

Semiclassical quantization of chaotic billiards: a scattering theory approach

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Received 9 September 1991, in final form 4 February 1992

Accepted by M V Berry

Abstract. We derive a semiclassical secular equation which applies to quantized (compact) billiards of any shape. Our approach is based on the fact that the billiard boundary defines two dual problems: the ‘inside problem’ of the bounded dynamics, and the ‘outside problem’ which can be looked upon as a scattering from the boundary as an obstacle. This duality exists both on the classical and quantum mechanical levels, and is therefore very useful in deriving a semiclassical quantization rule. We obtain a semiclassical secular equation which is based on classical input from a finite number of classical periodic orbits. We compare our result to secular equations which were recently derived by other means, and provide some numerical data which illustrate our method when applied to the quantization of the Sinai billiard.

PACS numbers: 0365, 0380, 0545

1. Introduction

One of the most challenging problems in the field of ‘quantum chaos’ is to find a consistent and convergent method by which the energy spectra of classically chaotic Hamiltonian systems could be calculated using input which is derived exclusively from classical dynamics. Such a procedure is referred to as ‘quantization’, and it should extend the Bohr–Sommerfeld (or more generally the EBK) quantization rules, which apply to integrable systems.

Gutzwiller’s trace formula for the density of states was for a long time the only theoretical approach to the quantization problem [1]. It served as a starting point for various interesting developments [2, 3], and found applications in many specific problems [4, 5]. However, the trace formula does not converge absolutely for real energies [6, 7]. Thus, the manipulations of the trace formula (such as, for example, the calculation of spectral correlation functions) are strictly forbidden if one takes the trace formula literally as a numerical expression. In spite of this intrinsic difficulty, these applications were very fruitful and explained important features in the observed spectra of classically chaotic systems [8].

After several years of frustrated efforts to cope with the basic problematics of the trace formula, a new theoretical approach seems to have emerged. This uses

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the semiclassical trace formula as a formal starting point, gets around the divergence problems in a systematic way, and provides a well defined semiclassical quantization rule [3, 9–11]. It is based on classical information derived from a limited number of periodic orbits, and its applicability was successfully tested in some numerical examples. These new developments proceeded along two independent lines. The common features of these theories is that they yield a semiclassical expression for a secular function—a function whose zeros are the spectrum of the quantum Hamiltonian. The mere shift of the focus from the spectral density to the secular function brings an enormous advantage—one replaces the discussion of a function whose poles are on the real axis, by a function which is regular (and even analytic) in a strip about the real axis. The two approaches previously mentioned differ in the way by which the secular functions are derived. We shall briefly review them and thus provide the background for the present derivation of the semiclassical secular function.

The first approach starts by considering the (properly regularized) spectral determinant. This function can be expressed semiclassically as a sum of contributions from classical periodic orbits. In particular, it can be written as a product of a phase factor $\exp(i\pi\bar{N}(E))$ and an infinite sum. $\bar{N}(E)$ is the mean spectral counting function. Each term in the infinite sum is due to a ‘pseudo-orbit’—a combination of periodic orbits whose total length defines the period of the pseudo-orbit [9, 12–14]. This sum is the semiclassical dynamical ζ function. Unfortunately, it does not converge absolutely on the real energy axis. It does, however, converge for complex energies with a sufficiently large imaginary part, and hence it represents the spectral determinant in this domain of the complex energy plane. The analytic continuation of the spectral determinant towards the real energy axis was the main obstacle which plagued the semiclassical theory till very recently. Lately, Keating proposed a procedure by which an approximate expression for the spectral determinant on the real energy can be obtained [15, 16]. This procedure is similar in many ways to the Riemann–Siegel approximation to the Riemann ζ function on the critical line (hence the name ‘Riemann–Siegel lookalike’). Keating [15] made use of the fact that the spectral determinant must be real valued on the real energy axis. This is a consequence of the Hermitian character of the Hamiltonian. One then proceeds by considering the semiclassical expressions for the spectral function and its complex conjugate as formal expressions and requiring that they be equal for real E . Some further formal manipulations (which will also be used in section 3) result in an approximate expression for the spectral determinant which is real on the real E axis, and which consists of a finite number of terms, with composite periods which are shorter than $\frac{1}{2}h$ times the mean level density. This result had earlier been conjectured in [9], by analogy with the actual Riemann–Siegel formula for the Riemann zeta function. Some numerical checks of this formula were carried out recently, with very encouraging results [12, 17].

The second approach, proposed by Bogomolny [11] starts out by expressing the eigenvalue problem as a boundary integral equation. A semiclassical expression for the secular equation is derived and expressed in the form

$$Z_R(E) = \det(I - T(E)) \quad (1.1)$$

where $T(E)$ is a matrix of a finite dimension $\Lambda(E)$. $T(E)$ is shown to be semiclassically unitary (that is when matrix multiplication is carried out by the stationary phase method). $T(E)$ is interpreted as the semiclassical approximation of the quantum version of a Poincaré mapping in the phase space of the bounded motion. The choice

of the particular phase-space Poincaré section is, within some bounds, not too crucial, and its phase area $A(E)$ (for systems in two dimensions) determines the dimension of $T(E)$ via $\Lambda(E) = A(E)/\hbar$. The functional form of the secular equation (1.1) is very important, and Bogomolny took advantage of it to derive a large number of interesting results. Since in our approach the secular functions are also expressed in the same way, we shall defer the discussion of the properties of such secular equations to section 3 in the present paper. At this stage it will suffice to mention, that this secular equation can be expressed in terms of closed orbits of the classical Poincaré map of period which does not exceed $\Lambda(E)/2$. Thus, a finite and well defined semiclassical secular equation is obtained directly, without having to rely on formal manipulations of non-convergent series.

In the present paper we would like to show that secular equations of the type (1.1) follow very naturally in the quantization of billiards by making use of the duality between the ‘inside’ and ‘outside’ dynamics—the billiard boundary defines both the bounded motion inside and the scattering dynamics outside. Thus, the quantization condition for the ‘inside’ problem can be expressed in the form

$$Z_S(E) = \det(I - S(E)) \quad (1.2)$$

where $S(E)$ is defined in terms of the scattering matrix $S(E)$ for the ‘outside’ problem. In section 2 of the present paper we shall derive two expressions of the type (1.2), which pertain to different scattering situations. We shall make use of the fact that the S matrix is the quantum analogue of the Poincaré scattering mapping [18] (PSM) to derive the semiclassical approximation of (1.2). For billiards, however, the PSM can be related to a Poincaré mapping of the bounded dynamics, due to the duality of the classical ‘inside’ and ‘outside’ problems. Thus, we shall provide an alternative derivation of the secular equations obtained originally by Bogomolny.

Section 3 will be dedicated to the consequences of the special form of the secular equation of the type (1.1, 1.2). In it we develop the semiclassical approximation to the secular function, and show that the secular function can be expressed as a sum over ‘pseudo-orbits’ (which we prefer to call ‘composite orbits’) of the PSM, whose composite periods are bounded. The relation between this bound and bounds on the period of orbits of the billiard is also discussed.

Section 4 deals with another aspect of the ‘inside’ and ‘outside’ duality. Random matrix theory is known to be intimately connected with quantum systems whose classical analogue is chaotic. In particular it governs the fluctuations of the energy spectrum of chaotic billiards [19] as well as the fluctuations in the eigenphase spectrum of S matrix corresponding to chaotic scattering [20]. The present developments allow us to investigate this phenomenon in greater depth.

In section 5 we demonstrate the applicability of our method by comparing some of its predictions with numerical calculations on the Sinai billiard. Section 6 will present a summary and a discussion of various issues related to the present work.

2. Derivation of the secular equation

In the present chapter we shall derive secular equations for billiards in the plane. We shall start by discussing non-concave billiards and obtain the secular equation in terms of the S matrix for the scattering in the plane. If the billiards have more complicated shapes, one has to define appropriate scattering problems to obtain the secular equation. This will be done in the second part of the present chapter.

2.1. Non-concave billiards

For the discussion of this class of billiards it is advantageous to choose an arbitrary point inside the billiard as the reference point (origin), and use a polar representation of phase space in terms of the coordinates (r, ϕ) and the conjugate momenta (k_r, l) (note that classical actions are expressed in units of \hbar).

A general scattering wavefunction which vanishes on the billiard boundary, takes the asymptotic form

$$\Psi^{(l)}(\mathbf{r}) = F_l^{(-)}(\mathbf{r}) + \sum_m \tilde{S}_{l,m} F_m^{(+)}(\mathbf{r}) \quad (2.1)$$

where $F_l^{(\pm)}(\mathbf{r})$ are the incoming and outgoing cylindrical waves with angular momentum l

$$F_l^{(\pm)}(\mathbf{r}) = [J_l(kr) \pm iN_l(kr)] \exp(il\phi).$$

\tilde{S} (the matrix whose elements are $\tilde{S}_{l,m}$) is defined as the scattering matrix.

If the incident wavefunction is constructed from plane waves, the form (2.1) converges absolutely outside the smallest circle, centred at the origin, which completely encloses the billiard. (This fact governs the optimal choice of the origin — the origin should be chosen to be at the centre of the smallest covering circle. For convex billiards this point is always inside the billiard.) Any possible divergence of (2.1) is due to the presence of the Newmann functions, which diverge as $r \rightarrow 0$. If a linear combination of the functions $F^{(l)}(\mathbf{r})$ could be found, so that the coefficients of the Newmann functions will all vanish, the resulting series will converge over the entire plane. It is then also a valid representation of a solution to the Schrödinger equation *inside* the billiard, which also satisfies the boundary conditions on the billiard boundaries. In other words, it yields an eigenfunction of the billiard. The coefficients of the Newmann functions can be made to vanish only if

$$Z(E) = \det(I - \tilde{S}(E)) = 0. \quad (2.2)$$

We therefore take (2.2) to be a secular equation for the ‘inside’ quantized billiard.

The matrix \tilde{S} is of infinite dimension. The fact that the billiard does not extend to infinity implies that we can find two angular momentum bounds, $L_+ > L_-$ such that for any l which is sufficiently far from the interval (L_-, L_+) , and for any m , we have

$$\tilde{S}_{l,m} \approx \delta_{l,m} \quad (2.3)$$

This means that the value of the determinant (2.2) is always negligible, and the secular equation is not very useful in its present form. A way to discard the contribution from the physically uninteresting domain is naturally suggested in the semiclassical domain, since there the approximate form (2.3) becomes progressively more exact. Thus we can truncate \tilde{S} to the domain

$$L_- \leq l \quad m \leq L_+ \quad (2.4)$$

and replace (2.2) by a semiclassical secular equation

$$Z_{sc}(E) = \det(I - S(E)) = 0 \quad (2.5)$$

where S is a matrix of dimension $\Lambda = L_+ - L_-$ which is obtained by restricting \tilde{S} to the domain (2.4), and which is semiclassically unitary, due to (2.3).

