  $\rho \cdot b(v_-, \chi_-) = b(v_+, \chi_+)$ . Let  $\kappa$  be the curvature of  $\partial Q$  at  $m$ , and let  $\theta$  be the angle between  $v_-$  and the positive tangent vector to  $\partial Q$  at  $m$ . Then  $v_+ = \rho_m(v_-)$ , and

$$\chi_+ = \chi_- + \frac{2\kappa}{\sin \theta}. \quad (2.2)$$

This formula is well known when  $M = \mathbf{R}^2$  [Si,Bu1], and extends to all surfaces of constant curvature.

Let now  $b = b(v, \chi)$  be an arbitrary infinitesimal beam, and set  $b' = \Phi^s \cdot b = b(v', \chi')$ , where  $v' = \Phi^s(v)$ . We will express  $\chi'$  separately for each surface.

a) Flat case ( $M = \mathbf{R}^2$ ). By elementary euclidean geometry, we have

$$\chi' = \frac{\chi}{1 - s\chi} = -s^{-1} + \frac{s^{-2}}{s^{-1} - \chi}. \quad (2.3)$$

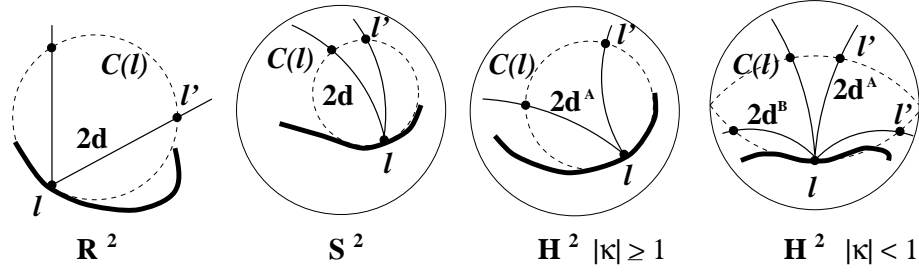


Fig. 3.

b) Curvature one case ( $M = \mathbf{S}^2$ ). By elementary spherical geometry:

$$\chi' = -\cot s + \frac{\sin^{-2} s}{\cot s - \chi}. \quad (2.4)$$

c) Curvature minus one case ( $M = \mathbf{H}^2$ ). The considerations depend on whether  $|\chi|$  is greater or less than one. However, the final expression is the same (we omit the details):

$$\chi' = -\coth s + \frac{\sinh^{-2} s}{\coth s - \chi}. \quad (2.5)$$

Note that in the limit  $s \rightarrow 0$  Eqs. (2.3–2.5) coincide. For  $v \in V$  set  $D(v) = \sin \theta / \kappa$ , so that Eq. (2.2) becomes

$$-\chi_- + \chi_+ = \frac{2}{D(v)}. \quad (2.6)$$

Using classical formulas for surfaces of constant curvature ([Vi], compare also Eq. (2.8) below with [Ta], for a different but related context), we will give a geometric interpretation of the function  $D(\cdot)$ . Let  $v \in V$ , and let  $m = m(l) \in \partial Q$  be the origin point of  $v$ . Let  $C(l) \subset M$  be the osculating circle (hypercycle if  $M = \mathbf{H}^2$  and  $|\kappa(l)| < 1$ ) of  $\partial Q$ . The geodesic,  $G(v)$ , corresponding to  $v$  intersects  $C(l)$  at  $m$  and another point,  $m' = m(l')$ . Let  $\tilde{d}(v)$  be one half of the signed distance between  $m$  and  $m'$ , along  $G(v)$ . If  $|\kappa(l)| < 1$ , the hypercycle  $C(l)$  consists of two components, see Fig. 3. Then there are two possibilities: the points  $l$  and  $l'$  belong to the same component (resp. different components) of  $C(l)$ , Fig. 3. The former (resp. the latter) case occurs if  $|D(v)| \leq 1$  (resp.  $|D(v)| > 1$ ).

*Remark.* When  $\kappa(l) = 0$  ( $D(v) = \infty$ ) and  $M = \mathbf{R}^2, \mathbf{S}^2$  there is ambiguity in the above definition of  $\tilde{d}(v)$ . In this case there are two different values  $\tilde{d}(v) = \pm \tilde{d}_0$  ( $\tilde{d}_0 = +\infty$  for  $M = \mathbf{R}^2$  and  $\tilde{d}_0 = \pi/2$  for  $M = \mathbf{S}^2$ ) satisfying the above definition (if  $M = \mathbf{H}^2$ , then  $\tilde{d}_0 = 0$  and two values coincide). In what follows we always choose in such case the negative value  $-\tilde{d}_0$  as the definition for  $\tilde{d}(v)$ , i.e., we consider the case of zero curvature boundary as a limiting case of a negative curvature boundary. Thus  $\tilde{d}(v) \in [-\infty, \infty)$  if  $M = \mathbf{R}^2$  and  $\tilde{d}(v) \in [-\pi/2, \pi/2)$  if  $M = \mathbf{S}^2$ .

Set

$$d(v) = \begin{cases} \tilde{d}(v) & \text{if } M = \mathbf{R}^2 \text{ or } M = \mathbf{S}^2 \\ \tilde{d}(v) & \text{if } M = \mathbf{H}^2 \text{ and } |\kappa(l)| \geq 1 \\ \tilde{d}(v) & \text{if } M = \mathbf{H}^2, |\kappa(l)| < 1, |D(v)| \leq 1 \\ \tilde{d}(v) + i\pi/2 & \text{if } M = \mathbf{H}^2, |\kappa(l)| < 1, |D(v)| > 1 \\ \tilde{d}(v) + i\pi/2 & \text{if } M = \mathbf{H}^2, |\kappa(l)| < 1, |D(v)| > 1. \end{cases} \quad (2.7)$$

Then we have

$$D(v) = \begin{cases} d(v) & \text{if } M = \mathbf{R}^2 \\ \tan d(v) & \text{if } M = \mathbf{S}^2 \\ \tanh d(v) & \text{if } M = \mathbf{H}^2. \end{cases} \quad (2.8)$$

For the case  $M = \mathbf{H}^2$  we will use the following classification of points of the phase space  $V$ . We say that  $v \in V$  is of type  $A$  (resp.  $B$ ) if  $|D(v)| \leq 1$  (resp.  $|D(v)| > 1$ ). Let  $V^A, V^B$  be the corresponding subsets of  $V$ . Then  $V = V^A \cup V^B$  is a partition. We will use the notation:

$$\tilde{d}(v) = \begin{cases} d^A(v) \in [-\infty, \infty] & \text{if } v \in V^A \\ d^B(v) \in (-\infty, \infty) & \text{if } v \in V^B. \end{cases} \quad (2.9)$$

### 3. Generalized Two-Periodic Trajectories (g.t.p.t.s)

Consider the billiard dynamics in an arbitrary table on a surface of constant curvature. Eqs. (2.2) and (2.3–2.5) describe the action of the billiard map on infinitesimal beams. Starting with an arbitrary  $b(v, \chi)$  and iterating the equations, we obtain for  $\chi$  after an infinite number of reflections a formal continued fraction

$$c \equiv \chi^\infty = a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 \cdots}}, \quad (3.1)$$

whose coefficients are determined by  $d_i = d(\phi^{i-1} \cdot v)$ , and by the lengths  $s_i$  of consecutive billiard segments, where  $i = 1, 2, \dots$ . The idea to associate a continued fraction (3.1) to a billiard orbit has been introduced by Y. Sinai in the seminal paper [Si], where he considered billiards in  $\mathbf{R}^2$ . Equation (3.1) is a direct extension of Sinai's idea to an arbitrary surface of constant curvature.

Let  $Q$  be a billiard table, and let  $v \in V$  be an arbitrary point in the phase space of the billiard map. Set  $v_1 = v, v_2 = \phi(v), d_i = d(v_i), i = 1, 2$ , and let  $s = s(v)$  be the distance between the origin points of  $v_1$  and  $v_2$ , respectively (Fig. 2). Let  $T(v) = T(d_1, d_2, s)$  be the associated g.t.p.t. (see Sect. 1). The g.t.p.t.  $T(v)$  can be realized as a trajectory in an artificial billiard table whose exact shape  $Q_v$  is not important (see Fig. 2). We denote by  $\phi_v$  the associated billiard map.

Let  $c(v)$  be the formal continued fraction Eq. (3.1), corresponding to  $T(v)$ . Note that  $c(v)$  is periodic. Proposition 1 below relates the convergence of  $c(v)$  with the stability type of  $T(v)$ . Recall that the standard definitions of elliptic, hyperbolic, and parabolic periodic points can be expressed in terms of the appropriate power of the differential of the transformation, i. e., a particular matrix associated with the periodic orbit, see, e. g., [KH]. Hence, these definitions straightforwardly extend to generalized periodic orbits, and we leave the details to the reader. In what follows we will talk about elliptic, parabolic, or hyperbolic g.t.p.t.s. We say that a g.t.p.t. is (exponentially) unstable if it is either hyperbolic or parabolic (resp. hyperbolic).

