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Scattering from a square obstacle

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Abstract

We use the scattering from a square obstacle as an example to illustrate the method of quantization of billiards from a scattering point of view. At the same time, we discuss rather surprising quantum and classical aspects of this simple scattering system. We attribute these peculiarities to the fact that the square presents to the outside obtuse angles at the corners. These properties are typical of the class of "quasi-integrable" scatterers.

1. Introduction

Billiards provide a most appropriate model for the understanding and discussion of classical chaotic Hamiltonian systems and their quantum counterparts [1–11]. Depending on the shape of the billiard the classical dynamics is regular, mixed or chaotic. For the corresponding quantum billiard the Schrödinger equation reduces to the Helmholtz equation in two dimensions and the reflection rules are replaced by appropriate homogeneous boundary conditions. The spectral statistics depends generically on the underlying classical motion.

The billiard can also be considered as an obstacle in a scattering problem. Since the boundary determines both the bounded motion inside the billiard and the scattering dynamics outside, it seems reasonable to investigate further the possible connections between the exterior and the interior dynamics, and use it as a means to gain better understanding of the billiard system. In the present paper we shall review some recent developments in this direction, and show how they find their application for the square billiard.

This paper is planned in the following way. The rest of the introduction chapter will be dedicated to a summary of some recent results which make use of the interior exterior connection. The last paragraph will explain why the square billiard was chosen to demonstrate these results, and why this seemingly trivial example has a few interesting surprises in store.

In the following we list the most important results obtained for convex billiards with differentiable boundaries. Most of them can be extended to billiards of arbitrary shapes. The results will be briefly summarized, detailed discussion and proofs can be found in Refs. [11–13].

- On the classical level, one can show that the classical Poincaré Scattering Map [14] is identical to the mapping which describes the

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dynamics in the interior. This is a simple consequence of the fact that the normal and the tangent at the point of reflection are common to the reflections from either the interior or the exterior.

In the domain of quantum mechanics, heuristic arguments were given in [12–16] to show that the knowledge of the energy dependent scattering matrix $S(E)$ is sufficient for the quantization of the motion inside a billiard. Recently Pillet proved rigorously for billiards with differentiable boundaries that, if at a certain energy E, 1 is an eigenvalue of multiplicity n of $S(E)$, then E is an eigenvalue of multiplicity n of the interior problem [17]. For systems like the circle and the ellipse one can also show the reverse, namely that $S(E)$ has an eigenvalue 1 if E is a Dirichlet eigenvalue, but the proof that this holds in general is not complete yet.

The inside–outside connection provides the basis for the semiclassical quantization which proceeds in two steps [11]. The first one consists of a truncation of the Hilbert space which is the domain of the scattering operator. This is possible because the billiard has a finite size, and to each incoming direction there exist two limiting values of the angular momentum such that no scattering occurs for angular momentum values below or above these limits. Correspondingly, in the semiclassical approximation, $S_{lm}(E) \simeq \delta_{lm}$ for $|l|$, $|m| \gg \Lambda/2$. The effective dimension Λ is determined by the extension of the billiard and the energy considered. In [11,13] its minimum value has been estimated as $[kL/\pi]$, where L is the circumference of the billiard and $[\cdot]$ stand for the integer value. The eigenvectors corresponding to rows with indices $|l|$ larger than $\Lambda/2$ are localized around $|l|$. That is, for the quantization of the billiard we may replace $S(E)$ by a finite dimensional matrix $\tilde{S}(E)$ defined as $\tilde{S}_{lm}(E) = S_{lm}(E)$, $|l|$, $|m| \leq \Lambda/2$. In the truncated Hilbert space, the quantization condition reads

$$\det(I - \tilde{S}(E)) = 0 . \tag{1.1}$$

This is the "Semiquantal" secular equation and it is the starting point for the semiclassical quantization of convex billiards with a differentiable boundary [11].

- The second step consists of deriving a semiclassical expression for the scattering matrix. Using Kirchhoff's approximation [26], one finds that the semiclassical S matrix equals the complex conjugate of the semiclassical 1-step evolution operator. The latter is the quantum counterpart of the billiard map which describes the classical dynamics inside the billiard. This equivalence is revealed by making use of the correspondence between scattering trajectories outside, and bounded trajectories inside which was mentioned above.
- The semiclassical secular equation is obtained from (1.1) by first writing it in terms of $\mathrm{Tr}\, S^n$, $n = 1, \ldots, \Lambda$. Then, one writes $\mathrm{Tr}\, S^n$ in terms of periodic orbits of the billiard, and upon substitution, one gets the secular equation as a sum over composite orbits [18] of the billiard map. The structure of the sum is analogous to that of the Riemann–Siegal lookalike [18,19] expression.
- Eq. (1.1) leads also to the Gutzwiller trace formula. The spectral density is obtained by calculating the logarithmic derivative of (1.1) and inserting the semiclassical expressions for $\mathrm{Tr}\, S^n$. The mean level density can be shown to be the phase of $\det(-S)$ divided by 2π, and smoothed over energy.
- The energy spectrum of the billiard is shown by this formalism to be intimately related to the spectrum (eigenphases) of the S matrix [13]. This in turn implies that there is a link between the statistical properties of the energy eigenvalues and the eigenphases. It was shown that both spectra display the statistics of the Orthogonal Ensemble if the Poincaré scattering map (and hence the billiard map) is chaotic.