To continue with the semiclassical analysis and point out the duality between the 'inside' and the 'outside' problems, we should digress and discuss the classical dynamics inside and outside the billiard.

It is possible to reduce the billiard dynamics to a mapping in a two-dimensional phase space. Usually, one chooses the arc length and the projection of the velocity on the tangent at the point of incidence as the coordinates. Here, it is advantageous to introduce another representation in which one coordinate is the impact parameter b which gives the directed distance of the straight segment from the origin: $b = (\mathbf{v} \times \mathbf{r})_z / |\mathbf{v}|$, where \mathbf{v} is the velocity vector. The impact parameter is related to the angular momentum via $l = bk$ and therefore we shall use l as the action variable. The conjugate coordinate is the angle θ which is the direction of the velocity vector relative to a fixed axis in space. For non-concave billiards, the boundary limits the range of θ and b (or l) values in the following way: For any value of θ there exist two parallel tangents to the billiard pointing in the θ direction. Denote by $b_-(\theta)$ and $b_+(\theta)$ their distance from the origin. The sign \pm refers to the side (left or right) of the origin of the point of tangency. Any classical trajectory inside the billiard is a straight chord in the direction θ at a distance $b = l/k$ away from the origin. Upon impact on the boundary the trajectory is specularly reflected. It then coincides with a new chord, which can again be characterized by its (θ, l) coordinates. Thus, the inside dynamics can be completely specified as a mapping of points on the (θ, l) cylindrical phase space. The functions $l_{\pm}(\theta) = kb_{\pm}(\theta)$ enclose a strip σ on the cylinder, defining the bounded phase space domain where the 'inside' dynamics takes place.

A scattering trajectory outside the billiard can also be parameterized in terms of the parameters (θ, l) . The scattering process can be considered as a mapping of the incoming (θ, l) to the outgoing (θ', l') . The functions $l_{\pm}(\theta)$ appears also in the description of the dual scattering problem. For any given θ there exist two limiting values of the angular momentum, $l_-(\theta)$ and $l_+(\theta)$ such that no reflection occurs for scattering trajectories which aim at the billiard at an angle θ with $l < l_-(\theta)$ or $l > l_+(\theta)$. Thus, the strip σ appears also as the domain where the billiard affects the scattering. For (θ, l) values outside σ , there will be no deflection, and the trajectories continue as straight lines (the extrema of σ along the angular momentum axis give the semiclassical values of L_{\pm} discussed above). The 'inside'-'outside' duality is completed by noting that any external reflection $(\theta, l) \rightarrow (\theta', l')$ corresponds to an internal trajectory which impinges on the billiard at the same point with opposite incoming and outgoing directions and the same values of the angular momenta but for a change of sign (see figure 1).

We now introduce the Poincaré scattering mapping (PSM) [21, 22]. In the present case it is constructed in the following way. A point $(\theta, l) \in \sigma$ corresponds to a straight line trajectory which is incident on the billiard at an angle θ and an impact parameter $b = l/k$ (see figure 1). The trajectory scatters into the direction θ' with an angular momentum l' . A mapping of σ onto itself mediated by scattering trajectories could be defined once a re-injection mechanism which turns outgoing trajectories into incoming ones could be found. This is done in the present context by considering (θ', l') as defining an incoming trajectory, which moves on the same line as the previous out-going trajectory, but impinges on the scatterer on a diametrically opposite point. Successive iterations of the PSM are conjugate to reflections inside the boundary, and the correspondence between the scattering dynamics and the internal dynamics is established.

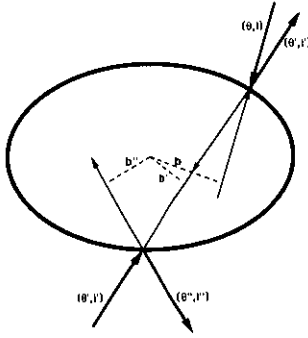


Figure 1. An example of the inner–outer duality for non-convex billiards. The internal trajectory $(\theta, l) \rightarrow (\theta', l') \rightarrow (\theta'', l'')$ is exactly mirrored by the equivalent external PSM trajectory. b , b' and b'' are the corresponding impact parameters.

The generating function which induces the mapping is given in terms of the reduced action

$$\Phi = - \left(\int r dk_r + \int \phi dl \right). \quad (2.6)$$

One can easily show that Φ is a sum of two terms, where each corresponds to (minus) the integral of the radial momentum from the point of least approach on the incoming or outgoing branch of the trajectory to the point of impact on the billiard. (In other words Φ is the sum of the WKB phase shifts incurred in the incoming and outgoing branches of the trajectory. The phase shift is negative due to the repulsive scattering potential.)

The quantization of classical canonical transformations of the type described earlier is well known [23]. Thus an explicit semiclassical expressions for the matrix elements of S in terms of classical scattering trajectories and the corresponding actions can be readily given. We now observe that the action $-\Phi$ serves also as the generating function for the internal mapping. Thus, the S matrix which appears in the secular equation can be interpreted in a dual way—either as the quantized version of the PSM (S in equation (2.3)), or as the quantized version of the classical mapping for the ‘inside’ problem, and both interpretations are semiclassically equivalent. In other words, Bogomolny’s T matrix (equation (1.1)) and the semiclassical S matrix (equation (1.2)) stand for the same semiclassical object.

It should be noted, however, that this equivalence is strictly justified for non-concave billiards only, where the domain of the ‘inside’ and the ‘outside’ Poincaré mappings coincide. More complicated billiard shapes may also, in principle, be treated similarly, but then not all ‘outside’ trajectories have valid ‘inside’ counterparts. These are the so called ‘ghost’ trajectories [24]. In the next section we shall show how the ideas formulated here can be extended to billiards with more complicated shapes, in a different manner.

2.2. Billiards of any shape

We now develop a secular equation for billiards of a more general shape. For the sake of simplicity we limit ourselves to billiards with the following properties:

(i) The billiard has to accommodate at least one chord Γ which is normal to the two tangents at its two ends (see figure 2). Γ divides the billiard into two parts to which we refer in the following as the 'left' (L for short) and 'right' (R) parts.

(ii) The tangents at the end points of Γ do not intersect L in their continuation towards R, and do not intersect R in their continuation towards L.

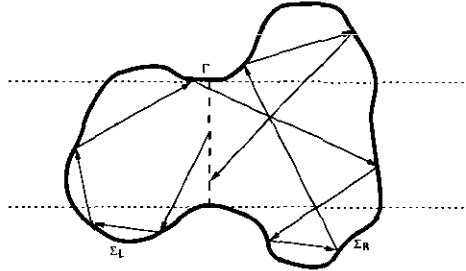


Figure 2. The division of a complex billiard into left and right scattering problems. The dividing chord Γ is shown as a heavy broken line, and the two tangents as thin broken lines. Also shown is a classical orbit contributing to the semiclassical approximation to S .

If there exist more than one chord with these properties, one can choose any chord for the purpose of the discussion and proof. There exists, however, an optimal chord, and the guide-lines for its choice will be explained.

It is possible to generalize the treatment that will be given in this section to billiards which do not possess such a chord Γ , by allowing curved chords. This generalization does not, however, bring new physical insight, and so will not be pursued further here.

We now turn the 'inside' billiard problem into two independent scattering systems. The L system is constructed by considering the opened billiard defined by the boundary of L (to be denoted by Σ_L) and the two parallel tangents which go towards R. The two tangents form a channel which matches smoothly to Σ_L . Imposing specular reflection conditions classically, or Dirichlet boundary conditions for the wave problem, the L system is completely specified. The R system is defined in exactly the same manner. Note that the R (L) part of the billiard is completely discarded when we consider the L (R) scattering system. To each of the two systems we attach a coordinate system where the chord Γ coincides with the y axis. The positive x axis coincides with the lower channel wall for the L system. The negative x axis for the R system coincides with the lower wall of its channel.

A scattering solution for the L (R) system must vanish on the boundary Σ_L (Σ_R), and asymptotically, for large values of x ($-x$),

$$\Psi_n^{(L,R)}(x, y) \rightarrow \sum_{l=1}^{\Lambda} \sqrt{(k_n/k_l)} \left[\delta_{n,l} e^{\mp i k_l x} + S_{n,l}^{(L,R)} e^{\pm i k_l x} \right] \phi_l(y). \quad (2.7)$$

The scattering solutions describe a wave propagating down (up) the positive (negative) axes in the mode n . After scattering from the L (R) boundary, the Λ modes which are energetically allowed to propagate in the channel are scattered with amplitudes given by the corresponding scattering matrix $S^L(S^R)$. The number of propagating modes Λ is the integer part of $[Dk/\pi]$. D is the length of the chord Γ and k is the wavenumber.

The functions $\phi_l(y)$ are a complete set of transverse mode functions with eigenenergies $(l\pi/D)^2$. The corresponding longitudinal wavenumbers are k_l , for $1 \leq l \leq \Lambda$.

At this point we would like to use the scattering functions (2.7) to construct an eigenfunction of the original billiard, namely, a function which vanishes on the entire boundary $\Sigma = \Sigma_L + \Sigma_R$. A possible candidate will be a function which is defined piece-wise as a linear combination of the Ψ_l^L inside the L region ($x \leq 0$) and a linear combination of the Ψ_l^R inside the R region ($0 \leq x$). Such a function will automatically satisfy the boundary conditions on Σ . The requirement that the function and its normal derivative be continuous at Γ , provides the quantization condition.

The main approximation which we are now about to introduce, consists of taking the form (2.7) as a proper representation of the exact wavefunctions not only in the limit $|x| \rightarrow \infty$ but also in the vicinity of Γ . That is, we neglect the contribution of the evanescent modes to the scattering wavefunctions at the interface. Such an approximation is justified since, classically, evanescent modes correspond to propagation with imaginary momenta. In the semiclassical approximation such trajectories give contributions which are exponentially small, and are therefore discarded. This approximation is expected to be at its worst at threshold k values, namely, when the longitudinal wavenumber is very small, so that an evanescent wave takes a large distance to decay. The reason for our particular choice of the chord Γ is now becoming clear. In this way the matching of the channels to the L and R billiards is smooth, and under such conditions, the role played by evanescent modes is minimized. We could make the same construction but with channels which are not smoothly matched. Then, the evanescent modes play a significant role at the interface, which may limit the applicability of the semiclassical approximation.