**Proposition 1.** *Let  $v \in V$  be arbitrary, and let the notation be as above. The g.t.p.t.  $T(v)$  is (exponentially) unstable if and only if the continued fraction  $c(v)$  converges (exponentially fast).*

We outline a proof of Proposition 1, referring to [Wa] for the standard material on continued fractions. With a periodic continued fraction one associates a fractional linear transformation, or, equivalently, a  $2 \times 2$  matrix, defined up to a scalar factor. For a  $c(v)$  this matrix essentially coincides with the linear transformation associated with the g.t.p.t.  $T(v)$ . The claim now follows from the standard facts [Wa] (we leave details to the reader).

Note that Proposition 1 (and its proof) straightforwardly extends to generalized periodic trajectories of any period.

*Remark.* Another approach to the stability of  $T(v)$  is to consider the linearization  $D\phi_v^2$ . Then  $T(v)$  is hyperbolic if  $|tr(D\phi_v^2)| > 2$ , parabolic if  $|tr(D\phi_v^2)| = 2$ , and elliptic if  $|tr(D\phi_v^2)| < 2$ .

**Lemma 1.** *Let  $v \in V$ , and let  $d_1, d_2, s$  be the associated data. Then the coefficients  $a_i, b_i, i \geq 1$  of the continued fraction  $c(v)$  are given by the following formulas:*

- a)  $M = \mathbf{R}^2$ . We have  $a_{2n+1} = -2s^{-1} + 2d_1^{-1}, a_{2n} = -2s^{-1} + 2d_2^{-1}, b_n = -s^{-2}$ ;
- b)  $M = \mathbf{S}^2$ . Then  $a_{2n+1} = -2 \cot s + 2 \cot d_1, a_{2n} = -2 \cot s + 2 \cot d_2, b_n = -\sin^{-2} s$ ;
- c)  $M = \mathbf{H}^2$ . Here we have  $a_{2n+1} = -2 \coth s + 2 \coth d_1, a_{2n} = -2 \coth s + 2 \coth d_2, b_n = -\sinh^{-2} s$ .

*Proof.* The formulas are obtained by direct computations from Eqs. (2.2–2.6).  $\square$

Since the g.t.p.t.  $T(v)$  and the continued fraction  $c(v)$  are essentially determined by the triple  $(d_1, d_2, s)$  corresponding to  $v$ , we will use the notation  $T(d_1, d_2, s)$  and  $c(d_1, d_2, s)$  in what follows. The formulas of Lemma 1 allow to compute the  $2 \times 2$  matrix associated with  $c(d_1, d_2, s)$ . Analyzing this matrix for each of the three surfaces, we obtain simple criteria for the convergence of  $c(d_1, d_2, s)$ .

**Proposition 2.** *The continued fraction  $c(d_1, d_2, s)$  converges if and only if the following inequalities are satisfied:*

a) If  $M = \mathbf{R}^2$ :

$$(s - d_1)(s - d_2)(s - d_1 - d_2)s \geq 0. \quad (3.2)$$

b) If  $M = \mathbf{S}^2$ :

$$\sin(s - d_1) \sin(s - d_2) \sin(s - d_1 - d_2) \sin s \geq 0. \quad (3.3)$$

c) If  $M = \mathbf{H}^2$ :

$$\sinh(s - d_1) \sinh(s - d_2) \sinh(s - d_1 - d_2) \sinh s \geq 0. \quad (3.4)$$

Taking into consideration that  $s \geq 0$  for  $\mathbf{R}^2$  and  $\mathbf{H}^2$ , and that  $0 \leq s \leq 2\pi$  for  $\mathbf{S}^2$ , we reformulate Proposition 2 in a more explicit form.

a) Let  $M = \mathbf{R}^2$ . Then  $T(d_1, d_2, s)$  is unstable if and only if

$$s \in \begin{cases} [d_1, d_2] \cup [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [0, d_1 + d_2] \cup [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \quad (3.5)$$

b) Let  $M = \mathbf{S}^2$ . Set

$$s \bmod \pi = \begin{cases} s & \text{if } s \leq \pi \\ s - \pi & \text{if } s > \pi. \end{cases}$$

Then  $T(d_1, d_2, s)$  is unstable if and only if

$$s \bmod \pi \in \begin{cases} [d_1 + d_2, \pi] \cup [d_1, d_2] & \text{if } d_1, d_2 \geq 0 \\ [0, d_1 + d_2 + \pi] \cup [\pi - d_1, \pi - d_2] & \text{if } d_1, d_2 \leq 0 \\ [d_2, \pi + d_1] \cup [0, d_1 + d_2] & \text{if } d_1 \leq 0, d_2 \geq 0, |d_2| \geq |d_1| \\ [d_2, \pi + d_1] \cup [\pi + d_2 + d_1, \pi] & \text{if } d_1 \leq 0, d_2 \geq 0, |d_2| \leq |d_1|. \end{cases} \quad (3.6)$$

c) Let  $M = \mathbf{H}^2$ . We say that  $T(d_1, d_2, s)$  is of type  $(A - A)$  if  $v_1 \in V^A$  and  $v_2 \in V^A$ . The other types:  $(A - B)$ ,  $(B - A)$ , and  $(B - B)$  are defined analogously. We formulate the criteria of instability for  $T(d_1, d_2, s)$  "type-by-type".

Type  $(A - A)$ :

$$s \in \begin{cases} [d_1^A, d_2^A] \cup [d_1^A + d_2^A, \infty) & \text{if } d_1^A, d_2^A \geq 0 \\ [0, \infty) & \text{if } d_1^A, d_2^A \leq 0 \\ [0, d_1^A + d_2^A] \cup [d_1^A, \infty) & \text{if } d_1^A \geq 0, d_2^A \leq 0. \end{cases} \quad (3.7a)$$

Type  $(B - B)$ :

$$s \in \begin{cases} [d_1^B + d_2^B, \infty) & \text{if } d_1^B + d_2^B \geq 0 \\ [0, \infty) & \text{if } d_1^B + d_2^B \leq 0. \end{cases} \quad (3.7b)$$

Types  $(A - B)$  or  $(B - A)$ :

$$s \in \begin{cases} [d_1^A, \infty) & \text{if } d_1^A \geq 0 \\ [0, \infty) & \text{if } d_1^A \leq 0. \end{cases} \quad (3.7c)$$

It is worth mentioning that in Eqs. (3.2–3.4) (resp. Eqs. (3.5–3.7)) the hyperbolicity of  $T(d_1, d_2, s)$  corresponds to strict inequalities (resp. inclusions in the interior). The equality case (resp. boundary case) corresponds to the parabolicity of  $T(d_1, d_2, s)$ . There are also two special cases when  $T(d_1, d_2, s)$  is parabolic independently of the value of  $s$ :  $M = \mathbf{R}^2$ ,  $d_1 = d_2 = -\infty$  and  $M = \mathbf{H}^2$ ,  $|d_1| = |d_2| = \infty$  (it means also that  $v_1, v_2 \in V^A$ ).

We say that the right-hand side in Eqs. (3.5–3.7) is the instability set of  $T(d_1, d_2, s)$ . In general, it is a union of two intervals, where one of them degenerates when  $|d_1| = |d_2|$ , while the other is always nontrivial. For want of a better name, we will say that the interval which persists is the "big interval", and the other one is the "small interval". This motivates the following terminology: We will say that  $T(d_1, d_2, s)$  is (strictly) B-unstable if  $s$  belongs to the (interior of the) big interval of instability. The proposition below makes this terminology explicit.

**Proposition 3.** *The g.t.p.t.  $T(d_1, d_2, s)$  is B-unstable if (and only if) the triple  $(d_1, d_2, s)$  satisfies the following conditions:*

a) Let  $M = \mathbf{R}^2$ . Then

$$s \in \begin{cases} [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \quad (3.8)$$

b) Let  $M = \mathbf{S}^2$ . Then

$$s \bmod \pi \in \begin{cases} [d_1 + d_2, \pi] & \text{if } d_1, d_2 \geq 0 \\ [0, d_1 + d_2 + \pi] & \text{if } d_1, d_2 \leq 0 \\ [d_2, \pi + d_1] & \text{if } d_1 \leq 0, d_2 \geq 0. \end{cases} \quad (3.9)$$

c) Let  $M = \mathbf{H}^2$ . Then:

In the case  $(A - A)$

$$s \in \begin{cases} [d_1^A + d_2^A, \infty) & \text{if } d_1^A, d_2^A \geq 0 \\ [0, \infty) & \text{if } d_1^A, d_2^A \leq 0 \\ [d_1^A, \infty) & \text{if } d_1^A \geq 0, d_2^A \leq 0, \end{cases} \quad (3.10a)$$

or  $|d_1^A| = |d_2^A| = \infty$  and arbitrary  $s$ .