In the present paper we will consider the quantization of the square billiard to check and demonstrate some of the results listed above. We chose this example for several reasons.

(i) The quantization of the interior in this case is trivial, and the eigenenergies are known analytically. The exterior problem is not simple at all and therefore, extracting the eigenenergies from the $S(E)$ matrix offers a good method to check the accuracy of some of our numerical methods.

(ii) In his proof, Pillet had to require a smooth boundary. By studying the square billiard, we looked for systematic deviations which could bear some evidence concerning the necessity of the smoothness requirement.

(iii) The exterior scattering problem is akin to interior problems which Berry and Richens [20] called quasi-integrable. They are concave polygon billiards with angles which are rational fractions of π. These systems derive their name from the fact that they possess two integrals of the motion, but because of topological reasons, their dynamics is not isomorphic to the motion on a torus. Similarly, in spite of the interior square billiard being integrable, the exterior problem is only quasi-integrable. This kind of behavior conflicts with our previous assertion that the exterior and interior dynamics are isomorphic. This affects not only the classical dynamics, but also the quantum description, and we shall show that in both cases the culprits are the corners which present obtuse angles to the outside.

2. The inside–outside duality for the square billiard

The classical scattering from a square obstacle is simple. Placing the square (whose side length is taken to be 2) symmetrically about the origin, with sides parallel to the (x, y) axes, we note that every front of rays (which is not parallel to the axes) will be split by the obstacle into two fronts which propagate in opposite directions. Their relative intensity is $\tan\theta_i$, where θ_i is the incidence angle. Those rays which miss the obstacle continue to propagate along the same line. In case of incidence along one of the axes, only back scattering will occur.

The classical inside–outside duality holds for any ray which does not hit the corners: the Poincaré Scattering Map is isomorphic to the inside billiard map. For rays which hit the corner, however, we have to distinguish between those directions which have a conjugate ray inside (these are the rays which would penetrate into the square if continued), and those directions for which there is no conjugate ray inside (rays which would remain outside after going through the corner). Thus we have identified a set (of zero measure) in the domain of the scattering map which does not have an analogue in the inside dynamics. The scattering map conserves the magnitude of both components of the velocity, but, because of the existence of these sets of measure zero, the phase space cannot be mapped onto a torus. The interior dynamics is certainly integrable. The quantum dynamics seems to be rather sensitive to classical peculiarities which occur on sets of zero measure, as will be shown in the sequel.

The quantum solution inside is trivial because the Helmholtz equation as well as the boundary conditions are separable. This is not the case for the outside problem. Here, one requires that the wave function vanishes at e.g., $|y| = 1$ for $|x| \leq 1$ but there is no restriction on the value of the wave function at $|y| = 1$ for $|x| > 1$. Thus, the boundary conditions introduce a coupling between the x and y degrees of freedom, and the problem becomes non-separable. This implies that the S matrix cannot be diagonalized by an energy independent matrix, and that there is no analytic expression for it. In Section 3 we shall derive a close form for the scattering amplitude in the semiclassical regime. We shall show that it reproduces successfully the exact (numerical) results, and that it approaches the classical expectations. The rest of the present section will be dedicated to numerical tests of the interior–exterior duality.

The scattering matrix for billiards with convex shape can be written as [11]:

$$\hat{S}(k) = -\hat{h}^-(k)\hat{h}^+(k)^{-1}, \tag{2.1}$$

where

$$\hat{h}^{\pm}_{ln}(k) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, i^l H^{\pm}_l(kR(\varphi))\, e^{i(l-n)\varphi}. \tag{2.2}$$

$R(\varphi)$ is the distance of the boundary from the origin of the coordinate system. For our square that is given by

$$\begin{aligned} R(\varphi) &= \frac{1}{\cos(\varphi - \Phi)}, \\ &\Phi \in \{0, \tfrac{1}{2}\pi, \tfrac{3}{2}\pi\}, \\ &\varphi \in [\Phi - \tfrac{1}{4}\pi, \Phi + \tfrac{1}{4}\pi], \end{aligned} \tag{2.3}$$

and the matrix elements (2.2) read

$$\begin{aligned} \hat{h}^{\pm}_{ln}(k) &= \frac{2}{\pi}\int_0^{\pi/4} d\varphi\, i^l H^{\pm}_l\left(\frac{k}{\cos\varphi}\right)\cos((l-n)\varphi) \\ &\times \left(\sum_{q=0}^{3}\cos((l-n)q\cdot\tfrac{1}{2})\right), \quad l,n \in\,]-\infty,\infty[\,. \end{aligned} \tag{2.4}$$

Due to the symmetries of the obstacle the S matrix can be reduced according to the six irreducible representations of the symmetry group of the rectangle. For example, in the space of totally antisymmetric functions (with respect to reflections along the axes and the diagonals) the S matrix is $\hat{S}_1(k) = -\hat{h}^-_1(k)\,\hat{h}^+_1(k)^{-1}$, where

$$\begin{aligned} \hat{h}^{\pm}_{1_{ln}}(k) &= \frac{2}{\pi}\int_0^{\pi/4} d\varphi\, H^{\pm}_{4l}\left(\frac{k}{\cos\varphi}\right) \\ &\times \sin(4l\varphi)\sin(4n\varphi), \\ &l,n \in [1,\infty[\,. \end{aligned} \tag{2.5}$$

The desymmetrized square billiard corresponding to $\hat{S}_1(k)$ is constructed by considering the triangle $0 \le x,\ y \le 1,\ x \le y$ and requiring the Dirichlet condition on its boundary.