For smooth matching at $x = 0$ we must require that there exist coefficients $a_l^{L,R}$, $l = 1, \dots, \Lambda$, which solve the linear equations

$$\sum_{n=1}^{\Lambda} a_n^L (k_n/k_l)^{1/2} [\delta_{n,l} + S_{n,l}^L] = \sum_{n=1}^{\Lambda} a_n^R (k_n/k_l)^{1/2} [\delta_{n,l} + S_{n,l}^R]. \quad (2.8a)$$

and

$$ik_l \sum_{n=1}^{\Lambda} a_n^L (k_n/k_l)^{1/2} [-\delta_{n,l} + S_{n,l}^L] = ik_l \sum_{n=1}^{\Lambda} a_n^R (k_n/k_l)^{1/2} [\delta_{n,l} - S_{n,l}^R]. \quad (2.8b)$$

This is a homogeneous set of 2Λ linear equations, which has a solution only if

$$Z_{sc}(E) = \det \begin{pmatrix} I & -S^R \\ S^L & I \end{pmatrix} = 0. \quad (2.9)$$

This is the secular equation for the problem. Simple algebra brings the secular function to its final form

$$Z_{sc}(E) = \det(I - S^L S^R) = \det(I - S(E)) \quad (2.10)$$

where $S = S^L S^R$ is a unitary matrix of dimension Λ . Thus, we have shown that the secular equation can also be written in the form (2.3) for the more general shapes of the billiard to be quantized.

This treatment can also be regarded as a variant of the 'inside'-'outside' duality, discussed in the previous section. If we connect a pipe to the L region, for example, the infinite phase space is divided into the 'inside', which is the (compact) interior of the region, and the 'outside', which is the pipe. The 'inside'-'outside' duality then consists of obtaining information concerning the 'inside' region through an S matrix, which is measured from outside. The different properties of the S matrices obtained by this and the former methods is a result of the different structure of the asymptotic domain (the 'outside') for both cases.

In the semiclassical approximation each of the matrices $S^{L,R}$ can be written in terms of classical orbits. The matrix element $S_{n,m}^L$, say, gets its contributions from all the classical orbits which enter L at Γ with a momentum component $k_y = n\pi/D$ and after being reflected arbitrarily many times *inside* L leave L through $\bar{\Gamma}$ with $k_y = m\pi/D$. The matrix elements of S^R are constructed in a similar way. To form the product S we perform the summation over the intermediate channel indices by using the stationary phase approximation. The resulting semiclassical expression for $S_{n,l}$ is constructed from trajectories which leave Γ towards L with $k_y = n\pi/D$. They are reflected from Σ_L till finally they escape L through Γ to R . There, they collide with the walls till they finally emerge through Γ with momentum $k_y = l\pi/D$ (see figure 2). These are the trajectories which affect the Poincaré mapping of the section (k_y, y) at $x = 0$, and they are the trajectories in terms of which one would construct the semiclassical approximation to the Poincaré mapping. Again, the secular equation by which we quantize the billiard is derived in terms of scattering theory, but can be semiclassically interpreted in terms of a quantized Poincaré mapping.

We would like to point out some important differences between the two classes of systems which were discussed in the preceding sections. As previously mentioned, and will be elaborated in the next chapter, the semiclassical evaluation of the secular equation requires classical input from periodic trajectories of the corresponding mapping with period up to $\Lambda/2$. For the case of convex billiards, these are orbits which strike the billiard boundaries exactly $\Lambda/2$ times. For the more general class of billiards these are trajectories which pass through the section Γ (from R to L) $\Lambda/2$ times, but within each of the parts of the original billiard, they may be arbitrarily long, and may strike the walls arbitrarily many times. As a matter of fact, if within either L or R there exists a strange set of trapped orbits, the classical scattering system displays chaotic scattering, of the type discussed e.g. in [25]. In other words, in the case of complicated billiards Λ does not determine the physical length of the trajectories which contribute to the secular function, nor does it determine their number. This is in striking contrast with the case of convex billiards. We would like to point out that in Bogomolny's approach one assumes that the quantized system displays properties which we attach here to the non-concave billiards. In this sense our derivation of the secular equation is not just an alternative derivation which starts from a different point of view, but also an extension to a much wider class of systems.

We would like to emphasize once again the main approximation which underlies our derivation of the secular equation. In both problems, we neglected the waves which spill out of the classical domain. In the case of the convex billiard these were the matrix elements of \tilde{S} which were not included in S . Strictly speaking S is not unitary, but it approaches a unitary limit as the system becomes more classical. In the alternative derivation, the matrix S is perfectly unitary, but the procedure by which we obtained the secular equation can only be justified by ignoring the evanescent modes. For k values far from threshold, this approximation seems to yield quite accurate results. A

numerical example will be discussed in section 5.

3. The secular equation

In the present chapter we shall discuss the properties of the secular equations which were derived in section 2. For most of the discussion we shall treat the two cases discussed above on the same footing. The cases in which the classical scattering system displays chaotic scattering (see end of the preceding chapter), will be specially dealt with.

We shall use the following notation. The eigenphases of the S matrix will be denoted by $\theta_l(E)$ for $l = 1, \dots, \Lambda$. (Λ itself is a function of E , but unless otherwise specified, we shall consider energy intervals between neighbouring thresholds, where it is a constant). Of particular importance will be the derivatives with respect to E of the eigenphases θ_l . We shall denote them by τ_l . But for a factor \hbar/Λ they are the eigenvalues of the time-delay matrix

$$T(E) = \frac{\hbar}{i\Lambda} S^\dagger S' \quad (3.1)$$

where the prime stands for differentiation with respect to E . $\tau(E) = \text{Tr}(T(E))$ is the Wigner delay time [26, 27].

The first result which we would like to quote, relates the spectral density of the 'inside' billiard

$$d(E) = \sum_n \delta(E - E_n) \quad (3.2)$$

to the trace of the time delay matrix,

$$d_R(E) = \frac{1}{2\pi} \frac{\partial}{\partial E} \sum_{l=1}^{\Lambda} \theta_l(E) = \frac{1}{2\pi} \sum_{l=1}^{\Lambda} \tau_l(E) = \frac{\Lambda}{2\pi\hbar} \tau(E) \quad (3.3)$$

which is a quantity characterizing the scattering. We will show that the difference between these two quantities is given by the logarithmic derivative of the secular function. To this end we rewrite the secular function in the form,

$$Z_{\text{sc}}(E) = \exp\left(\frac{i}{2} \sum_{l=1}^{\Lambda} \theta_l(E) - \frac{i}{2} \Lambda \pi\right) 2^\Lambda \prod_{l=1}^{\Lambda} \sin \frac{\theta_l(E)}{2}. \quad (3.4)$$

Since the last product on the right-hand side of (3.4) is real on the real E axis, the imaginary part of its logarithmic derivative is a sum of delta functions. Therefore, we can write the level density of the billiard as

$$d(E) = -\frac{1}{\pi} \text{Im} \lim_{\epsilon \rightarrow 0} \frac{Z'_{\text{sc}}(E + i\epsilon)}{Z_{\text{sc}}(E + i\epsilon)} + d_R(E). \quad (3.5)$$

It is easy to show that in situations where resonances dominate the scattering process, $d_R(E)$ is a superposition of normalized Lorentzians, positioned at the resonance energies, and having a width which is equal to the distance of the pole from the real

axis. This situation prevails in the case of chaotic scattering and in [28] we called this quantity the resonance density. The main difference between $d(E)$ and $d_R(E)$ is that the former is a sum of δ spikes, whereas the latter is a continuous function. One can show that

- (i) the two densities have a common smooth average $\langle d(E) \rangle$; and
- (ii) the function $d_R(E)$ converges to $\langle d(E) \rangle$ in the semiclassical limit.

To prove the first statement we evaluate (3.5) at a distance ϵ from the real E axis. $d(E + i\epsilon)$ and $d_R(E + i\epsilon)$ so evaluated are the Lorentzian-weighted averages of their respective values on the real E axis. However, the S matrix (and therefore also Z'_{sc}) decays exponentially to zero as $\epsilon \rightarrow +\infty$, and so from (3.5) we see that $\langle d_R(E) \rangle \rightarrow \langle d(E) \rangle$.

The second statement follows from the observation that in the semiclassical limit, and for chaotic scattering, the S matrix is dominated by overlapping resonances, whose width is much larger than the average level spacing. The time delay is therefore smooth over an interval which contains many levels. Indeed, this argument was used in [28] to derive Weyl's formula. We complete the argument in section 4, where we show that $|d_R(E) - \langle d(E) \rangle|$ approaches zero for $E \rightarrow \infty$, when the underlying dynamics is chaotic. In section 5 we shall provide some numerical evidence to illustrate these results, and in section 6 we shall show how this discussion applies in a simple problem which can be solved analytically.

It is convenient to consider the secular function at a given energy as the characteristic polynomial of the matrix S ,

$$\det(I - S) = \det(\lambda I - S)|_{\lambda=1} = \left[\sum_{l=0}^{\Lambda} f_l \lambda^l \right]_{\lambda=1} = \sum_{l=0}^{\Lambda} f_l. \tag{3.6}$$

In general, f_l , expressed in terms of the eigenvalues of S , is the homogeneous symmetric polynomial of degree $\Lambda - l$ which can be constructed from the Λ eigenvalues of S . The unitarity of S implies an important symmetry among the f_l , namely

$$\exp \left[-i \frac{1}{2} \sum_{i=1}^{\Lambda} \theta_i(E) \right] f_l = (-1)^\Lambda \exp \left[i \frac{1}{2} \sum_{i=1}^{\Lambda} \theta_i(E) \right] f_{\Lambda-l}^*. \tag{3.7}$$

At this point we introduce the resonance counting function

$$N_R(E) = \frac{1}{2\pi} \sum_{i=1}^{\Lambda} \theta_i(E) - \frac{\Lambda}{2} = \int_0^E dE' d_R(E'). \tag{3.8}$$

Using (3.3) we can rewrite (3.7) in the form

$$\exp [-i\pi N_R(E)] f_l = \exp [i\pi N_R(E)] f_{\Lambda-l}^*. \tag{3.9}$$

We shall show below that this set of equations plays the role of the 'functional equations' which appear in the dynamical ζ function approach [9]. Note that the phase factors in (3.9) involve the resonance counting function $N_R(E)$. Recalling the relations between $d(E)$, $d_R(E)$ and $\langle d(E) \rangle$, we find that in the semiclassical limit $N_R(E) \rightarrow \langle N(E) \rangle$. This fact brings the analogy between (3.9) and the ζ functional equation into sharper focus.

Utilizing relation (3.9) one may now write for the secular function

$$\det(I - S) = \sum_{l=0}^{\Lambda} f_l = \exp [i\pi N_R(E)] \operatorname{Re} \left\{ \exp[-i\pi N_R(E)] \sum_{l=l_0}^{\Lambda} (1 + \epsilon(\Lambda, l_0)) f_l \right\}. \quad (3.10)$$

Here $l_0 = \frac{1}{2}(\Lambda + 1)$ for odd Λ . If Λ is even, $l_0 = \frac{1}{2}\Lambda + 1$. $\epsilon(\Lambda, l_0) = 1$ except when Λ is even and $l = l_0$, in which case $\epsilon(\Lambda, l_0) = 0$. The advantage of writing the secular function in the form (3.10) is that the zeros of the resulting function manifestly lie either on the real energy axis, or are placed symmetrically about it. This form is therefore amenable to approximation, without fear of compromising this important property.