In the case  $(B - B)$

$$s \in \begin{cases} [d_1^B + d_2^B, \infty) & \text{if } d_1^B + d_2^B \geq 0 \\ [0, \infty) & \text{if } d_1^B + d_2^B \leq 0. \end{cases} \quad (3.10b)$$

In the cases  $(A - B)$  or  $(B - A)$

$$s \in \begin{cases} [d_1^A, \infty) & \text{if } d_1^A \geq 0 \\ [0, \infty) & \text{if } d_1^A \leq 0. \end{cases} \quad (3.10c)$$

#### 4. Proofs of Theorem 1 and Theorem 2

*Proof of the main theorem (Theorem 1).* We will define a cone field on the phase space of the billiard map. A cone in  $T_v V$  corresponds to an interval in the projectivization,  $B_v$ . In Sect. 2 we have explicitly identified each space  $B_v$  with the standard projective line  $\mathbf{R} \cup \infty$ . Therefore, a cone field,  $\mathcal{W}$ , is determined by a function,  $W(\cdot)$ , on  $V$ , where each  $W(v) \subset \mathbf{R} \cup \infty$  is an interval in the projective coordinate  $\chi$ .

We introduce an auxiliary coordinate  $f$  on  $B_v$ , which has a simple geometric meaning. Let  $b(v, \chi)$  be an infinitesimal beam, and let  $G(v)$  be the corresponding oriented geodesic. Consider the beams  $\Phi^t \cdot b(v, \chi)$ , obtained by the action of the geodesic flow. Suppose, that  $M = \mathbf{R}^2$  or  $M = \mathbf{S}^2$ , or  $M = \mathbf{H}^2$  and  $|\chi| \geq 1$ . Then there is  $t \in \mathbf{R} \cup \infty$ , such that the beam  $\Phi^t \cdot b(v, \chi)$  has infinite curvature. If  $M = \mathbf{R}^2$  or  $M = \mathbf{H}^2$  ( $|\chi| \geq 1$ ), then  $t$  is unique, and we set  $f(\chi) = t$ . If  $M = \mathbf{S}^2$ , then  $t$  is unique modulo  $\pi$ , and let  $f(\chi) \in [-\pi/2, \pi/2)$  be the one with the smallest absolute value. We denote by  $o(v, \chi) \in M$  the origin point of  $\Phi^{f(\chi)} \cdot v$ . This is the focusing point of the infinitesimal

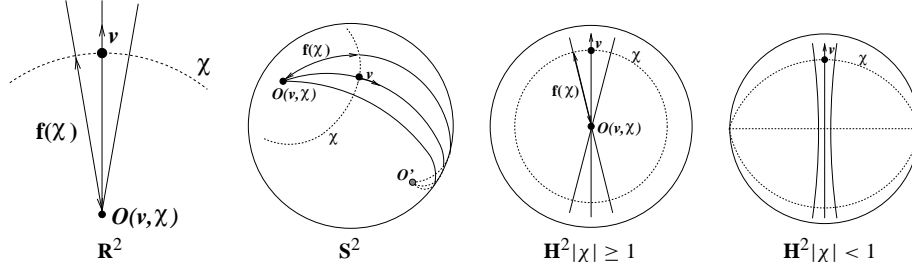


Fig. 4.

beam  $b(v, \chi)$ , see Fig. 4a, b, c. If  $M = \mathbf{H}^2$ , and  $|\chi| < 1$  then the beam  $b(v, \chi)$  has no focusing point Fig. 4d.

While the focusing point,  $o(v, \chi)$ , depends on both  $v$  and  $\chi$ , the signed focusing distance is determined by the curvature of the beam alone,  $f = f(\chi)$ . The explicit relations between  $f$  and  $\chi$  depend on  $M$ .

- a) When  $M = \mathbf{R}^2$ , we have  $\chi = 1/f$ ;
- b) If  $M = \mathbf{S}^2$ , we have  $\chi = \cot(f)$ ;
- c) If  $M = \mathbf{H}^2$  and  $|\chi| \geq 1$ , we have  $\chi = \coth(f)$ .

We will define the cone field  $\mathcal{W}$  using the projective coordinate  $\chi$ .

- a) Let  $M = \mathbf{R}^2$ . Set

$$W(v) = \begin{cases} [-\infty, D^{-1}(v)] & \text{if } D(v) \leq 0 \\ [D^{-1}(v), +\infty] & \text{if } D(v) > 0. \end{cases}$$

- b) Let  $M = \mathbf{S}^2$ . Set

$$W(v) = \begin{cases} [-\infty, D^{-1}(v)] & \text{if } D(v) \leq 0 \\ [D^{-1}(v), +\infty] & \text{if } D(v) > 0. \end{cases}$$

- c) Let  $M = \mathbf{H}^2$ . We consider two cases.

- 1) If  $v \in V^A$ , we set

$$W(v) = \begin{cases} [-\infty, D^{-1}(v)] & \text{if } D(v) \leq 0 \\ [D^{-1}(v), +\infty] & \text{if } D(v) > 0, . \end{cases}$$

- 2) If  $v \in V^B$ , then

$$W(v) = [-\infty, D^{-1}(v)].$$

In terms of the auxiliary coordinate  $f$  the cone field  $\mathcal{W}$  is given for  $M = \mathbf{R}^2$  and  $M = \mathbf{S}^2$  by the following intervals:

$$W(v) = \begin{cases} [d(v), 0] & \text{if } d(v) \leq 0 \\ [0, d(v)] & \text{if } d(v) > 0. \end{cases}$$

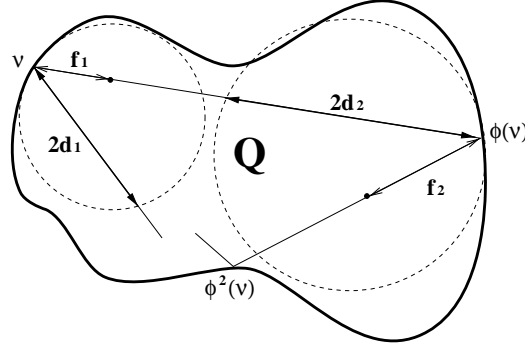


Fig. 5.

In what follows, we will use the cone field  $\mathcal{W}$  in one form or the other, whichever is more convenient.

We recall the classification of points in the phase space of the billiard map. A point  $v \in V$  is  $B$ -hyperbolic (we will also say strictly  $B$ -unstable) if the corresponding g.t.p.t.  $T(v)$  is  $B$ -unstable and hyperbolic. A point is  $B$ -parabolic if  $T(v)$  is  $B$ -unstable and parabolic. Putting the two definitions together, we will say that  $v \in V$  is  $B$ -unstable if the corresponding g.t.p.t.  $T(v)$  is  $B$ -unstable (i. e., either  $B$ -parabolic or  $B$ -hyperbolic). We will say that  $v \in V$  is eventually strictly  $B$ -unstable if there exists  $n \geq 0$  such that the points  $\phi^i(v)$  are  $B$ -unstable for  $0 \leq i < n$  and  $\phi^n(v)$  is strictly  $B$ -unstable.

**Lemma 2.** *Let  $M$  be a surface of constant curvature, let  $Q \subset M$  be an arbitrary billiard table, and let  $\mathcal{W}$  be the cone field defined above. Let  $v \in V$  be such that the g.t.p.t.  $T(v)$  is (strictly)  $B$ -unstable.*

*Then  $\phi(W(v)) \subseteq W(\phi(v))$  (resp. the strict inclusion  $\phi(W(v)) \subset W(\phi(v))$  holds).*

*Proof.* Let  $(d_1, d_2, s)$  be the triple, associated to  $v$ . We will prove the claim separately for each of the three surfaces.

a) Let  $M = \mathbf{R}^2$  (Fig. 5). We rewrite Eq. (2.6) as

$$\frac{(s - f_1 - d_2)}{s - f_1} = \frac{(d_2 - f_2)}{f_2}. \quad (4.1)$$

Since  $(d_1, d_2, s)$  satisfies Eq. (3.8), we obtain  $(d_2 - f_2)/f_2 \geq 0$ . The inequality is strict if  $T(v)$  is strictly  $B$ -unstable. This implies the claim.

b) Let  $M = \mathbf{S}^2$  (Fig. 5). Equation (2.6) and the relation between  $\chi$  and  $f$  on  $\mathbf{S}^2$  imply

$$\frac{\sin(s - f_1 - d_2)}{\sin(s - f_1)} = \frac{\sin(d_2 - f_2)}{\sin f_2}. \quad (4.2)$$

Since the triple  $(d_1, d_2, s)$  satisfies Eq. (3.9),  $\sin(d_2 - f_2)/\sin f_2 \geq 0$  (strict inequality if  $T(v)$  is strictly  $B$ -unstable). Simple considerations, which we leave to the reader, yield the claim.

c) Let  $M = \mathbf{H}^2$ . From Eqs. (2.5) and (2.6) we have

$$\chi_2 = \frac{2}{D(v)} - \coth s + \frac{\sinh^{-2} s}{\coth s - \chi_1}. \quad (4.3)$$

Recall that  $V = V^A \cup V^B$ , a partition of  $V$  into the sets of points of type  $A$  and type  $B$ . Hence, depending on the type of  $v_i$ ,  $i = 1, 2$ , we have four cases to consider. We will prove the claim case-by-case.