To make use of the above relations, we must truncate the Hilbert space which so far is of infinite dimension. For any truncation which keeps the unitarity of the S matrix within the desired accuracy, the condition that S has a unit eigenvalue is expressed by the secular equation (1.1). By inserting the truncated $\hat{S}_1(k)$ into the secular equation (1.1) it is easily verified that it is equivalent to the requirement

$$\det(\hat{j}_1(k)) = 0, \quad \hat{j}_1(k) = \frac{\hat{h}^+_1(k) + \hat{h}^-_1(k)}{2}. \tag{2.6}$$

The most efficient truncation is found as a compromise between two conflicting requirements. On the one hand, if the dimension is too large, the determinant will be vanishingly small independently of k, because $\hat{S}_{1_{ln}}(k) \simeq \delta_{ln}$ for $|l|, |n| \to \infty$. On the other hand, if the dimension is too small, the unitarity of the truncated S matrix is violated. Denoting the dimension by Λ_1, one can find a semiclassical lower bound $\Lambda_1 > \Lambda_{1,\min} = [k/\pi]$. To get a quantitative estimate of the best choice of Λ_1, we define k_i^{ex} as the exact Dirichlet eigenvalues, k_i^{num} the numerically obtained solutions of the truncated secular equation. In Fig. 1 we show the mean values of the relative deviations $\delta k = \langle |(k_i^{\mathrm{num}} - k_i^{\mathrm{ex}})|/k_i^{\mathrm{ex}}\rangle$ as a function of $\delta\Lambda = \Lambda_1 - \Lambda_{1,\min}$. (The triangular brackets in the definition of δk stand for an average over the interval $0 < k < 20$.) Already

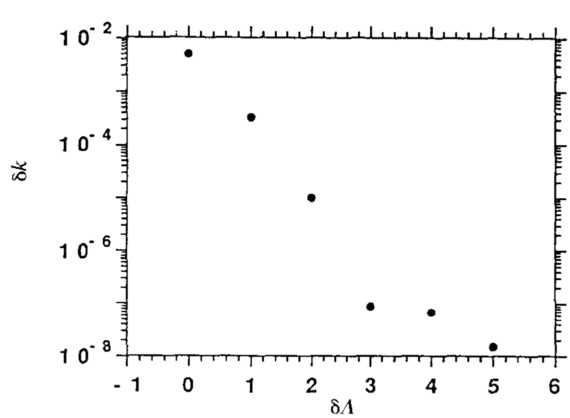

Fig. 1. The average of the absolute deviations δk as a function of the excess dimension $\delta\Lambda$.

for $\delta\Lambda = 2$ the zeros of the secular equation deviate by almost 10^{-5} from the exact Dirichlet eigenvalues. No parasitic zeros appear and no zeros are missing up to $k = 60$, where the numerical accuracy of the secular equation becomes insufficient.

Next, we numerically evaluated the integrals (2.5) in order to obtain $\tilde{S}_1(k)$. In Fig. 2 we show the behavior of the eigenphases $\theta_l(k)$ of the truncated $\tilde{S}_1(k)$ as a function of k. The truncation is performed by omitting those eigenphases which are less than $10^{-3}\pi$ or greater than $(2 - 10^{-3})\pi$. As expected from the results of the preceeding paragraph, the dimension of the scattering matrix increases by 1 at values of $k \simeq n\pi$, $n = 1, 2, \ldots$. The crosses mark the values of k which correspond to eigenvalues of the interior Dirichlet problem. They coincide with those values of k for which an eigenphase of $\tilde{S}_1(k)$ approaches 0. For $k = 25.328$ and $k = 28.964$, where the Dirichlet eigenvalue is twofold degenerate, two eigenphases of $\tilde{S}_1(k)$ vanish. These degeneracies have a number theoretical origin: they come about whenever an integer can be decomposed into a sum of squares in several ways, e.g., $65 = 8^2 + 1^2 = 7^2 + 4^2$.

In summary, we find that (at least for $k \le 60$), if $\tilde{S}_1(k)$ has an eigenvalue 1 of multiplicity n, then k corresponds to a Dirichlet eigenvalue of the same multiplicity. Hence, within this range and numerical accuracy, we could not identify any deviation from the quantization condition which might be due to the existence of corners in the square billiard.

To study the effects of the classical quasi-integrability on the quantum description of the scattering process, we investigated some statistical properties of the eigenphases of the scattering matrix. In Fig. 3 we compare the total phase $\Theta(k)/2\pi = (\sum_{l=1}^{\Lambda_1} \theta_l(k) - \Lambda_1\pi)/2\pi$ of $\det(-\tilde{S}_1(k))$