The f_l coefficients can be expressed in terms of $\operatorname{Tr}(S^n)$ with $n = 1, \dots, \Lambda$. This follows from Newton's identities

$$\operatorname{Tr}(S^k) + f_{\Lambda-1} \operatorname{Tr}(S^{k-1}) + \dots + f_{\Lambda-k+1} \operatorname{Tr}(S) + k f_{\Lambda-k} = 0 \quad (3.11)$$

which are valid for $1 \leq k \leq \Lambda$. This is a set of Λ linear equations for the f_l , in triangular form, with $\operatorname{Tr}(S^n)$ as coefficients. These equations can therefore be solved using a simple iterative algorithm. The result takes the form

$$f_{\Lambda-l} = \sum_{p,r} \phi_{(p,r)}^{(l)} (\operatorname{Tr}(S^{p_1}))^{r_1} \dots (\operatorname{Tr}(S^{p_n}))^{r_n} \quad (3.12)$$

where p and r are vectors of non-negative integers, and the sum is over all vectors which satisfy

$$\sum_{i=1}^{N(p)} p_i r_i = p \cdot r = l \quad p_1 > p_2 > \dots > p_{N(p)} > 0 \quad p_{i > N(p)} = 0. \quad (3.13)$$

We denote by $N(p)$ the number of non-zero p_i s. The coefficients $\phi_{(p,r)}^{(l)}$ are combinatorial factors, resulting from the triangular nature of (3.11). As an illustration, we quote the results for the first few f_l :

$$f_{\Lambda-1} = -\operatorname{Tr}(S) \quad (3.14a)$$

$$f_{\Lambda-2} = \frac{1}{2} \{ (\operatorname{Tr}(S))^2 - \operatorname{Tr}(S^2) \} \quad (3.14b)$$

$$f_{\Lambda-3} = -\frac{1}{6} \{ (\operatorname{Tr}(S))^3 + 2 \operatorname{Tr}(S^3) - 3 \operatorname{Tr}(S) \operatorname{Tr}(S^2) \}. \quad (3.14c)$$

The semiclassical approximation for the secular equation is now obtained by expressing $\operatorname{Tr}(S^n)$ in terms of periodic orbits of the PSM. It was shown previously [18] that

$$\operatorname{Tr}(S^n) \approx \sum_{\substack{\alpha \\ \eta_\alpha \rho_\alpha = n}} C_{\alpha, \rho_\alpha} \exp(i\rho_\alpha \Phi_\alpha). \quad (3.15)$$

The summation is over all primitive periodic orbits α of period η_α which, if repeated ρ_α times, will perform a period of length n . The action Φ_α is assumed to include the Maslov index as well as the phase factor, which gets an addition of π for every reflection

from the billiard boundary (for Dirichlet boundary conditions). The amplitudes C_{α, ρ_α} are given by

$$C_{\alpha, \rho_\alpha} = \frac{\eta_\alpha}{\sqrt{|\det(I - M_\alpha^{\rho_\alpha})|}} \tag{3.16}$$

where M_α is the monodromy matrix calculated for the primitive periodic orbit α .

Substituting the semiclassical expression (3.15) in (3.16) we get

$$f_{\Lambda-l} \approx \sum_{(\mathbf{p}, \mathbf{r})} \phi_{(\mathbf{p}, \mathbf{r})}^{(l)} \prod_{j=1}^{N(\mathbf{p})} \left[\sum_{\eta_\alpha \rho_\alpha = p_j} C_{\alpha, \rho_\alpha} \exp(i \rho_\alpha \Phi_\alpha) \right]^{r_j} \tag{3.17}$$

where $N(\mathbf{p})$ is the length of the vector \mathbf{p} (and \mathbf{r}). This complicated sum can be viewed as a sum over objects which Berry and Keating called ‘pseudo-orbits’ and which we prefer to name *composite orbits*: For each composite period l one considers all *groups* of primitive periodic orbits with periods η_α and repetitions ρ_α such that

$$l = \sum_{\alpha} r_{\alpha} \eta_{\alpha} \rho_{\alpha}. \tag{3.18}$$

The non-negative integers r_{α} and the corresponding $p_{\alpha} = \eta_{\alpha} \rho_{\alpha}$ identify the partition labels (\mathbf{p}, \mathbf{r}) . The composite action is obtained by summing the actions of the periodic orbits together with their multiplicities, and the composite amplitude is obtained by taking the product of the amplitudes (including also the combinatorial factor $\phi_{(\mathbf{p}, \mathbf{r})}^{(l)}$). Thus, expression (3.17) for the coefficients $f_{\Lambda-l}$ can be written as a sum over composite orbits

$$f_{\Lambda-l} = \sum_s C_s^{(l)} \exp(i \Phi_s^{(l)}) \tag{3.19}$$

where the index s labels the composite orbits.

The semiclassical expressions for f_l and $f_{\Lambda-l}$ do not manifestly obey the symmetry relation (3.19)—they are constructed from composite trajectories of different length and the corresponding semiclassical series (3.19) are not guaranteed to satisfy the proper symmetries. These observations emphasize the basic and most severe limitations of the semiclassical approximation—it does not manifestly preserve the basic requirement that the S matrix must be unitary for real energies. However, the secular function, in the form of (3.10), *does* obey the symmetry relation (3.9). Inserting (3.19) into (3.10) we get a semiclassical expression for the secular function,

$$\begin{aligned} \det(I - S(E)) &= \exp [i\pi N_R(E)] 2 \operatorname{Re} \left\{ \exp[-i\pi N_R(E)] \sum_{l=l_0}^{\Lambda} f_{\Lambda-l} \right\} \\ &= \exp [i\pi N_R(E)] 2 \operatorname{Re} \left\{ \exp[-i\pi N_R(E)] \sum_{l=0}^{\Lambda-l_0} \sum_s C_s^{(l)} \exp(i \Phi_s^{(l)}) \right\} \end{aligned} \tag{3.20}$$

(note that we have written (3.20) for the case that Λ is odd to facilitate the notation). (3.20) has the following properties:

(i) It manifestly possesses zeros which are either on the real E axis, or placed symmetrically about it. The latter could correspond to the case where there is a 'near miss' of the real axis by the secular function, caused by the approximation.

(ii) It is constructed from composite orbits with period less than $\Lambda/2$. This limits the periods of the original periodic orbits by the same bound. Notice that so far the period is counted by the number of applications of the PSM and not by the physical length of the period. We return to this point in the following paragraphs.

(iii) The f_i are expressed as cumulants in terms of the various powers of $\text{Tr}(S^n)$. Bogomolny has shown that because of the shadowing property of periodic orbits [29] there are effective cancellations between various terms in the semiclassical expression, and most of the contributions come from a subset of the possible composite orbits.

(iv) The structure of (3.20) is similar to that of the Riemann–Siegel lookalike secular equation derived by Keating by a completely different method [15].

So far we discussed the secular equation from the scattering point of view. We shall now make use of the 'inside'–'outside' connection. It enables us to interpret the above expressions in terms of the 'inside' dynamics, so that the contributing orbits are the periodic trajectories inside the billiard.

For convex billiards, when using the 'inside'–'outside' duality developed in section 2.1, iterating the PSM consists of following a classical orbit from one collision with the boundary to the next. The physical length of an orbit between two reflections from the boundary is bounded by the maximal diameter of the billiard, and therefore the maximal physical length of a periodic orbit is proportional to the number of reflections from the boundary (the period of the orbit in the PSM). We reach the important conclusion that for convex billiards the secular equation is expressed in terms of a *finite* number of periodic orbits whose physical length is *bounded*. We can estimate the bound on the period in the following way. The dimension of the S matrix is $\Lambda = Bk$, where B is the largest diameter in the billiard. Periodic orbits which scatter $\frac{1}{2}\Lambda$ times from the wall will travel trajectories of mean length $\lambda = \frac{1}{2}\Lambda \langle c \rangle = \frac{1}{2}B \langle c \rangle k \approx \frac{1}{2}Ak$, where A is the area of the billiard and $\langle c \rangle$ is the length of a mean chord in the billiard. The time it takes to traverse such orbits is $\langle t \rangle = \lambda/v$. Using Weyl's formula for the mean level spacing in the billiard we get $\langle t \rangle \approx \frac{1}{2}h \langle d(E) \rangle$, which is similar to the time bound for periodic orbits in Keating's Riemann–Siegel lookalike secular equation.

The situation is much more complex when one wants to apply the same ideas for the general case, discussed in section 2.2. Now, there may appear periodic orbits which are localized in space and never intersect the section Γ . If in particular the scattering into the L or the R parts of the billiard is chaotic, there is no relation between the physical length of the orbits and the period of the orbit of the PSM to which they contribute. The sums on periodic orbits (3.15) extend over an infinity of orbits, and their convergence is not *a priori* guaranteed. Thus the expression (3.20) preserves the formal structure imposed by unitarity, but might be meaningless as a numerical representation of the secular function. We shall show now that the difficulty which arises because of the presence of arbitrary long trajectories can be removed. Here also, a proper Riemann–Siegel lookalike secular function can be written in terms of periodic orbits, whose physical length is bounded by the same bound as was given for the case of convex billiards.

Following the ideas proposed by Keating [15], we regard the semiclassical approximation (3.19) to the f_i as formal expressions and impose on them the symmetry relations (3.9). We get

$$\exp(-i\pi N_R(k)) \sum_s C_s^{(l)} \exp(i\Phi_s^{(l)}(k)) = \exp(i\pi N_R(k)) \sum_{s'} \left(C_{s'}^{(\Lambda-l)}\right)^* \exp(-i\Phi_{s'}^{(\Lambda-l)}(k)) \tag{3.21}$$

where for convenience we use the wavenumber k instead of the energy label. We choose a mean value \bar{k} and expand the classical actions as well as $N_R(k)$ about it. We denote the average value of the k derivative of $2\pi N_R(k)$ by λ_{av} . Note that λ_{av} is just Λ times the average physical length of a trajectory between successive passages through the Poincaré section at Γ . We can use (3.3) to write an explicit expression for λ_{av}

$$\lambda_{av} = 2\pi k \frac{\hbar^2}{m} \left\langle \frac{dN_R(E)}{dE} \right\rangle = \Lambda v \langle \tau(E) \rangle \tag{3.22}$$

where v is the velocity. Using (3.3) one can also write (3.22) in the form

$$\lambda_{av} = \hbar v \langle d_R(E) \rangle = \hbar v \langle d(E) \rangle. \tag{3.23}$$

The k derivative of the reduced actions $\Phi_s^{(l)}$ give $\lambda_s^{(l)}$ —the composite length of the composite periodic orbit. We proceed now with formal manipulations and multiply both sides of (3.21) by $\exp(ikx)$ (with an as yet arbitrary x) and integrate term by term both sides of (3.21) over an interval of size a about \bar{k} . We choose a to be large on the scale of the mean level spacing, but smaller than the distance between successive thresholds (this is easily achieved for large \bar{k}). Using the linearized form of the phase factors in (3.21), and assuming that the amplitudes $C_s^{(l)}$ do not vary rapidly with k we get

$$\begin{aligned} \exp(-i\pi N_R(\bar{k})) \sum_s C_s^{(l)} \exp(i\Phi_s^{(l)}(\bar{k})) \delta_a(\lambda_s^{(l)} + x - \lambda_{av}/2) \\ = \exp(i\pi N_R(\bar{k})) \sum_{s'} \left(C_{s'}^{(\Lambda-l)}\right)^* \exp(-i\Phi_{s'}^{(\Lambda-l)}(\bar{k})) \delta_a(-\lambda_{s'}^{(\Lambda-l)} + x + \lambda_{av}/2) \end{aligned} \tag{3.24}$$

where $\delta_a(x)$ is a smoothed δ function with a width of a^{-1} .