Case  $B - B$ . From Eq. (4.3) and Eq. (3.10b), we obtain  $\chi_2 \leq \tanh d_2^B$ , which implies the claim.

Case  $B - A$ . From Eq. (4.3) and Eq. (3.10c), we have  $\chi_2 \in [-\infty, \coth d_2^A]$  if  $d_2^A \leq 0$ , and  $\chi_2 \in [\coth d_2^A, \infty]$  if  $d_2^A > 0$ . The claim follows.

Case  $A - A$ . From Eq. (4.3) and Eq. (3.10a), we obtain  $\chi_2 \in [-\infty, \coth d_2^A]$  if  $d_2^A \leq 0$ , and  $\chi_2 \in [\coth d_2^A, \infty]$  if  $d_2^A > 0$ , which implies the claim.

Case  $A - B$ . From Eq. (4.3) and Eq. (3.10c), we have  $\chi_2 \leq \tanh d_2^B$ , implying the claim. This proves Lemma 2.  $\square$

Now we finish the proof of the main theorem. Since, by assumption, almost every point of the phase space is eventually strictly B-unstable, Lemma 2 implies that the cone field  $\mathcal{W}$  is eventually strictly invariant. The claim now follows from a theorem of Wojtkowski [Wo1, Wo2].  $\square$

*Proof of Theorem 2.* Let  $l(v)$  and  $r(v)$  be the left and the right endpoints of the interval  $W(v)$  defined in terms of the projective coordinate (for the cone fields defined above  $l(v)$  and  $r(v)$  are either  $\infty$  or  $D^{-1}(v)$ ). Let  $l_1(v)$  and  $r_1(v)$  be the left and the right endpoints of the interval  $\phi(W(v))$ . Applying Theorem 2 in [Wo2] to the billiards, satisfying the assumptions of the main theorem, we obtain

$$\int_V \lambda_+ d\mu \geq \int_V \log \frac{\sqrt{\zeta} + 1}{\sqrt{\zeta} - 1} d\mu, \quad (4.4)$$

where

$$\zeta(v) = \frac{r(\phi(v)) - l_1(v)}{r(\phi(v)) - r_1(v)} \frac{l(\phi(v)) - r_1(v)}{l(\phi(v)) - l_1(v)}.$$

Let  $\phi_v$  be the map associated with the g.t.p.t.  $T(v)$ . By straightforward calculations

$$\left( \frac{\sqrt{\zeta} + 1}{\sqrt{\zeta} - 1} \right)^2 + \left( \frac{\sqrt{\zeta} + 1}{\sqrt{\zeta} - 1} \right)^{-2} = |\text{tr}(D\phi_v^2)|.$$

The claim now follows from the inequality (4.4).  $\square$

## 5. Applications and Examples

There are many classes of planar domains with hyperbolic billiard dynamics [Wo2], [Bu3,4, Ma]; see also [Tab] and the references there. In Subsect. 5.1 we will apply the Main Theorem to obtain convenient sufficient conditions of hyperbolicity for elementary billiard tables on all surfaces of constant curvature. In Subsect. 5.2 we will use these conditions (as well as the Main Theorem directly) to construct several classes of examples of billiard tables with chaotic dynamics on  $\mathbf{S}^2$  and  $\mathbf{H}^2$ . In Subsect. 5.3, expanding the ideas of [Wo2] for billiards in  $\mathbf{R}^2$ , we obtain a simple set of principles for constructing billiard tables with hyperbolic dynamics on arbitrary surfaces of constant curvature.

*5.1. Elementary billiard tables: Conditions for hyperbolicity.* We shall use the term “elementary billiard tables” to denote billiard tables  $Q$ , such that  $\partial Q$  is a finite union of arcs,  $\Gamma_i$ , of constant geodesic curvature,  $\kappa(\Gamma_i) = \kappa_i$ . We will use the notation  $\Gamma_i^+$  (resp.  $\Gamma_i^-$ , resp.  $\Gamma_i^0$ ) to indicate that  $\kappa_i > 0$  (resp.  $\kappa_i < 0$ , resp.  $\kappa_i = 0$ ). Let  $C_i$  be the curve of constant curvature containing  $\Gamma_i$ . Let  $D_i \subset M$  be the smallest region such that  $C_i = \partial D_i$ . The representation  $\partial Q = \cup_{i=1}^N \Gamma_i$  is unique, and we call  $\Gamma_i$  the components. We will refer to  $\Gamma_i^+$  (resp.  $\Gamma_i^-$ , resp.  $\Gamma_i^0$ ) as the components of type plus (resp. of type minus, resp. of type zero).

Applying the Main Theorem to elementary billiard tables in  $\mathbf{R}^2$ , we recover a classical result of L. Bunimovich [Bu1].

**Corollary 1.** *Let  $Q \subset \mathbf{R}^2$  be an elementary billiard table with at least two boundary components, and assume that not all of them are neutral. If for every  $\Gamma_i^+$  we have  $D_i \subset Q$ , then the billiard in  $Q$  is hyperbolic.*

The extension of this result for  $M = \mathbf{S}^2$  and  $M = \mathbf{H}^2$  will be given below. For this purpose we introduce the following terminology: If  $R \subset S \subset M$  are regions with piecewise  $C^1$  boundaries, we call the inclusion  $R \subset S$  *proper* if  $\partial R \cap \text{int } S \neq \emptyset$ .

Consider now an elementary billiard table  $Q \subset \mathbf{S}^2$ . For any domain  $D \subset \mathbf{S}^2$  we denote by  $-D \subset \mathbf{S}^2$  the domain obtained by the reflection of  $D$  about the center of the sphere (polar domain).

**Condition S1.** The table  $Q$  satisfies  $D_i \subset Q$  for every boundary component  $\Gamma_i^+$ . Besides, either  $-D_i \subset Q$ , or  $-D_i \subset \mathbf{S}^2 \setminus Q$ , and the inclusions are proper.

**Condition S2.** For every  $\Gamma_j^-$  we have  $D_j \subset \mathbf{S}^2 \setminus Q$ , and the inclusions  $-D_j \subset \mathbf{S}^2 \setminus Q$ , or  $-D_j \subset Q$  are proper.

**Corollary 2.** *Let  $Q \subset \mathbf{S}^2$  be an elementary billiard table with at least two boundary components of nonzero type. If  $Q$  satisfies Conditions S1 and S2, then the billiard in  $Q$  is hyperbolic.*

*Outline of proof.* Straightforward analysis shows that  $Q$  satisfies the conditions of the Main Theorem.

*Remark.* Suppose  $Q' = \mathbf{S}^2 \setminus Q$  is connected. If  $Q$  satisfies Conditions S1 and S2, then  $Q'$  also does, and hence the billiard in  $Q'$  is hyperbolic.

Let  $Q \subset \mathbf{H}^2$  be an elementary billiard table. We use the notation  $\Gamma_i^A$  (resp.  $\Gamma_i^B$ ) if  $|\kappa_i| \geq 1$  (resp.  $|\kappa_i| < 1$ ). In combination with the previous conventions, this yields the self-explanatory notation  $\Gamma_i^{A+}$ ,  $\Gamma_i^{A-}$ ,  $\Gamma_i^{A0}$ ,  $\Gamma_i^{B+}$ , etc. We will call them the components of type A plus, B minus, etc.

**Condition H1.** For every component  $\Gamma_i^{A+}$  of  $\partial Q$ , we have  $D_i \subset Q$ .

**Condition H2.** There are no components of type B+.

**Corollary 3.** *Let  $Q \subset \mathbf{H}^2$  be an elementary billiard table with at least two boundary components. If  $Q$  satisfies conditions H1 and H2, then the billiard in  $Q$  is hyperbolic.*

*Outline of proof.* The assumptions of Corollary 3 imply those of the Main Theorem.