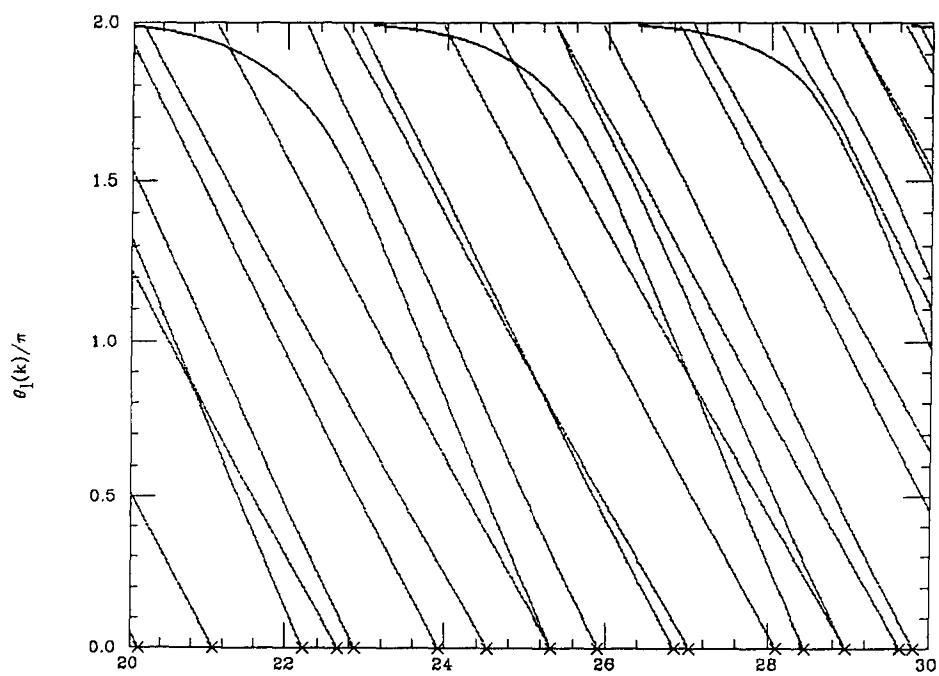

Fig. 2. The eigenphases $\theta_l(k)$ of $\tilde{S}_1(k)$ as a function of k for $k \in [20,30]$. Only those eigenphases of $\tilde{S}_1(k)$ are plotted which obey the inequality $10^{-3}\pi \le \theta_l(k) \le (2 - 10^{-3})\pi$. The crosses mark those values of k which correspond to an eigenvalue of the interior Dirichlet problem.

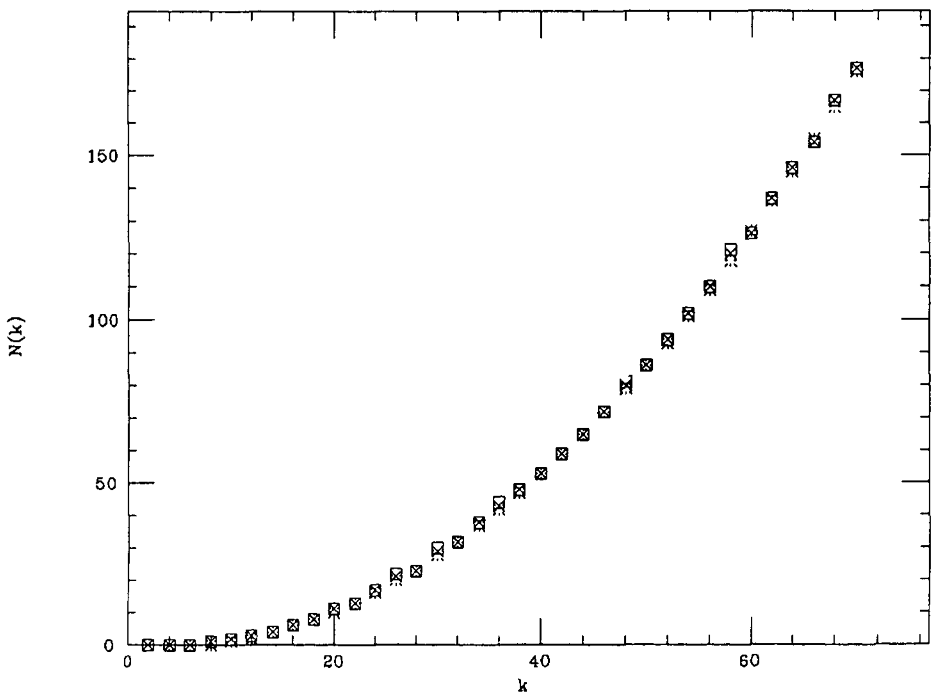

Fig. 3. The exact number of Dirichlet eigenvalues below k (square), the approximate number of eigenvalues below k as obtained by evaluating formula (2.7) (star) and the total phase $\Theta(k)/2\pi = (\Sigma_{l=1}^{\Lambda} \theta_l(k) - \Lambda\pi)/2\pi$ of $\det(-\tilde{S}_1(k))$ (cross).

(cross) with the number of Dirichlet eigenvalues $N(k)$ (square) with wavenumber less than or equal to k. These are compared with the smooth number function (star) of a triangular billiard.

$$N(k) \simeq \frac{1}{8\pi} k^2 - \frac{2+\sqrt{2}}{4\pi} k + \frac{3}{8}. \qquad (2.7)$$

In [22,23] it has been proven that for billiards with starlike shape $\Theta(k)/2\pi$ approaches the first two terms in (2.7) asymptotically as $k \to \infty$. Fig. 3 shows that the agreement between the three functions is very good even for rather moderate k values.