(3.24) is now integrated with respect to x over the interval $[-\infty, 0]$, where we approximate the smoothed δ functions by their sharp counterparts (this approximation is discussed and justified by Keating [15]). This gives us

$$\begin{aligned} \exp(-i\pi N_R(\bar{k})) \sum_{\lambda_s^{(l)} \geq \frac{1}{2}\lambda_{av}} C_s^{(l)} \exp(i\Phi_s^{(l)}(\bar{k})) \\ = \exp(i\pi N_R(\bar{k})) \sum_{\lambda_{s'}^{(\Lambda-l)} \leq \frac{1}{2}\lambda_{av}} \left(C_{s'}^{(\Lambda-l)}\right)^* \exp(-i\Phi_{s'}^{(\Lambda-l)}(\bar{k})). \end{aligned} \tag{3.25a}$$

Note that the summation on the left-hand side of this expression is over composite orbits with length larger than $\frac{1}{2}\lambda_{av}$, whereas the summation on the right-hand side is over composite orbits of length smaller than $\frac{1}{2}\lambda_{av}$. Note also that this bound is identical to the one derived earlier for convex billiards.

Another relation is obtained by replacing x by $-x$ in (3.24) and integrating:

$$\begin{aligned} \exp(-i\pi N_R(\tilde{k})) \sum_{\lambda_s^{(l)} \leq \frac{1}{2}\lambda_{av}} C_s^{(l)} \exp(i\Phi_s^{(l)}(\tilde{k})) \\ = \exp(i\pi N_R(\tilde{k})) \sum_{\lambda_s^{(\Lambda-l)} \geq \frac{1}{2}\lambda_{av}} \left(C_s^{(\Lambda-l)}\right)^* \exp(-i\Phi_s^{(\Lambda-l)}(\tilde{k})). \end{aligned} \tag{3.25b}$$

We have thus obtained the important result that the imposition of the functional equation symmetries on the semiclassical approximation result in the relations (3.25), which enable us to express the contribution from all the long orbits to $f_{\Lambda-l}$, in terms of the contribution from the short orbits to f_l^* , and *vice versa*. The dividing line between ‘long’ and ‘short’ orbits is given in terms of $\frac{1}{2}\lambda_{av}$. We can now partition expression (3.19) for $f_{\Lambda-l}$ into its short orbits and long orbits contributions. For the long orbit sum we can use (3.25) and write an expression for $f_{\Lambda-l}$ which is based exclusively on short (or composite) orbits:

$$f_{\Lambda-l} = \sum_{\lambda_s^{(l)} \leq \frac{1}{2}\lambda_{av}} C_s^{(l)} \exp(i\Phi_s^{(l)}(\tilde{k})) + \exp(2i\pi N_R(\tilde{k})) \sum_{\lambda_s^{(\Lambda-l)} \leq \frac{1}{2}\lambda_{av}} \left(C_s^{(\Lambda-l)}\right)^* \exp(-i\Phi_s^{(\Lambda-l)}(\tilde{k})). \tag{3.26}$$

This expression is now to be substituted in (3.10) and it provides the desired expression for the secular equation which is based on periodic orbits of a *finite period*, and of a *finite physical length*. The resulting expression is now completely equivalent to the Riemann–Siegel lookalike secular equation derived by Keating. It has a structural advantage in keeping the contributions from groups of composite trajectories with the same composite period l in the form of cumulants. This might be exploited by using the techniques developed by Cvitanovic and Eckhardt [29]. We are studying this possibility now.

As a last point, we would like to give an alternative interpretation of the bound $\frac{1}{2}\Lambda$ on the contributing periodic orbits to the secular equation. We have shown that this limit does not restrict the physical length of the orbits when chaotic scattering prevails. However, it imposes an interesting bound on the average length of the contributing orbits. Suppose the length of the segment dividing the billiard is taken to be D , and suppose also for the sake of simplicity that the division is into two approximately equivalent pieces. The distribution of physical lengths of trajectories between two consecutive passes through the segment is [30]

$$P(t) \sim \gamma \exp(-\gamma t). \tag{3.27}$$

The mean length of a trajectory between two consecutive passes is therefore γ^{-1} , which by simple geometrical considerations can be shown to be $\langle t \rangle \approx Dv/\pi A$, where v is the velocity and A is the area of the L (or R) half of the billiard. The dimension of the S matrix can be approximated by

$$\Lambda \approx \frac{kD}{\pi} \approx \frac{mvD}{\pi\hbar}. \tag{3.28}$$

Inserting the expression for $\langle t \rangle$ into (3.28) we get

$$\frac{1}{2}\Lambda \langle t \rangle \approx \frac{1}{2}h \langle d(E) \rangle. \tag{3.29}$$

where the average level density is $\langle d(E) \rangle = mA/2\pi\hbar^2$ by virtue of Weyl’s relation. This again is reminiscent of Keating’s time bound, only this time applied to the *average* length of trajectories used.

4. The relation between eigenphases and eigenenergies

The last important issue that we would like to discuss is the relation between the distribution of the eigenphases of the S matrix on the unit circle, and the distribution of the energies obtained as solutions of the semiclassical secular equation $\det(I - S(E)) = 0$. It is well known that the distribution of eigenenergies of chaotic billiards may be characterized by one of the Hermitian random matrix ensembles [31, 32]. Also, the distribution of eigenphases of the S matrix of a scattering system can be described by means of one of the circular random matrix ensembles, if the PSM of the scattering is chaotic [18, 22] (this class of systems comprises all chaotic scattering systems, plus systems which would become chaotic if closed). We present our results using a formulation introduced by Bogomolny [33] in a different context. It is more transparent and general than our original formalism, and we are indebted to Dr Bogomolny for allowing us to use it here.

A zero of the secular equation occurs each time that any of the eigenphases of the S matrix $\theta_l(E)$ equals an integer multiple of 2π . Hence, the spectral density $d(E)$ can be expressed as

$$d(E) = \sum_l \delta(E - E_l) = \sum_{l=1}^{\Lambda} \tau_l \delta_p(\theta_l(E)) \quad (4.1)$$

where $\tau_l = \theta'_l(E)$, and the prime denotes differentiation with respect to E ; $\delta_p(x)$ is the periodic δ function. Let us consider the spectral density at an interval of size Δ about a mean energy E_0 . Δ is taken to be large on the scale of the mean level separation, which is given by Weyl's formula to be $2\pi\hbar^2/mA$, where A is the area enclosed by the billiard. Then, for $\epsilon = (E - E_0) < \Delta$

$$d(\epsilon) \approx \sum_{l=1}^{\Lambda} \tau_l \delta_p(\theta_l(E_0) + \epsilon\tau_l). \quad (4.2)$$

If we were able to show that the fluctuations in the distribution of the τ_l about their mean τ are small, we could replace all the τ_l in (4.2) by τ . Defining $\theta = 2\pi - \epsilon\tau$ we would then get

$$d(\epsilon) \approx \tau \sum_{l=1}^{\Lambda} \delta_p(\theta_l(E_0) - \theta) \quad (4.3)$$

which would have established the correspondence between the energy spectral density (the left-hand side of (4.3)) and the eigenphase spectral density (the right-hand side of (4.3)). Consider now a billiard, whose 'inside' dynamics is chaotic. The corresponding PSM for the scattering problem is also chaotic, and therefore both the energy spectrum and the S matrix eigenphase spectrum can be described in terms of a random matrix theory (RMT) which correspond to the same symmetry (the fact that the Hamiltonian is a Hermitian operator while the S matrix is unitary is irrelevant if Λ is larger than the number of energy levels considered). This correspondence suggests that the conditions under which (4.3) is valid are met for chaotic systems. In other words, we conjecture that for scattering systems which have a chaotic PSM, the distribution of the τ_l is peaked about their mean, and the distribution becomes more narrow as Λ increases. We now proceed to show why such behaviour is indeed plausible.

It was shown in [25] that the correlation length of the elements of the S matrix, with respect to change of energy, is approximately given by the inverse mean time delay:

$$\gamma \approx \langle \tau(E) \rangle^{-1} \approx \frac{\Lambda}{2\pi} \langle d(E) \rangle^{-1}. \tag{4.4}$$

The correlation length is therefore, in the semiclassical limit, much larger than the mean level spacing. The correlation length of $\tau(E)$ will be similar, since it is composed of Lorentzians whose width is γ (this is similar to the argument that explains Ericson fluctuations [34]). We now make the central assumption, which is that the correlation length of the *individual* τ_i is also of the order of γ . This is a reasonable assumption, since $\tau(E)$ is the average of the τ_i , and a different result would entail strong correlations between them.

Consider the matrix $\hat{S}(E) = \exp(-iE/\gamma)S(E)$. This is in effect the original S matrix, normalized so that its average time delay vanishes. Since matrices belonging to the circular orthogonal ensemble (COE) exhibit eigenphase repulsion, eigenphases do not cross, and so $\langle \hat{\tau}_i \rangle$ must also vanish. We now make two approximations:

- (i) The two matrices $S_1 = \hat{S}(E)$ and $S_2 = \hat{S}(E + \gamma)$ are statistically independent.
- (ii) The $\hat{\tau}_i$ are constant over the range $(E, E + \gamma)$.