*Remark.* The purpose of the assumptions that  $\partial Q$  has at least two boundary components, and that the inclusions are proper is to exclude degenerate situations, where each  $v \in V$  is B-parabolic. For instance, this is the case if  $Q$  is a disc, or an annulus between concentric circles.

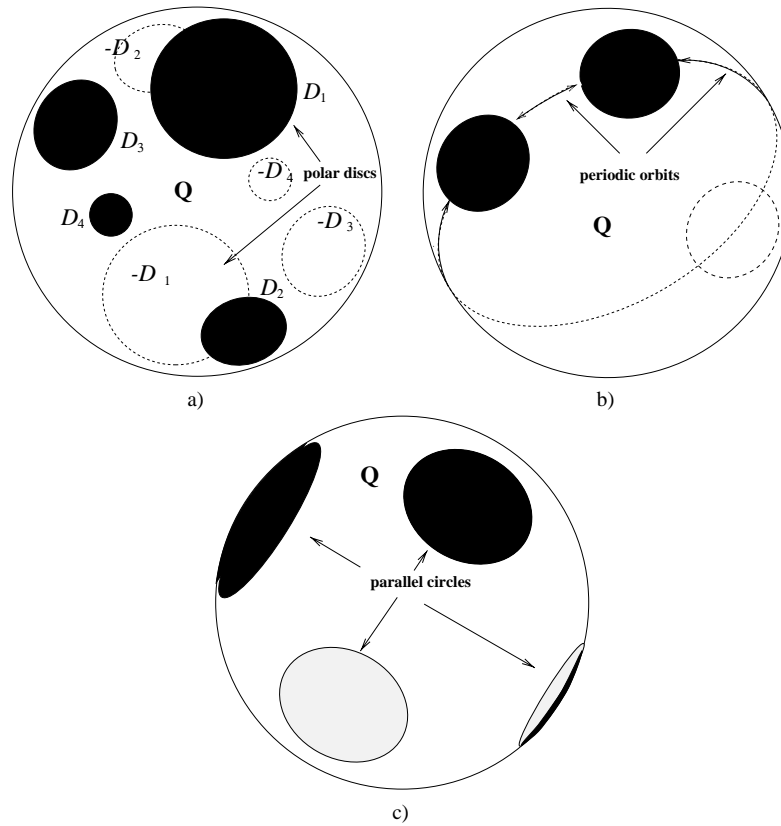


Fig. 6.

5.2. *Elementary hyperbolic billiard tables: Examples.* Using Corollaries 2 and 3, we will produce examples of elementary billiard tables with hyperbolic dynamics in  $\mathbf{S}^2$  and  $\mathbf{H}^2$ . Besides, we will give examples of elementary billiard tables that do not satisfy the assumptions of Corollaries 2 and 3, but have hyperbolic dynamics. We will prove the hyperbolicity of these billiards from the Main Theorem.

### 5.2.1. Examples on the sphere.

*Spherical Lorenz gas.* One of the first examples of hyperbolic billiards was the flat torus with a round hole, i. e., the Sinai billiard. This dynamical system is the simplest special case of the Lorenz gas, which is still actively investigated. The natural analog of the Lorenz gas on the sphere is the billiard table, obtained by removing a finite number of disjoint discs, see Fig. 6a.

Removing one disc, or a pair of parallel discs, we obtain an integrable billiard [Ve]. Let  $D_i$ ,  $1 \leq i \leq n$ , be the removed discs, so that  $Q = \mathbf{S}^2 \setminus \cup D_i$ , and  $n > 1$ . If all intersections  $D_i \cap \pm D_j$ ,  $i \neq j$ , are empty, then the billiard in  $Q$  is hyperbolic, by Corollary 2, see Fig. 6b for  $n = 2$ . For these billiards the non-intersection condition above is also necessary for hyperbolicity. If it is not satisfied, then  $Q$  has stable periodic

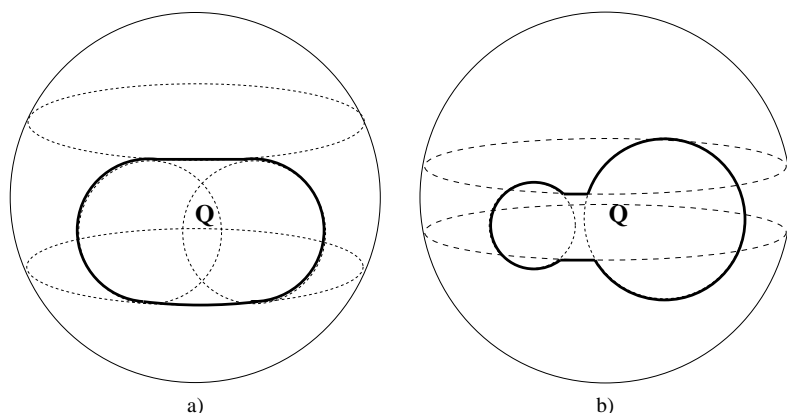


Fig. 7.

orbits of period two. They go along the large circle which connects the centers of the two removed discs.

Let now  $Q$  be obtained by removing  $m$  pairs of parallel discs,  $P_i$ ,  $1 \leq i \leq m$ , and  $n$  single discs,  $D_j$ ,  $1 \leq j \leq n$ , where  $m + n > 1$ . Consider the configuration  $(\cup_{i=1}^m \pm P_i) \cup (\cup_{j=1}^n \pm D_j)$ . Suppose that the only nonempty intersections are the trivial ones:  $P_i \cap -P_i \neq \emptyset$ , see Fig. 6c. Corollary 2 does not apply, however a direct analysis shows that almost every point of the phase space is eventually strictly B-unstable. By the Main Theorem, these billiard tables are hyperbolic.

*Pseudo-stadia.* A pseudo-stadium on  $\mathbf{S}^2$  is an elementary billiard table  $Q$ , such that  $\partial Q$  has four components: Two of them are parallel, and of negative type, and the other two are of positive type, see Fig. 7. The two positive components may have the same curvature, Fig. 7a, or different curvatures, Fig. 7b. If  $Q$  satisfies the conditions of the Main Theorem (like the pseudo-stadia in Figs. 7a, 7b), then  $Q$  is hyperbolic.

*Flowers.* Figures 8a,b,c are examples of elementary billiard tables, that belong to the class of “flowers”. Some flowers satisfy the conditions of Corollary 2, and hence, are hyperbolic. Note that the dual tables  $Q' = \mathbf{S}^2 \setminus Q$  satisfy the conditions of Corollary 2 as well (see Figs. 8a,b,c). Hence, they are also hyperbolic,

*Billiard tables with flat components.* Let a billiard table  $Q \subset \mathbf{S}^2$  (not necessarily elementary) have a flat component,  $\Gamma^0 \subset \partial Q$ . We apply to  $Q$  the method of reflections, widely used to study billiards in polygons [Ge]. In a nutshell, we associate with  $Q$  the table  $Q_1$ , which is the union of  $Q$  and its reflection about  $\Gamma^0$ , see Fig. 9a. The billiard dynamics in  $Q$  and  $Q_1$  are essentially isomorphic. (We leave it to the reader to extend the argument of [Ge] from  $\mathbf{R}^2$  to all surfaces of constant curvature.) Hence, if  $Q_1$  satisfies the conditions of the Main Theorem, then the billiard in  $Q$  is hyperbolic.

Sometimes the method of reflections yields an easy proof of hyperbolicity. Figure 9a illustrates this point: The table  $Q$  in Fig. 9a does not satisfy the conditions of Corollary 2, but  $Q_1$  does. The preceding discussion implies that  $Q$  is hyperbolic.

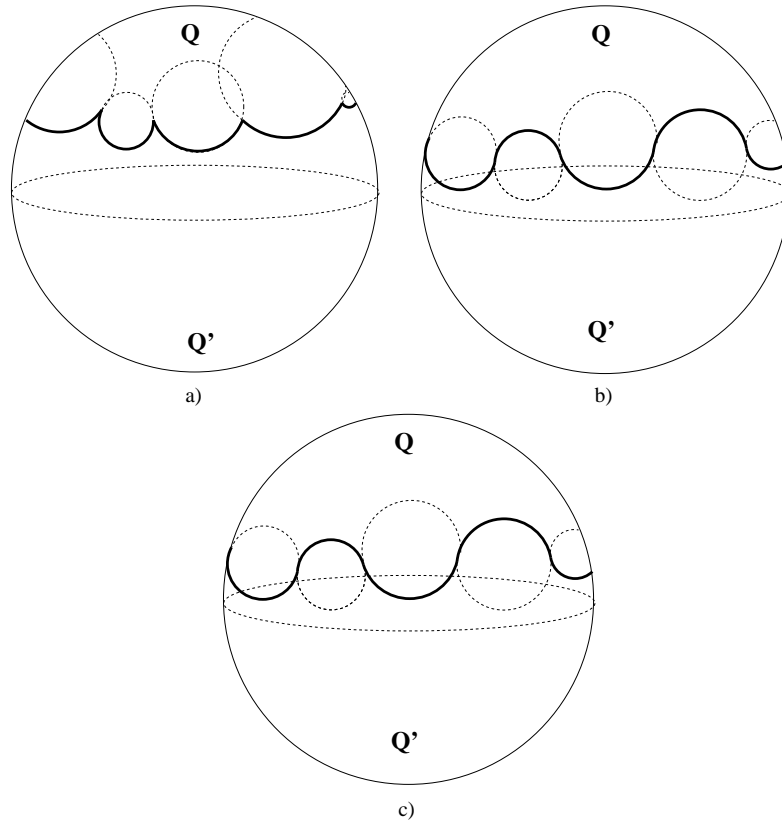


Fig. 8.