Next we studied the local fluctuation properties in the spectrum of the eigenphases of $\tilde{S}_1(k)$ and the scattering matrices corresponding to the other five symmetries of the square. We restricted to the nearest-neighbor level-spacing distribution $P(s)$ and the mean square deviation of the number of eigenvalues in a given interval L from its mean value, $\Sigma_2(L)$. First, we calculated the k-average of $P(s)$ and $\Sigma_2(L)$ for each of the six different scattering matrices for $30 < k < 60$. In Fig. 4 we show the results obtained by averaging over the different symmetries. $P(s)$ and $\Sigma_2(L)$ approach the nearest-neighbor level-spacing distribution and the Σ_2-statistics of random unitary matrices that are typical members of the circular orthogonal ensemble (COE) [24]. This is rather surprising since it is well known, that the eigenvalues statistics of the interior Dirichlet problem is Poissonian for each of the six symmetries. Thus, the spectral statistics of the Dirichlet eigenvalues differs from that of the eigenphases of the scattering matrix. This is a direct consequence of the breakdown of the correspondence between the Poincaré Scattering map and the billiard interior map.

In [13] it was argued that the local fluctuation properties in the eigenphase spectrum of the scattering matrix equal those in the eigenvalue spectrum of the interior Dirichlet problem. This follows from the observation that the spectral density $d(E)$ can be written as

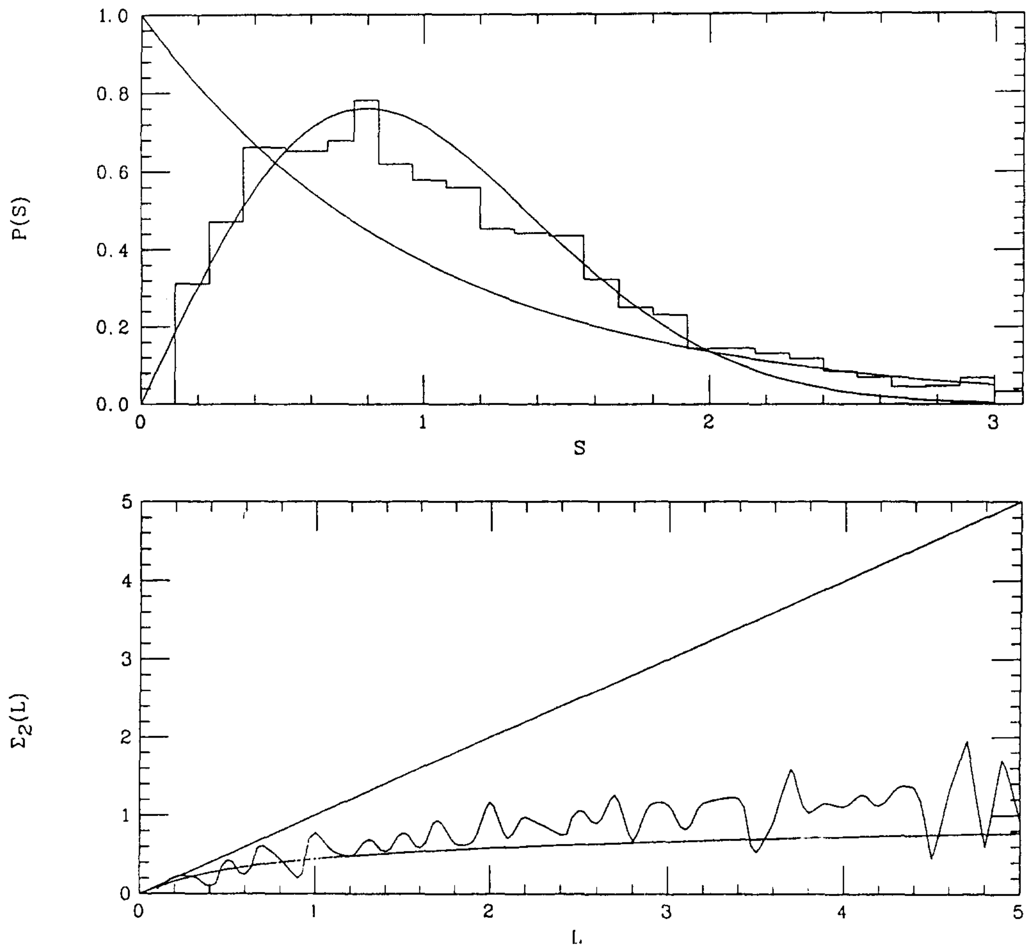

Fig. 4. The nearest-neighbor level-spacing distribution and the Σ_2-statistics for the eigenphases of $\tilde{S}(k)$. They are obtained by first calculating their k-averages for each of the 6 scattering matrices corresponding to the different symmetries of the square, where $k \in [30,60]$, and then taking their average.

$$d(E) = \sum_i \delta(E - E_i) = \sum_{l=1}^{\Lambda} \tau_l(E)\, \delta_p(\theta_l(E)) , \quad (2.8)$$

where $\theta_l(E)$ are the eigenphases of $S(E)$, $\tau_l(E) = \mathrm{d}\theta_l(E)/\mathrm{d}E$, and $\delta_p(x)$ is the 2π-periodic δ-function. When considering the spectral density in an interval Δ, which is small compared to the mean level-spacing, about an energy E_0 then (2.8) may be written as

$$\delta(\epsilon) = \sum_{l=1}^{\Lambda} \tau_l(E)\, \delta(\theta_l(E) + \epsilon\tau_l(E)) , \quad (2.9)$$