We arrange the eigenphases $\theta_i^{(1,2)}$ of $S_{1,2}$ in ascending order over the interval $[0, 2\pi)$. Since $\langle \hat{\tau}_i \rangle = 0$, and since eigenphases cannot cross, we can assume that the n th eigenphase of S_1 evolves into the n th eigenphase of S_2 . According to assumption 2, the $\hat{\tau}_i$ must therefore be

$$\hat{\tau}_i \approx \frac{\theta_i^{(2)} - \theta_i^{(1)}}{\gamma} = \left(\theta_i^{(2)} - \theta_i^{(1)} \right) \langle \tau_i \rangle. \tag{4.5}$$

The variance of τ_i has in (4.5) been transformed into the variance of $\delta\theta = \theta_i^{(2)} - \theta_i^{(1)}$. It is obvious that as Λ grows the distribution of $\delta\theta$ is more and more sharply peaked around 0. The simplest model, which assumes a ‘neutral gas’ behaviour for the eigenphases (no correlations, except for the fact that eigenphases are not allowed to cross) gives the intuitive result $\text{Var}(\delta\theta) \propto \Lambda^{-1}$. A more realistic model is one that takes into consideration the level repulsion, as exhibited by the COE.

In order to estimate $\text{Var}(\delta\theta)$ for the COE we approximate it by

$$\text{Var}(\delta\theta) \approx \tilde{V} = \frac{1}{\Lambda} \sum_{n=1}^{\Lambda} \left\langle \left(\theta_n - \left[\left(n - \frac{1}{2} \right) \frac{2\pi}{\Lambda} \right] \right)^2 \right\rangle_{\text{COE}} \tag{4.6}$$

where $\langle \cdot \rangle_{\text{COE}}$ signifies the COE ensemble average. Note that the quantity in the square brackets is approximately $\langle \theta_n \rangle_{\text{COE}}$. The COE average (4.6) was worked out analytically by B Dietz [35], and we are grateful for her help. The result is

$$\tilde{V} = \frac{4\pi^2}{3\Lambda} - \frac{\pi^2}{3\Lambda^2} - \frac{8}{\Lambda} \sum_{n=1}^{\Lambda} \frac{1}{n^2} + \frac{8}{\Lambda^2} \sum_{n=1}^{\Lambda} \frac{1}{n} - \frac{8}{\Lambda^2} \sum_{n=1}^{\Lambda/2} \frac{1}{(2n-1)^2} \tag{4.7}$$

where for the sake of simplicity Λ has been taken to be even. We can further approximate (4.7) by using the relations

$$\sum_{n=1}^{\Lambda} \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{1}{\Lambda} + O(\Lambda^{-2}) \tag{4.8a}$$

$$\sum_{n=1}^{\Lambda/2} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} + O(\Lambda^{-1}) \quad (4.8b)$$

$$\sum_{n=1}^{\Lambda} \frac{1}{n} = \gamma + \log \Lambda + O(\Lambda^{-1}) \quad (4.8c)$$

where $\gamma = 0.577215\dots$ is Euler's constant. Inserting (4.8) into (4.7) yields

$$\tilde{V} \rightarrow \frac{8}{\Lambda^2} \left[1 + \gamma - \frac{\pi^2}{6} + \log \Lambda \right] \approx \frac{8 \log \Lambda}{\Lambda^2} \quad \text{for } \Lambda \rightarrow \infty. \quad (4.9)$$

In the next chapter we present some numerical tests, which substantiate the applicability of the COE variance (4.7) as a good approximation for $\text{Var}(\tau_i)$.

In section 3 we stated that, in the semiclassical limit, $d_R(E) \rightarrow \langle d(E) \rangle$. This can now easily be shown from (4.9). $d_R(E)$ can be written as

$$d_R(E) = \langle d(E) \rangle + \frac{1}{2\pi} \sum_{l=1}^{\Lambda} \hat{\tau}_l(E). \quad (4.10)$$

Assuming the $\hat{\tau}_i$ to be uncorrelated, we then get approximately

$$\text{Var} \{d_R(E) - \langle d(E) \rangle\} \propto \frac{\log \Lambda}{\Lambda} \langle d(E) \rangle \rightarrow 0 \quad \text{for } \Lambda \rightarrow \infty. \quad (4.11)$$

5. Numerical simulations

In this chapter we bring the results of numerical simulations concerning the application of the secular function to the spectra of chaotic billiards. The model we chose to work with is the Sinai billiard. This billiard has a high degree of symmetry, which as a consequence is also present in the solutions. For that reason we use the symmetry reduced version of the billiard only (see inset of figure 5(a)).

The Sinai billiard is an example of a concave billiard, and so the relevant secular equation formulation is the one given in section 2.2. The natural choice of the section Γ is the 'neck' of the billiard. The L part of the billiard is then the whole of the billiard, minus the wall closing the neck. We call this scattering system the 'elbow', for obvious reasons. The R part consists of the wall only. Note that the area of the elbow is the area of the whole billiard, while the area of the R part is zero. It is easy to see that the S matrix of the R part is $S^R = -I$, and therefore the secular function equation takes the form

$$Z_{sc}(E) = \det(I + S^L(E)) = 0 \quad (5.1)$$

where $S^L(E)$ is the scattering matrix of the elbow. The superscript L will be dropped in the following, as we will always be referring to $S^L(E)$ as the S matrix. For the sake of convenience we will use the k dependence, instead of the E dependence, of S. The dimension of S is the integer part of $[kD/\pi]$, where D is the width of the neck.

The secular function Z_{sc} is an approximation, since the contribution of the evanescent modes is neglected. It is possible to write an exact secular function in a form which is very similar to (5.1):

$$Z(k) = \det(I + \tilde{S}(k)) \quad (5.2)$$

where $\tilde{S}(k)$ is now the scattering matrix for *all* the modes, both open and evanescent, and so is of infinite dimension. $S(k)$ can be extracted from $\tilde{S}(k)$ by truncating $\tilde{S}(k)$ to its first Λ rows and columns. Note that $\tilde{S}(k)$ is not unitary—on the contrary, its determinant vanishes, since its elements decrease exponentially as the mode number increases beyond Λ . Consequently,

$$I + \tilde{S}(k) \approx \begin{pmatrix} I + S(k) & 0 \\ 0 & I \end{pmatrix} \quad (5.3)$$

and therefore $Z(k) \approx Z_{sc}(k)$.

Note the difference between the case of a convex billiard, in which we define the ‘inner’–‘outer’ problem duality (see section 2.1), and the more general case described above. In the former, the infinite dimensional \tilde{S} is unitary, while its semiclassical truncation $S(k)$ is not. In the latter, the infinite dimensional \tilde{S} is singular, while its semiclassical truncation $S(k)$ is exactly unitary.

5.1. The accuracy of the semiclassical eigenvalues

The first result we show concerns the shift of the positions of the zeros of $Z_{sc}(k) = \det(I - S(k))$ relative to the zeros of $Z(k) = \det(I - \tilde{S}(k))$ (see above). The latter secular function is exact, while the former is an approximation, resulting from the neglect of the contributions of the evanescent modes.

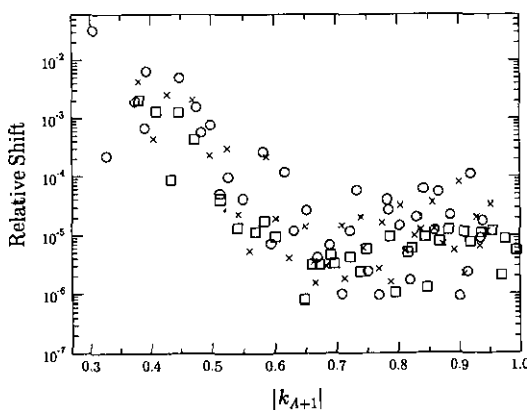


Figure 3. The shifts in the positions of the zeros of Z_{sc} relative to the true eigenenergy positions, for $\Lambda = 1$ (squares), $\Lambda = 2$ (crosses) and $\Lambda = 3$ (circles). The errors are in units of the mean level spacing, and the abscissa is taken to be the magnitude of the wavenumber of the lowest evanescent mode, $|k_{\Lambda+1}|$, measured in units of $(\pi/D)\sqrt{2\Lambda+1}$.

In figure 3 we show the errors in the positions of the zeros of Z_{sc} relative to the exact eigenvalues. The errors are measured in units of the mean level spacing, while the abscissa is proportional to $|k_{\Lambda+1}|$, which is the magnitude of the longitudinal wavenumber of the lowest evanescent mode

$$k_{\Lambda+1} = i \left[\left(\frac{(\Lambda + 1)D}{\pi} \right)^2 - k^2 \right]^{1/2}. \tag{5.4}$$

This representation was chosen since the neglect of the lowest evanescent mode is expected to contribute most to the error (see later). We normalize $|k_{\Lambda+1}|$ by $(\pi/D)\sqrt{2\Lambda + 1}$, so that we can display the errors for different Λ on the range $[0, 1]$. The errors are given for S matrices of dimension $\Lambda = 1, 2, 3$. We see that the errors are very small, averaging about 10^{-5} – 10^{-4} of the mean level spacing over most of the energy range. The errors start increasing exponentially as $|k_{\Lambda+1}|$ drops below ~ 0.6 . A rough extrapolation of this trend shows that the errors will be of the order of the mean level spacing for $|k_{\Lambda+1}| = 0$, which is the point at which the lowest evanescent mode becomes propagating.

We will now evaluate the dependence of the errors on the magnitude of the first evanescent mode. Let us take \tilde{S} to include the first evanescent mode, in addition to the open modes. \tilde{S} is therefore

$$\tilde{S} = \begin{pmatrix} & \mu_1 \\ (S) & \vdots \\ & \mu_\Lambda \\ \hat{\mu}_1 \dots \hat{\mu}_\Lambda & \mu_{\Lambda+1} \end{pmatrix} = \begin{pmatrix} (S) & \boldsymbol{\mu} \\ \hat{\boldsymbol{\mu}}^T & \mu_{\Lambda+1} \end{pmatrix}. \tag{5.5}$$

Time reversal symmetry implies here $\hat{\boldsymbol{\mu}} = i\boldsymbol{\mu}$. We will use this in the following.