Let  $\partial Q$  have two or more flat components. Then, typically,  $Q$  does not satisfy conditions of the Main Theorem. Let  $Q_1$  be the table, obtained by “reflecting and unfolding”  $Q$  about the flat components any number of times (including infinity). Often,  $Q_1$  is not a subset of  $\mathbf{S}^2$  because of self-intersections. Then we think of  $Q_1$  as a billiard table located in a branched covering of  $\mathbf{S}^2$ . Unfolding  $Q$  infinitely many times, we can always assume that  $Q_1$  has no flat components in its boundary. However, typically, the phase space of  $Q_1$  will have points  $v$  such that in the corresponding triple  $(d_1, d_2, s)$  the distance  $s$  is near  $\pi$ . Therefore,  $Q_1$  does not satisfy the conditions of the Main Theorem. See, for example, the stadium in Fig. 9b. If  $Q_1$  is located strictly inside a hemisphere (possibly with self-intersections), then this problem does not arise. In particular, if  $Q_1$  satisfies the conditions of Corollary 2, then the billiard dynamics in  $Q$  is hyperbolic. For instance, in Figs. 9c and 9d,  $Q_1$  is inside the upper hemisphere, and satisfies the conditions of Main Theorem. Hence, the “stadia” in Figs. 9c and 9d have hyperbolic billiard dynamics.

### 5.2.2. Examples on the hyperbolic plane.

*Analogs of the Sinai billiard.* Consider the billiard tables  $Q \subset \mathbf{H}^2$  (not necessarily elementary) such that  $\partial Q$  has components of nonpositive curvature only (Fig. 10a). Let

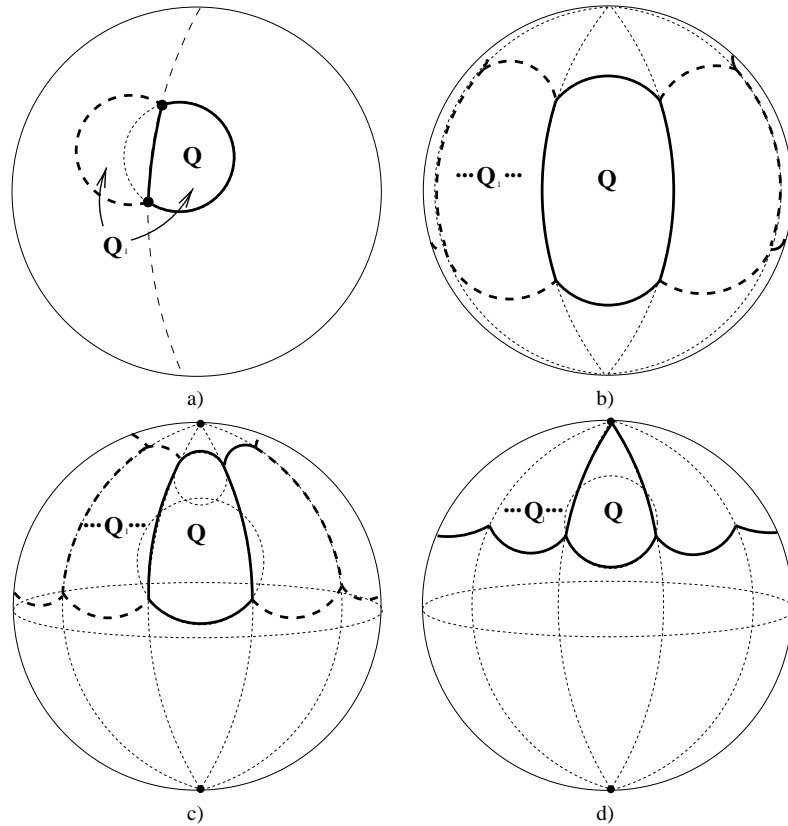


Fig. 9.

$v \in V$ . If  $v \in V^A$ , then  $d^A(v) \leq 0$ , and for  $v \in V^B$  we also have  $d^B(v) \leq 0$ . By Eq. (3.10),  $Q$  satisfies conditions of Main Theorem, hence these billiard tables have hyperbolic dynamics.

*Polygons.* Let  $Q$  be a geodesic polygon in  $\mathbf{H}^2$ , see Fig. 10b. Then  $V = V^B$ , and  $d^B(v) = 0$  for every  $v \in V$ . By Eq. (3.10b),  $Q$  satisfies the assumptions of Main Theorem. Thus, geodesic polygons in  $\mathbf{H}^2$  have hyperbolic dynamics. In fact, polygons are a special case of the Sinai billiards in  $\mathbf{H}^2$ .

*Stadia.* Let  $Q \subset \mathbf{H}^2$  be an analog of the stadium:  $\partial Q$  has four components, two of type zero, and two of positive type (Fig. 11). Let  $Q$  be any stadium, and let  $Q_1$  be the table obtained by unfolding  $Q$  about the flat components infinitely many times, see Fig. 11. If  $Q_1$  satisfies the conditions of Corollary 3, then, applying the method of reflections [Ge], extended to the hyperbolic plane, we obtain that the billiard in  $Q$  is hyperbolic. Figure 11 illustrates this point.

*Flowers.* This is another class of elementary billiard tables in  $\mathbf{H}^2$  (Fig. 12). If  $\partial Q$  satisfies conditions H1 and H2 (see Figs. 12a and 12b), then, by Corollary 3, the billiard in  $Q$  is hyperbolic.

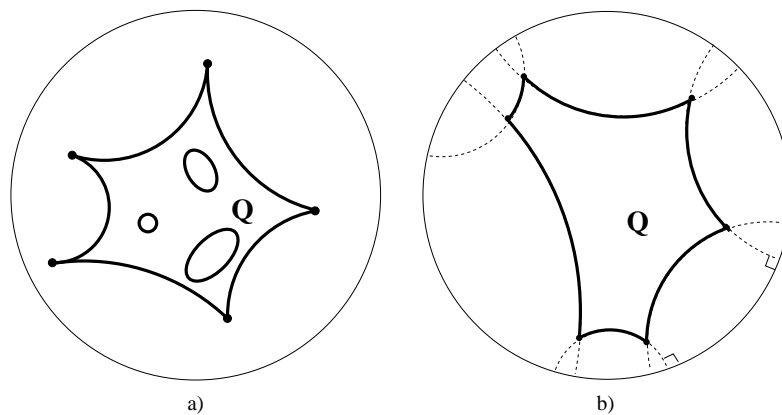


Fig. 10.

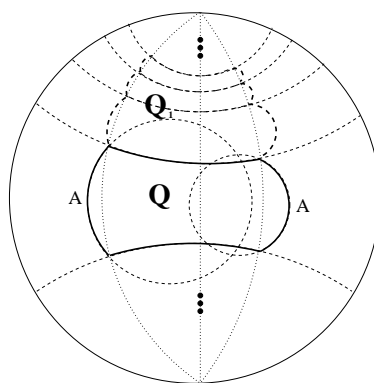


Fig. 11.

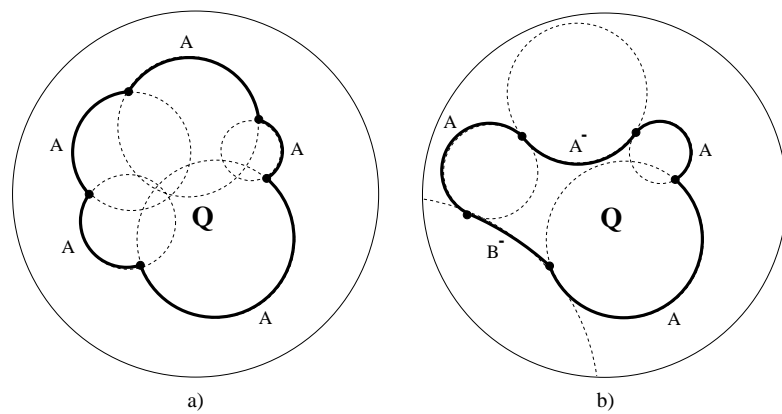


Fig. 12.