where $\epsilon = (E - E_0)$. If all the $\tau_l(E)$ could be replaced by their mean value, $\tau(E) = (1/\Lambda)\Sigma_{l=1}^{\Lambda} \tau_l(E)$, then the correspondence between the spectral density inside the billiard and the eigenphase density, which is given as $\Sigma_{l=1}^{\Lambda} \delta(\theta_l - \theta)$, would be established. $\tau(E)$ is the Wigner delay time [25]. The $\tau_l(E)$ may be set equal to $\tau(E)$, if we can show that the variance of the τ_l distribution is small compared to the mean value. In Fig. 5 we show the $\tau_l(k)$, $\tau(k)$ and the ratio $\mathrm{var}(\tau_l(k))/\tau(k)$ for $k \in [30,60]$ for $\tilde{S}_1(k)$. Whenever the dimension of the truncated $\tilde{S}_1(k)$ increases by 1 the deviation of that $\tau_l(k)$ which corresponds to the 'new' eigenphase is very large and decreases only slowly with k. $\mathrm{var}(\tau_l(k))/\tau(k)$ is still large for $k = 60$ and decreases only slowly.

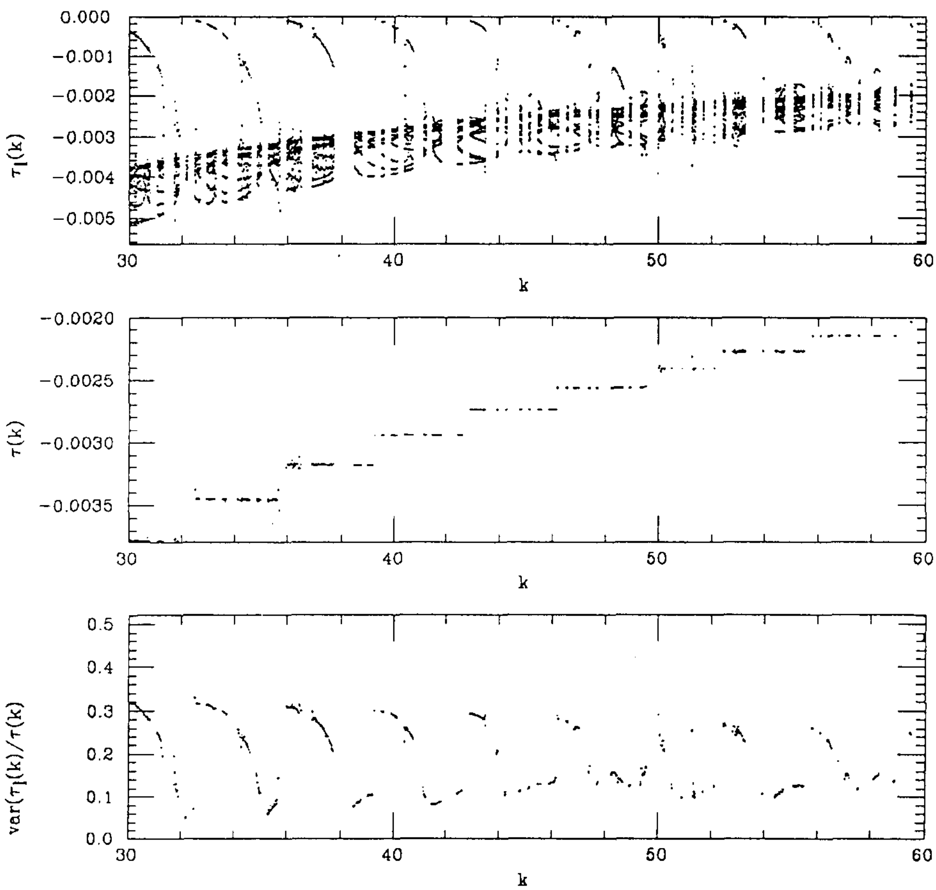

Fig. 5. The derivative of the lth eigenphase $\theta_l(k)$ of $\tilde{S}_1(k)$ with respect to E, $\tau_l(k) = \mathrm{d}\theta_l(E)/\mathrm{d}E = 1/2k\, \mathrm{d}\theta_l(k)/\mathrm{d}k$, their average $\tau(k) = (1/\Lambda)\sum_{l=1}^{\Lambda} \tau_l(k)$ and the ratio of the variance of the $\tau_l(k)$ about their mean and of $\tau(k)$.

Accordingly, the argumentation given in [13] does not apply to the square and is not in contradiction to our results for the spectral statistics of the eigenphases of the scattering matrix.

3. Semiclassical theory for the scattering matrix

The semiclassical theory of scattering is based on Kirchhoff's approximation [26] for the scattering amplitude

$$f(\theta_i, \theta_f) = \sum_l \sum_n \mathrm{e}^{-\mathrm{i}l\theta_i} (\hat{S}_{ln}(k) - \delta_{ln})\, \mathrm{e}^{\mathrm{i}n\theta_f} . \qquad (3.1)$$

The wavefunction is written as an incoming plane wave in the direction θ_i and an outgoing scattered wave,

$$\Psi(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\boldsymbol{r}} + \Psi_{\mathrm{scat}}(\boldsymbol{r}) . \qquad (3.2)$$