Let U be the matrix of eigenvectors of S . Since S is unitary and symmetric, U is an orthogonal (real) matrix. We can therefore write

$$U^T S(k) U = \Theta \tag{5.6}$$

where Θ is the diagonal matrix of eigenvalues $\Theta_{nn} = \exp(i\theta_n)$. Defining

$$\tilde{U} = \begin{pmatrix} (U) & 0 \\ 0 & 1 \end{pmatrix} \tag{5.7}$$

we get

$$Z(k) = \det(I + \tilde{S}(k)) = \det(1 + \tilde{U}^T \tilde{S}(k) \tilde{U}) = \begin{vmatrix} (\Theta + I) & \boldsymbol{\eta} \\ i\boldsymbol{\eta}^T & \mu_{\Lambda+1} + 1 \end{vmatrix} \tag{5.8}$$

where $\boldsymbol{\eta} = U^T \boldsymbol{\mu}$. We also define $Z^{(n)}(k)$ to be

$$Z^{(n)}(k) = \begin{vmatrix} (\Theta^{(n)} + I) & \boldsymbol{\eta}^{(n)} \\ i(\boldsymbol{\eta}^{(n)})^T & \mu_{\Lambda+1} + 1 \end{vmatrix} \tag{5.9}$$

where $\Theta^{(n)}$ is a diagonal matrix whose diagonal elements are $\exp(i\theta_n) \dots \exp(i\theta_\Lambda)$, and $\boldsymbol{\eta}^{(n)}$ is a vector composed of the last $\Lambda - n + 1$ elements of $\boldsymbol{\eta}$. Clearly, $Z^{(1)}(k) = Z(k)$. We now evaluate (5.8) using the recursion relation

$$Z^{(n)}(k) = (e^{i\theta_n} + 1) Z^{(n-1)}(E) - i\eta_n^2 \prod_{m=n+1}^{\Lambda} (e^{i\theta_m} + 1). \tag{5.10}$$

Iterating (5.10) we finally get

$$Z(k) = (\mu_{\Lambda+1} + 1) \prod_{m=1}^{\Lambda} (e^{i\theta_m} + 1) - \sum_{n=1}^{\Lambda} \left[i\eta_n^2 \prod_{m \neq n} (e^{i\theta_m} + 1) \right]. \tag{5.11}$$

We can now evaluate the shift in the zeros of $Z(k)$ relative to the zeros of $Z_{sc}(k)$. Let us expand (5.11) around a point k_0 , at which $\theta_1 = \pi$ (and therefore $Z_{sc}(k_0) = 0$). We get

$$Z(k_0) = -i\eta_1^2 \prod_{m=2}^{\Lambda} (e^{i\theta_m} + 1) \tag{5.12a}$$

and

$$\left. \frac{\partial Z(k)}{\partial k} \right|_{k_0} = i \left\{ (\mu_{\Lambda+1} + 1)\theta_1' e^{i\theta_1} - 2\eta_1 \eta_1' - \sum_{n=2}^{\Lambda} \eta_n^2 \sum_{\substack{m>1 \\ m \neq n}}^{\Lambda} \frac{i\theta_m' e^{i\theta_m}}{e^{i\theta_m} + 1} \right\} \prod_{m=2}^{\Lambda} (e^{i\theta_m} + 1) \tag{5.12b}$$

where the prime denotes differentiation with respect to k . We note that $\eta_n \ll 1$, far enough from the threshold, and therefore we take only the leading term in (5.12b) into account. The first-order correction to the zero is determined by

$$0 = Z(k_0) + \delta k \left. \frac{\partial Z(k)}{\partial k} \right|_{k_0} \tag{5.13}$$

or

$$\delta k = \frac{\eta_1^2}{\theta_1'} \exp(-i\theta_1) = -\frac{\eta_1^2}{\theta_1'}. \tag{5.14}$$

We recall that $\theta_n' \rightarrow \hbar^{-1} \langle \tau(k) \rangle$, where $\tau(k)$ is the time delay, and that the mean level density is

$$\langle d(k) \rangle \approx \frac{\Lambda}{2\pi\hbar} \langle \tau(k) \rangle. \tag{5.15}$$

The shift in k , in units of the mean level spacing, is therefore given by

$$\delta \bar{k} = \delta k \langle d(k) \rangle \approx -\frac{\Lambda}{2\pi} \eta_1^2. \tag{5.16}$$

In order for the one evanescent channel approximation to be meaningful, the zero it yields must lie on the real energy axis, and so $\delta \bar{k}$ must be real. We will now show that this is indeed the case. In fact, we show that the quantity $\eta_n^2 \exp(-i\theta_n)$ is always real, and that this is a basic symmetry of our system.

Let us inject flux into the waveguide through the n th eigenchannel of S . The wavefunction in the waveguide will then be given by

$$\begin{aligned} \Psi(x, y) = 2 \exp(i\frac{1}{2}\theta_n) & \left\{ \sum_{m=1}^{\Lambda} \left[U_{mn} \cos\left(k_m x + \frac{\theta_n}{2}\right) \sin\left(\frac{\pi m}{D} y\right) \right] \right. \\ & \left. + \exp(-i\frac{1}{2}\theta_n) \eta_n \exp(-|k_{\Lambda+1}|x) \sin\left(\frac{\pi(\Lambda+1)}{D} y\right) \right\} \end{aligned} \tag{5.17}$$

where $k_m = \sqrt{k^2 - (\pi m/D)^2}$ is the longitudinal wavenumber of channel m .

Inside the billiard, $\Psi(\mathbf{r})$ must satisfy the following boundary conditions. On the boundary Σ of the scatterer

$$\Psi(\mathbf{r} \in \Sigma) = 0. \tag{5.18a}$$

On the opening Γ , we see from (5.17) that it should satisfy the impedance boundary conditions

$$\left. \frac{\int_0^D dy (\partial\Psi(\mathbf{r})/\partial x) \sin(\pi my/D) \partial x}{\int_0^D dy \Psi(\mathbf{r}) \sin(\pi my/D)} \right|_{x=0} = -|k_m| \tag{5.18b}$$

for all m .

Conditions (5.18) determine $\Psi(x = 0, y)$ up to an arbitrary factor. Since both conditions are real, as is the Helmholtz equation itself, the wavefunction can always be written as a real function (up to a complex factor). (5.17) implies that $\Psi(x = 0, y)$ satisfies this requirement if

$$\text{Im} \left[\eta_n \exp \left(-i \frac{1}{2} \theta_n \right) \right] = 0. \tag{5.19}$$

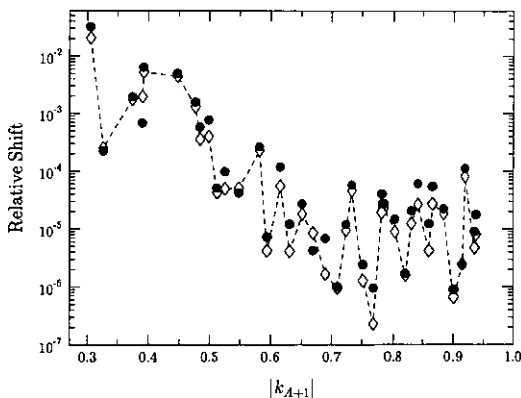


Figure 4. The shifts in the positions of the zeros of Z_{sc} relative to the true eigenenergy positions (circles), and the first-order approximation given in equation (5.14) (diamonds), for $\Lambda = 3$. The errors are in units of the mean level spacing, and the abscissa is taken to be the magnitude of $|k_{\Lambda+1}|$, measured in units of $(\pi/D)\sqrt{2\Lambda + 1}$. The broken curve is intended to guide the eye only.

In figure 4 we show the magnitude of the errors in the position of the zeros of $Z_{sc}(k)$, relative to $Z(k)$, for the Sinai billiard, with three open channels. The scales are the same as for figure 3. Also shown is the prediction of (5.14). We see that the agreement is good, and that, for the range of parameters shown, one can indeed attribute a large part of the error to the first evanescent mode.

The results shown here may be different for different systems. Equation (5.14) demonstrates that the important parameter is the magnitude of the first evanescent

mode, which is very low in the Sinai billiard. This parameter is therefore the dominant one governing the choice of the division of the billiard into two parts. The smallness of the magnitude of the first evanescent mode in the Sinai billiard is a consequence of the smooth (adiabatic) matching of the virtual waveguide (the broken lines in the inset of figure 5(a)) to the billiard. This smooth matching ensures a strong decay of the evanescent modes in the vicinity of the neck, and therefore the corresponding elements of \tilde{S} are suppressed. This emphasizes the statement, made in section 2.2, that the chord Γ dividing the billiard into its left and right parts must be chosen in such a way that the 'left channels' and the 'right channels' match smoothly.

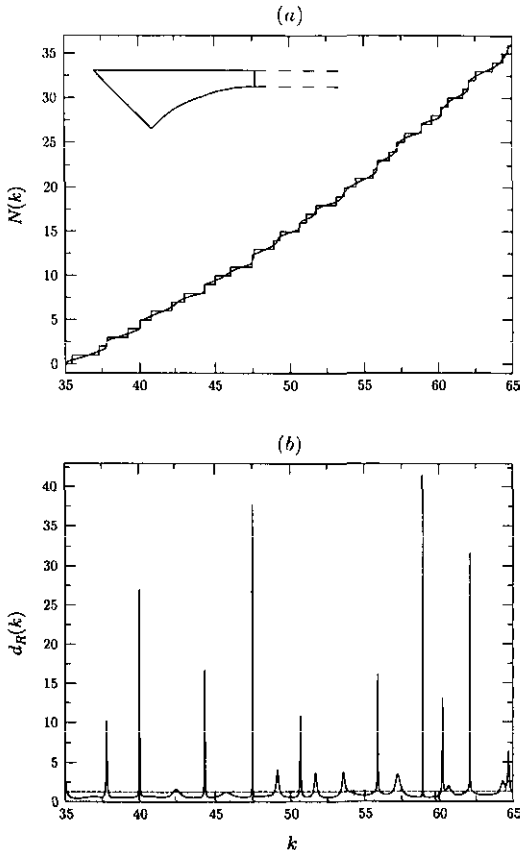


Figure 5. (a) The resonance counting function $N_R(k)$ (smooth line) against the spectral staircase, for $\Lambda = 1$, (b) The resonance density $d_R(k)$. The broken line represents the average value. Inset: the symmetry reduced Sinai billiard. The broken lines represent the opening of the billiard into a scattering system.

5.2. The resonance counting function

We shall now demonstrate the relation between the level density $d(k)$ and the resonance density $d_R(k)$. In sections 3 and 4 we derived a result saying that $d_R(k) \rightarrow \langle d(k) \rangle$ in the semiclassical ($\Lambda \rightarrow \infty$ in this case) limit. We illustrate this result in figures 5 and 6, in which we show the resonance counting function $N_R(k)$ overlaid on the exact spectral staircase $N(k)$ (figures 5(a) and 6(a)), and also the resonance density $d_R(k)$ (figures 5(b)

and 6(b)). Figure 5 shows the extreme case of $\Lambda = 1$. We see that both $N_R(k)$ and $N(k)$ oscillate around the same mean, the difference being that $N_R(E)$ displays smoothed steps, in contrast to the sharp steps which appear in $N(E)$. This is also reflected in the behaviour of $d_R(k)$, which fluctuates widely on the scale of the mean level spacing, and is consequently not a good approximation for $\langle d(k) \rangle$. Indeed, in the one channel case $d_R(k)$ exhibits sharp, separated resonances, which correspond closely to the positions of the eigenenergies of the billiard. For larger Λ , however, the situation changes. In figure 6 we show the case of $\Lambda = 6$. Here we already see that $N_R(k)$ is a smooth function, and $d_R(k)$ fluctuates slowly (on the scale of the mean level spacing) around $\langle d(k) \rangle$ (shown as a broken line). Note that there is now no connection between the peaks of $d_R(k)$ and the eigenenergies of the billiard.