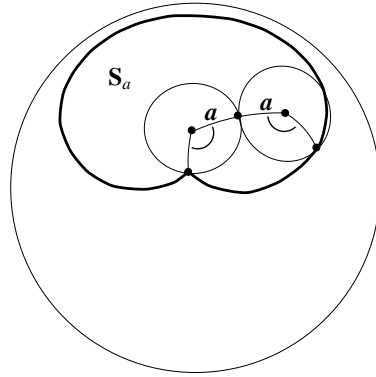


Fig. 13.

5.3. *Convex scattering for billiards on surfaces of constant curvature.* Let  $M$  be a surface of constant curvature. In this subsection we consider billiard tables in  $M$  with piecewise smooth boundary,  $\partial Q = \cup_i \gamma_i$ . We will investigate the conditions on the components  $\gamma_i$  which ensure that the billiard in  $Q$  is hyperbolic.

In [Wo2] Wojtkowski introduced the notion of convex scattering. By definition, a convex arc  $\gamma \subset \mathbf{R}^2$  is convex scattering, if it can be used as a component of a billiard table, for which the cone field defined in [Wo2] is invariant. Using the notion of convex scattering, Wojtkowski introduced three “principles of design of billiards (in  $\mathbf{R}^2$ ) with hyperbolic dynamics”, and constructed several examples of such tables.

In our notation,  $\gamma \subset \mathbf{R}^2$  is convex scattering if for any  $v \in V$ , such that the origin points of  $v$  and  $\phi(v)$  belong to  $\gamma$  the corresponding g.t.p.t.  $T(v)$  is B-unstable. Such condition is equivalent (see eq. 3.8) to the inequality  $d_1 + d_2 \leq s$  as it appears in [Wo2]. Let  $l$  be the arclength parameter on  $\gamma$ , and let  $r(l)$  be the radius of curvature. A convex arc  $\gamma$  is convex scattering if and only if  $r'' \leq 0$ , as it has been shown in [Wo2]. In what follows we generalize the notion of convex scattering to  $\mathbf{S}^2$  and  $\mathbf{H}^2$ . We call a convex curve  $\gamma \subset M$  convex scattering if for any  $v \in V$ , such that the origin points of  $v$  and  $\phi(v)$  belong to  $\gamma$  the corresponding g.t.p.t.  $T(v)$  is B-unstable. Using Proposition 3 we will obtain geometric criteria for convex scattering. Then we will extend to  $\mathbf{S}^2$  and  $\mathbf{H}^2$  Wojtkowski’s principles of design of billiards with hyperbolic dynamics.

*Convex scattering and hyperbolic billiard tables in  $\mathbf{S}^2$ .* A convex curve  $\gamma \subset \mathbf{S}^2$  is convex scattering if for every pair of the points  $\gamma_0, \gamma_1 \in \gamma$ , such that the arc of  $\gamma$  between  $\gamma_0$  and  $\gamma_1$  lies entirely on one side of the geodesic passing through  $\gamma_0$  and  $\gamma_1$ , we have

$$d_1 + d_2 \leq s \leq \pi \quad (5.1)$$

(compare with condition (3.9)). For simplicity of exposition, we will restrict our attention to piecewise convex billiard tables. The Main Theorem yields the following principles for the design of billiard tables in  $\mathbf{S}^2$  with hyperbolic dynamics:

- P1: All components of  $\partial Q$  are convex scattering.
- P2: Every component of  $\partial Q$  is sufficiently far, but not too far, from the other components.

More precisely, condition P2 means that any two consecutive bouncing points of the billiard ball satisfy Eq. (5.1), even if they belong to different components of the boundary. In particular, the interior angles between consecutive components of  $\partial Q$  are greater than  $\pi$ .

Let  $\kappa(l)$  be the geodesic curvature of  $\gamma$ . In Appendix A we will show that the differential inequality  $(\kappa^{-1})'' \leq 0$  is necessary, but, in general, not sufficient for convex scattering. However, a sufficiently short arc satisfying  $(\kappa^{-1})'' < 0$  is convex scattering.

Let  $S_a$  be the spherical analog of the cardioid. It is the curve obtained by rotating a circle of radius  $a$  on another circle of the same radius, see Fig. 13. For small  $a$  the curve  $S_a$  is well approximated by the cardioid  $R_a$ . Since  $R_a$  is (strictly) convex scattering [Wo2], the curvature,  $\kappa_a$ , satisfies the inequality  $\lim_{a \rightarrow 0} (\kappa_a^{-1})'' < 0$ . Since  $\tan r \sim a$ ,  $(\tan r)' \sim a^0$  and  $(\tan r)'' \sim a^{-1}$ , as  $a$  goes to zero, condition (A.5) is satisfied for sufficiently small  $a$ . Thus, there is a critical value,  $a_{cr}$ , such that for  $a < a_{cr}$  the curve  $S_a$  is convex scattering, and the billiard in it is hyperbolic. This approach generalizes to any curve on the sphere whose planar counterpart is strictly convex scattering.

Finally, let us mention here, that the application of the Main Theorem to the concave billiards on the sphere leads to the hyperbolicity criterion, which is closely related to the results of Vetier [Vet1, Vet2] (see also [KSS]). In fact, if concave billiard on the sphere satisfies Vetier conditions Cconditions 1.2–1.4 in [KSS]) it satisfies also the conditions of the Main Theorem.

*Convex scattering and hyperbolic billiard tables in  $\mathbf{H}^2$ .* A convex curve  $\gamma \subset \mathbf{H}^2$  is convex scattering if for each  $v \in V$ , such that the origin points of  $v$  and  $\phi(v)$  belong to  $\gamma$ , we have  $v, \phi(v) \in V^A$  and

$$d_1 + d_2 \leq s \quad (5.2)$$

(compare with Eq. (3.10)). The differential inequality  $(\kappa^{-1})'' \leq 0$  is necessary but, in general, not sufficient for Eq. (5.2), see Appendix B.  $(\kappa^{-1})'' < 0$  implies that every sufficiently short arc is convex scattering.

The Main Theorem yields the following principles for the design of billiard tables in  $\mathbf{H}^2$  with hyperbolic dynamics:

- P1: All convex components of  $\partial Q$  are convex scattering.
- P2: Every convex component of  $\partial Q$  is sufficiently far from any other component and satisfies  $\kappa(l) \geq 1$ .

More precisely, Condition P2 means that any two consecutive bouncing points of the billiard ball which belong to different components satisfy Eq. (3.10). This implies the following conditions on the angles between adjacent components of  $\partial Q$ .

- P3: Let  $\gamma', \gamma'' \subset \partial Q$  be two adjacent components. If they are both convex, then the angle between them is greater than  $\pi$ . If one of them is convex and the other is concave, then the angle is greater than or equal to  $\pi$ .

*Remark.* Comparing the principles of the design of hyperbolic billiard tables for the three types of surfaces of constant curvature, we see the same pattern. There are, however, important differences. For instance, on  $\mathbf{S}^2$ , we need to complement the requirement “to be far from each other” for the components of  $\partial Q$ , by the one “to be not too far”. The other important difference is that on  $\mathbf{S}^2$  and  $\mathbf{H}^2$  the differential inequality  $(\kappa^{-1})'' \leq 0$  is necessary, but not sufficient for convex scattering, see the appendix below.

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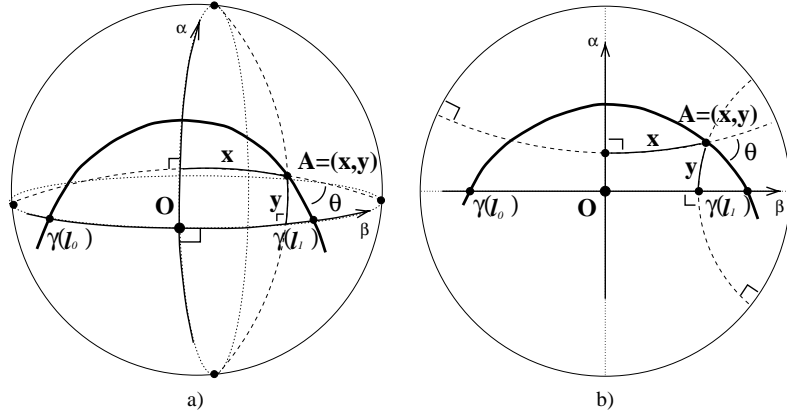


Fig. 14.

### Appendix: Geometry of Convex Scattering on $\mathbf{S}^2$ and $\mathbf{H}^2$

We will investigate when a convex arc on the sphere or the hyperbolic plane is convex scattering.