$\Psi(\boldsymbol{r})$ is assumed to obey the Dirichlet condition on the boundary Γ of the billiard. The scattering amplitude is obtained from the relation

$$\Psi_{\mathrm{scat}}(\boldsymbol{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{2\pi\mathrm{i}kr}}\, f(\theta_i, \theta_f) , \qquad (3.3)$$

where $\boldsymbol{r}$ is a point far from the boundary of the billiard. By applying Green's theorem to $\Psi_{\mathrm{scat}}(\boldsymbol{r})$, $f(\theta_i, \theta_f)$ may be expressed entirely in terms of $\Psi_{\mathrm{scat}}(\boldsymbol{r} \in \Gamma)$ and its normal derivative on the boundary Γ,

$$f(\theta_i, \theta_f) = \tfrac{1}{2} \int_{\Gamma} \mathrm{d}\Gamma(\boldsymbol{r}) [\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f \boldsymbol{r}'} \partial_{\boldsymbol{n}'} \Psi_{\mathrm{scat}}(\boldsymbol{r}') + \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i \boldsymbol{r}'} \partial_{\boldsymbol{n}'} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f \boldsymbol{r}'}] , \tag{3.4}$$

with $\boldsymbol{k}_f = k(\boldsymbol{r}/r)$. The Kirchhoff approximation is a short wavelength approximation, and it is introduced as follows:

- Every incoming direction divides the boundary Γ uniquely into two parts, the illuminated and the shaded part.
- In the shaded part near the boundary (Γ_S) the total wavefunction should be vanishingly small. Consequently, not only $\Psi(\boldsymbol{r} \in \Gamma_S) = 0$ but also $\partial_{\boldsymbol{n}} \Psi(\boldsymbol{r} \in \Gamma_S) \approx 0$.
- In the illuminated part near the boundary (Γ_I) the scattered and incoming components coherently enhance each other so that $\partial_{\boldsymbol{n}} \Psi(\boldsymbol{r} \in \Gamma_I) \approx \partial_{\boldsymbol{n}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i \boldsymbol{r}}$.

The resulting approximation for the scattering amplitude

$$f(\theta_i, \theta_f) = f_I(\theta_i, \theta_f) + f_S(\theta_i, \theta_f) \tag{3.5}$$

reads

$$f_I(\theta_i, \theta_f) = \tfrac{1}{2} \int_{\Gamma_I} \mathrm{d}\Gamma(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_i - \boldsymbol{k}_f)\boldsymbol{r}} (\boldsymbol{k}_i - \boldsymbol{k}_f)\boldsymbol{n} ,$$
$$f_S(\theta_i, \theta_f) = -\tfrac{1}{2} \int_{\Gamma_S} \mathrm{d}\Gamma(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_i - \boldsymbol{k}_f)\boldsymbol{r}} (\boldsymbol{k}_i + \boldsymbol{k}_f)\boldsymbol{n} . \tag{3.6}$$

For $\theta_i \neq \theta_f$ $F_I(\theta_i, \theta_f)$ provides the leading contribution to the semiclassical approximation of the scattering amplitude. In [11] the corresponding integral has been evaluated by the stationary phase approximation. We have shown that for convex billiards with differentiable shape it equals the complex conjugate of the semiclassical expression for the 1-step evolution operator which is the quantum counterpart of the billiard map.

For the square billiard one can integrate (3.6) analytically. The contribution of the shaded part is obtained by realizing that the flux of the vector field $(\boldsymbol{k}_i + \boldsymbol{k}_f)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_i - \boldsymbol{k}_f)\boldsymbol{r}}$ through Γ_S is divergent free. Hence the second of the integrals may be replaced by an integral along that diagonal of the square, which separates the shaded from the illuminated part. The result reads

$$f_I(\theta_i, \theta_f) = n_1 q_x\, \mathrm{e}^{\mathrm{i}n_1 q_x} \frac{\sin q_y}{q_y} + n_2 q_y\, \mathrm{e}^{\mathrm{i}n_2 q_y} \frac{\sin qx}{q_x} ,$$
$$f_S(\theta_i, \theta_f) = (n_1 K_x + n_2 K_y) \frac{\sin(n_1 q_x - n_2 q_y)}{(n_1 q_x - n_2 q_y)} ,$$
$$\begin{aligned} n_1 &= -1 \quad \text{for } \theta_i \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi[, \\ n_1 &= 1 \quad \text{otherwise} , \\ n_2 &= -1 \quad \text{for } \theta_i \in [0, \pi[, \\ n_2 &= 1 \quad \text{otherwise} , \end{aligned} \tag{3.7}$$

where

$$\boldsymbol{q} = \boldsymbol{k}_i - \boldsymbol{k}_f , \quad \boldsymbol{K} = \boldsymbol{k}_i + \boldsymbol{k}_f . \tag{3.8}$$

In Fig. 6 we show the logarithm of $|f(\theta_i, \theta_f)|^2$ for a few fixed values of θ_i; the lines correspond to the semiclassical approximation (3.7), while the dotted lines are the results obtained by inserting the numerical values for $\tilde{S}(k)$ into (3.1). For each value of θ_i, $|f(\theta_i, \theta_f)|^2$ has a narrow peak in the forward direction, where $\theta_i = \theta_f$ and for $\theta_i \neq \frac{1}{2}\pi$ two broader peaks to the left and to the right. The positions of the peaks correspond to those values of θ_f, for which classically specular reflection from θ_i to θ_f occurs. (3.7) reproduces them and the structure around them very well.