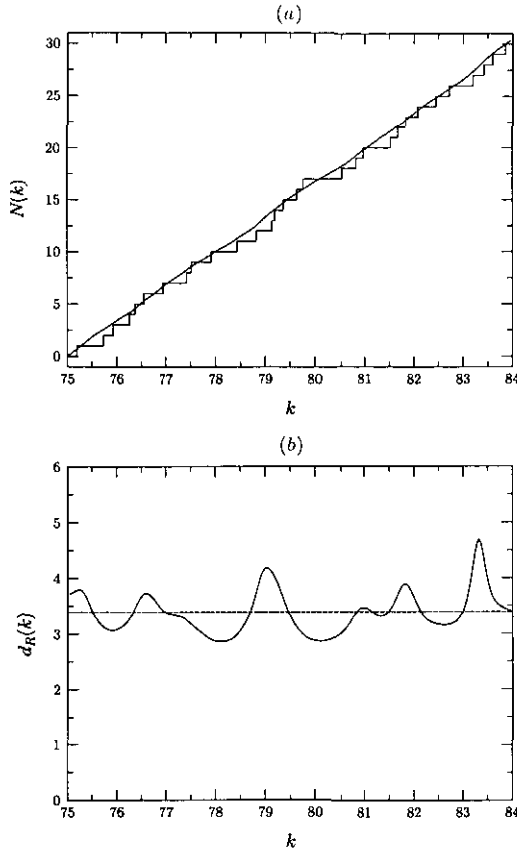


Figure 6. (a) The resonance counting function $N_R(k)$ (smooth line) against the spectral staircase, for $\Lambda = 6$. (b) The resonance density $d_R(k)$. The broken line represents the average value.

5.3. The distribution of τ_l

In section 4 we showed that the statistics of the eigenvalues of a chaotic billiard are intimately connected to the statistics of the eigenphases of the corresponding S matrix. This connection depended on the fact that the distribution of the τ_l (energy derivatives of the eigenphases θ_l) becomes more and more sharply peaked around $\langle \tau \rangle$, in the

semiclassical ($\Lambda \rightarrow \infty$) limit. In this section we demonstrate this result on the opened, symmetry reduced Sinai billiard.

We computed the S matrix as a function of k for $\Lambda = 3$ to 12. The eigenphases $\theta_l(k)$ were extracted, and the energy derivatives $\tau_l(k)$ were obtained by numerical differentiation. The results are shown in figure 7, where $\text{Var}(\tau_l)$ is given in units of $\langle \tau \rangle^2$ (full circles). We see that τ_l do indeed seem to cluster more and more around their average.

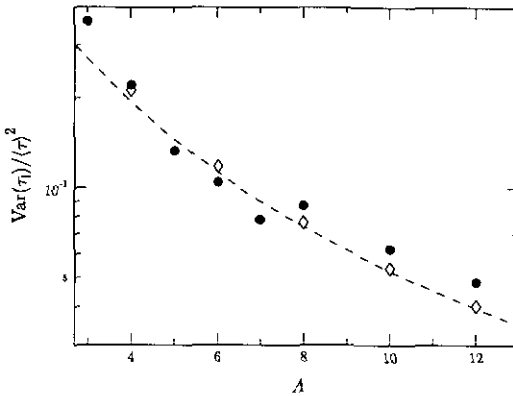


Figure 7. $\text{Var}(\tau_l)$ (in units of $\langle \tau \rangle^2$) as a function of Λ , (full circles) for the opened Sinai billiard. Also shown is $\text{Var}(\theta_{\text{COE}})$ (diamonds) and the asymptotic result (4.9) (broken curve).

In section 4 a simple model for the variance of τ_l was proposed. It was conjectured that the fluctuations of τ_l can be modelled by the fluctuations of the n th eigenphase θ_n around its mean position $\bar{\theta}_n$, where $0 \leq \theta_1 \leq \dots \leq \theta_\Lambda \leq 2\pi$. The variance of these fluctuations was then approximately given by (4.7), and asymptotically by (4.9). In figure 7 we also show the results of (4.7). Since the proposed model does not supply a proportionality factor, we normalized (4.7) by 3.4, in order to fit the data best. We see that the Λ dependence of $\text{Var}(\tau_l)$ is well modelled by the Λ dependence of $\text{Var}(\theta_{\text{COE}})$. We also show in a broken line the asymptotic formula (4.9). We conclude that the mechanism proposed in section 4 gives a reasonably good explanation for the statistical behaviour of τ_l .

6. Summary and conclusions

The main purpose of the present paper was to derive a semiclassical expression for the secular equation whose zeros are the energies of billiards in the plane. We have achieved this goal by taking advantage of the intimate relationship between the dynamics inside and outside the billiard boundary. The particular form of the semiclassical secular equation (1.2) enabled us to write the secular equation in a form which guarantees that unitarity imposed symmetries are preserved, even when the semiclassical approximation is used. The imposition of these symmetries was also the key element in enabling us to use the methods developed by Keating to express the secular equation in terms

of periodic orbits whose periods are bounded by the 'Heisenberg' time, namely, $\frac{1}{2}\hbar$ divided by the mean level spacing.

Similar equations were recently derived by Bogomolny [11], whose starting point was rather different from the one used in the present work. We tried to underline and explain the similarities as we developed our formalism. We have shown that the duality between the 'inside' and the 'outside' billiard problems establishes the correspondence between the two approaches, which is complete for the case of non-concave billiards.

Here we would like to make a few comments about this quantization method, which apply not only to the scattering approach developed here, but also to Bogomolny's approach.

The quantization procedure depends on the actual choice of the Poincaré section (e.g. the chord Γ). Its length determines the dimension Λ of the S matrix at a given energy. The orbits which are considered for the semiclassical secular equation also depend on the actual choice of the section. If this quantization procedure is at all valid, there should be a mechanism which removes the arbitrariness in the choice of the section, at least to leading order in \hbar . In a recent publication [28] we showed that the semiclassical level density (3.5) is nothing but a decomposition of Gutzwiller's level density formula into two disjoint classes of periodic orbits: those which contribute to $d_R(E)$ consist of periodic orbits which are trapped and never cross Γ , while the term on the right-hand side of (3.5) come from periodic orbits which do cross it. Thus, the choice of Γ does affect the definition of both terms but not their sum.

The relation between $d_R(E)$ and $d(E)$ (or the corresponding counting functions) is best illustrated by working them out for simple integrable billiards. Consider the quantization of a rectangular billiard with the four corners at $(0, 0)$, $(\alpha, 0)$, $(\alpha, 1)$, $(0, 1)$. We shall write the secular equation in two ways. We shall first chose the line Γ as the side of unit length along the y axis. We construct the channels by extending the two sides which are perpendicular to it. The S matrices are of dimension $\Lambda^{(1)} = [k/\pi]$ where $[x]$ stands for the integer part of x . It is easy to see that for Dirichlet boundary conditions,

$$S_{n,m}^L = -\delta_{n,m} \exp\left(2i\alpha(k^2 - (n\pi)^2)^{1/2}\right) \quad (6.1a)$$

and

$$S_{n,m}^R = -\delta_{n,m}. \quad (6.1b)$$

Hence the effective S matrix is given by

$$S_{n,m} = \delta_{n,m} \exp\left(2i\alpha(k^2 - (n\pi)^2)^{1/2}\right). \quad (6.1c)$$

The quantization condition can be readily shown to give the known spectrum

$$k_{n,m} = \pi(n^2 + (m/\alpha)^2)^{1/2} \quad (6.2)$$

since $\det(I - S) = 0$ is an exact quantization condition in the present example. The resonance counting function is given by

$$N_R^{(1)}(k) = \sum_{l=1}^{\Lambda^{(1)}} \alpha((k/\pi)^2 - l^2)^{1/2} - \frac{1}{2}\Lambda^{(1)}. \quad (6.3)$$

We now choose for the section Γ the side of length α along the x axis. One can repeat the calculation for this choice and find the same spectrum (6.2) as was previously derived. However, the dimension of the S matrix is $\Lambda^{(\alpha)} = [\alpha k/\pi]$, and the resonance counting function is now

$$N_R^{(\alpha)}(k) = \sum_{l=1}^{\Lambda^{(\alpha)}} \frac{1}{\alpha} ((\alpha k/\pi)^2 - l^2)^{1/2} - \frac{1}{2} \Lambda^{(\alpha)}. \quad (6.4)$$

It is clear that the two resonance counting functions are different. A closer inspection shows that the leading terms in the two expressions are the same, and they give the first (area) term in the Weyl formula,

$$N_R^{(1)}(k) \approx N_R^{(\alpha)}(k) \approx k^2 \frac{\alpha}{4\pi}. \quad (6.5)$$

Note that α is the area of the rectangle.

One can write down the exact spectral counting function for the rectangular billiard [36]

$$N(k) = \left[\alpha \sum_{l=1}^{\Lambda^{(1)}} ((k/\pi)^2 - l^2)^{1/2} \right]. \quad (6.6)$$

Comparing (6.3) and (6.6) we get

$$N_R^{(1)}(k) - N(k) = \sum_{l=1}^{\Lambda^{(1)}} (\{\xi_l\} - \frac{1}{2}) \quad (6.7)$$

where $\{\xi_l\}$ is the fractional part of $\xi_l = \alpha((k/\pi)^2 - l^2)^{1/2}$. The fractional part is a function which oscillates periodically about its mean value $\frac{1}{2}$. Hence, the sum represents a function with a vanishing mean. A similar result can be obtained for the difference $N_R^{(\alpha)}(k) - N(k)$. Thus, the resonance counting functions and the spectral counting functions share a common mean.

The resonance counting function was compared to the spectral counting function for the circular billiard [37]. There, the first two leading terms agree, while the next terms are the same up to an alternating sign.

We may conclude from these examples that different choices of the section Γ not only give S matrices with different dimensions, but also different counting functions $N_R(E)$. The example of the rectangular billiard shows that, in spite of these differences, the resulting spectrum is independent of the actual choice, confirming the general arguments given in the beginning of this discussion.

Another important issue which was discussed in section 4 is the relation between the energy and eigenphase spectra. Here, the fluctuations of the θ_l about their mean play an important role, and we brought strong evidence to support the conjecture that the distribution has a width which tends to zero in the semiclassical limit. This fact is intimately connected with the questions concerning the smoothness of the function $N_R(E)$ discussed in the previous paragraph.

Acknowledgments

It is a special pleasure to thank Dr J Keating, Dr E Bogomolny, Professor M V Berry, Dr B Dietz, Mr P Boasman and Professor S Agmon for stimulating discussions and help. US is obliged to the theoretical physics group at the H H Wills Physics Laboratory for the hospitality extended to him during his stay. This work was supported in part by grants from the US–Israel Binational Science Foundation (BSF) and the German–Israel Science Foundation (GIF).

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