Let  $M$  be any surface of constant curvature. Let  $\gamma \subset M$  be any smooth curve, and let  $\kappa(l)$  be the geodesic curvature of  $\gamma$  (as a function of arclength). Let  $r(l)$  be the radius of the osculating circle (hypercycle if  $M = \mathbf{H}^2$ , and  $|\kappa(l)| < 1$ ). Then  $\kappa = r^{-1}$  in  $\mathbf{R}^2$ , and  $\kappa = \cot r$  for  $\mathbf{S}^2$ . On  $\mathbf{H}^2$  we will modify the definition of  $r(l)$ . There are two cases, A and B (compare with Sect. 2), where  $|\kappa(l)| > 1$  in case A, and  $|\kappa(l)| \leq 1$  in case B. We will denote by  $r^A$  and  $r^B$  respectively the radius of the osculating circle (hypercycle). In the case A (resp. B) we have  $\kappa = \coth r^A$  (resp.  $\kappa = \tanh r^B$ ). We set  $r = r^A$  and  $r = r^B + i\pi/2$  respectively. Then  $\kappa = \coth r$ .

#### A. The Sphere

Let  $\alpha$  and  $\beta$  be a pair of orthogonal oriented geodesics on  $\mathbf{S}^2$ . For  $A \in \mathbf{S}^2$  let  $x$  and  $y$  be the oriented distances from  $A$  to  $\alpha$  and  $\beta$ . Then  $(x, y)$  is a coordinate system in  $\mathbf{S}^2$ .

Let now  $\gamma(l_0)$  and  $\gamma(l_1)$  be two points on  $\gamma$  such that the arc of  $\gamma$  between  $\gamma(l_0)$  and  $\gamma(l_1)$  lies on one side of the geodesic passing through these points, see Fig. 14a. Let  $\alpha$  be that geodesic, and let  $\beta$  be such that in the parameterization  $\gamma(l) = (x(l), y(l))$ ,  $l_0 < l < l_1$ , the coordinate  $y$  takes its maximal value when  $x = 0$ , see Fig. 14a. Let  $\theta(l)$  be the angle between  $\gamma$  and the orthogonal to  $\beta$  geodesic passing through  $\gamma(l)$ . By elementary geometry:

$$\frac{dx}{dl} = \cos \theta, \quad \frac{dy}{dl} = \frac{\sin \theta}{\cos x}, \quad (\text{A.1a})$$

$$\frac{d\theta}{dl} = \sin \theta \tan x - \cot r. \quad (\text{A.1b})$$

Since  $\gamma$  is convex, the inequality  $s < \pi$  in Eq. (5.1) is satisfied for any two points of  $\gamma$ .

It remains to consider the inequality  $s \geq d_1 + d_2$ . Set  $\Delta = s - d_1 - d_2$ . Then

$$\begin{aligned}\Delta &= \int [d(\arctan(\tan r \sin \theta)) + dx] \\ &= \int dy \frac{((\tan r)' + \cos \theta \tan r (\tan x + \sin \theta \tan r)) \cos x}{1 + \tan^2 r \sin^2 \theta}.\end{aligned}$$

Since  $y(l_0) = y(l_1) = 0$ , we obtain

$$\Delta = - \int dl \left( (\tan r)'' + F(\theta(l), r(l), x(l)) \right) \frac{y \cos x}{1 + \tan^2 r \sin^2 \theta}, \quad (\text{A.2})$$

where we have set for brevity

$$\begin{aligned}F(\theta, r, x) &= \tan r \sin^2 \theta (1 - \tan^2 x) - \sin^3 \theta \tan x \tan^2 r + \sin \theta \tan x \\ &+ (\tan r)' \tan r \sin 2\theta - \frac{(\tan r)' + \cos \theta \tan r (\tan x + \sin \theta \tan r)}{1 + \tan^2 r \sin^2 \theta} \\ &\times ((\tan^2 r)' \sin^2 \theta + \tan^2 r \sin \theta \sin 2\theta \tan x - \tan r \sin 2\theta).\end{aligned}$$

Set  $L = l_1 - l_0$ . From Eq. (A.2) we have

$$\Delta = - \frac{(\tan r)''}{12 \tan r} L^3 + O(L^4). \quad (\text{A.3})$$

Thus, if the curve  $\gamma$  is convex scattering, then the condition  $(\tan r(l))'' \leq 0$  holds everywhere on  $\gamma$ . Recall that  $\tan r = \kappa^{-1}$ . If the strict inequality  $(\kappa^{-1}(l_0))'' < 0$  holds, then, by Eq. (A.3), there is  $L^{cr}$  such that the arc  $\gamma(l) : l \in [l_0, l_0 + L^{cr}]$  is convex scattering. Thus, any sufficiently short curve satisfying the condition  $(\kappa^{-1})'' < 0$  is convex scattering.

By the choice of the coordinate system we have  $|x(l)| \leq \max(r)$  for the corresponding quantities on  $\gamma(l)$ ,  $l_0 \leq l \leq l_1$ . Then, we can obtain for  $F(\theta(l), r(l), x(l))$ ,  $l_0 \leq l \leq l_1$  the estimate

$$F < (\tan r)_{\max} (1 + 3(\tan r)_{\max}^2 + 5|(\tan r)'|_{\max}), \quad (\text{A.4})$$

where  $(\tan r)_{\max}$ ,  $|(\tan r)'|_{\max}$  are the maxima of the respective quantities on  $\gamma$  between the points  $\gamma(l_0)$  and  $\gamma(l_1)$ . Equation (A.2) implies that if the inequality

$$-(\tan r)'' > (\tan r)_{\max} (1 + 3(\tan r)_{\max}^2 + 5|(\tan r)'|_{\max}), \quad (\text{A.5})$$

holds everywhere, then  $\gamma$  is convex scattering.

## B. The Hyperbolic Plane

Let  $\alpha$  and  $\beta$  be a pair of geodesics in  $\mathbf{H}^2$ , intersecting orthogonally. Just like in part A, we associate with this a coordinate system  $(x, y)$  on the hyperbolic plane. For a convex curve,  $\gamma$ , and two points,  $\gamma(l_0)$  and  $\gamma(l_1)$  of  $\gamma$ , we choose the geodesics  $\alpha$  and  $\beta$  like in part A, see Fig. 14b. Then the curvature  $\kappa = \coth r$  of  $\gamma$  satisfies

$$\frac{dx}{dl} = \cos \theta, \quad \frac{dy}{dl} = \frac{\sin \theta}{\cosh x}, \quad (\text{B.1a})$$

$$\frac{d\theta}{dl} = -\sin\theta \tanh x - \coth r, \quad (\text{B.1b})$$

where  $\theta(l)$  is the angle between the geodesic through the point  $A$ , orthogonal to  $\beta$ , and  $\gamma$ . By straightforward calculations, we obtain

$$\begin{aligned} \Delta &= s - d_1 - d_2 = \int d(\operatorname{arctanh}(\tanh r \sin\theta)) + dx \\ &= \int dy \frac{\left( (\tanh r)' - \cos\theta \tanh r (\tanh x + \sin\theta \tanh r) \right) \cosh x}{1 - \tanh^2 r \sin^2 \theta}. \end{aligned}$$

Set

$$\begin{aligned} F(\theta, r, x) &= -\tanh r \sin^2 \theta (1 + \tanh^2 x) - \sin^3 \theta \tanh x \tanh^2 r - \sin\theta \tanh x \\ &\quad - (\tanh r)' \tanh r \sin 2\theta + \frac{(\tanh r)' - \cos\theta \tanh r (\tanh x + \sin\theta \tanh r)}{1 - \tanh^2 r \sin^2 \theta} \\ &\quad \times \left( (\tanh^2 r)' \sin^2 \theta + \tanh^2 r \sin\theta \sin 2\theta \tanh x + \tanh r \sin 2\theta \right). \end{aligned}$$

Then, since  $y(l_0) = y(l_1) = 0$ , we have

$$\Delta = - \int dl \left( (\tanh r)'' + F(\theta(l), r(l), x(l)) \right) \frac{y \cosh x}{1 - \tanh^2 r \sin^2 \theta}. \quad (\text{B.2})$$

Let  $L = l_1 - l_0$ . By Eq. (B.2), we obtain

$$\Delta(L) = - \frac{(\tanh r)''}{12 \tanh r} L^3 + O(L^4). \quad (\text{B.3})$$

This leads to the necessary condition for convex scattering curve on the hyperbolic plane:  $(\kappa^{-1})'' \leq 0$ . Just like in part A, Eq. (B.3) implies that any sufficiently short arc satisfying  $(\kappa^{-1})'' < 0$  is convex scattering.

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