The classical limit of (3.7) is obtained by replacing $\sin q_y/q_y$, $\sin q_x/q_x$ and $\sin(n_1 q_x + n_2 q_y)/(n_1 q_x + n_2 q_y)$ by δ-functions; for example

$$q_x \frac{\sin q_y}{q_y} \overset{k\to\infty}{\rightarrow} \frac{\pi}{k} \frac{\cos\theta_i}{|\cos\theta_i|} \{\delta(\theta_i + \theta_f - \pi) + \delta(\theta_i + \theta_f - 3\pi)\} . \tag{3.9}$$

It is given as

$$f_I(\theta_i, \theta_f) \overset{k\to\infty}{\rightarrow} \pi \exp\{-\mathrm{i}2k \left|\sin\left(\frac{\theta_f - \theta_i}{2}\right)\right| \times \delta\left(\frac{\theta_i + \theta_f}{2} - \frac{\pi}{2} - \Phi\right) ,$$
$$f_S(\theta_i, \theta_f) \overset{k\to\infty}{\rightarrow} -2\pi\delta(\theta_i - \theta_f) , \tag{3.10}$$

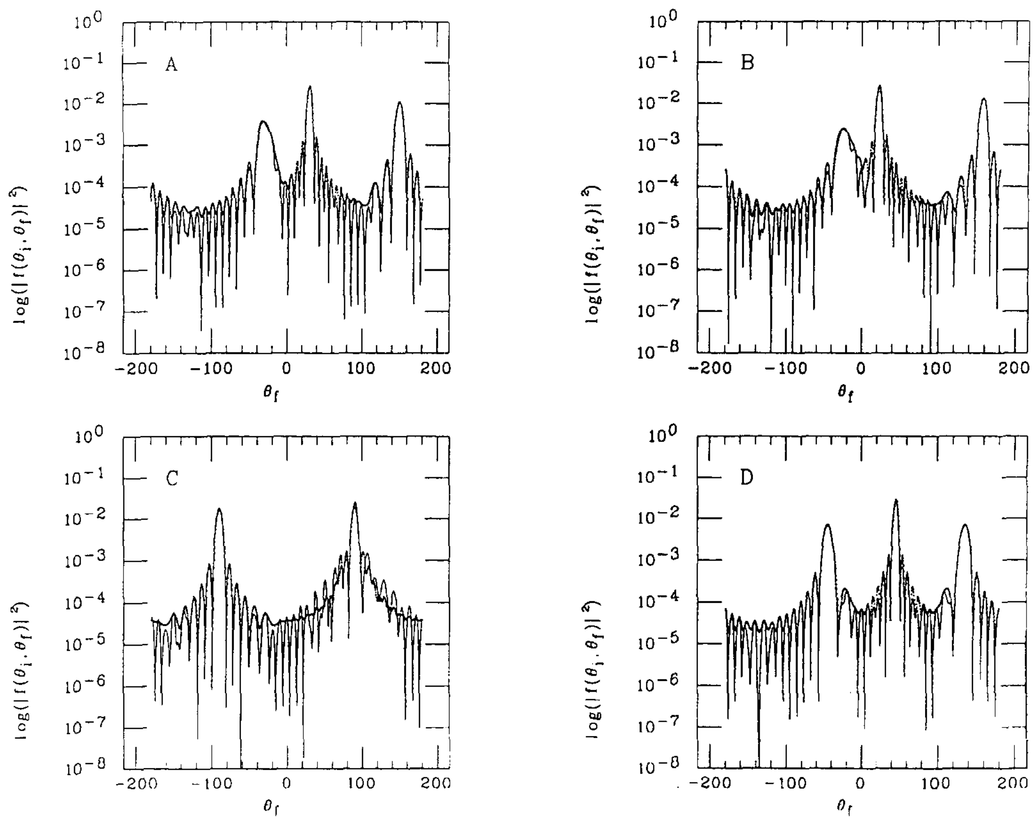

Fig. 6. The logarithm of $|f(\theta_i,\theta_f)|^2$ for a few fixed values of θ_i((A): $\theta_i=\frac{1}{6}\pi$, (B) $\theta_i=\frac{1}{8}\pi$, (C) $\theta_i=\frac{1}{2}\pi$, (D) $\theta_i=\frac{1}{4}\pi$) as a function of θ_f. The dotted lines are obtained by inserting $S(k)$ (see (2.1), (2.4)) into (2.10) and the continuous lines are obtained by evaluating (2.16).

where Φ is defined in (2.3). By inserting (3.10) into (3.1) it is easily verified that in the classical limit the scattering matrix equals the Fourier transform of $f_I(\theta_i,\theta_f)$. $f_I(\theta_i,\theta_f)$ is nonvanishing only for those values of $\theta_i+\theta_f$, for which specular reflection from the sides of the square occurs. The only contributing part to $f_S(\theta_i,\theta_f)$ originates from the forward direction.

In summary, the expression (3.6) for the scattering amplitude which has been derived using very simple arguments provides a very good approximation to the exact scattering amplitude at least around those values of $\theta_i+\theta_f$ where classically specular reflection occurs. In the classical limit it is nonvanishing only for these values of $\theta_i+\theta_f$.

4. Summary and conclusions

The scattering approach for the quantization of billiards was applied here for the case of the square billiard. This example turns out to be rather interesting, in spite of its being integrable as an interior problem. We emphasized certain properties which follow from the breakdown of the interior–exterior correspondence, which, in the present case occurs because of the discontinuity of the boundary at the corners.

One of the main results of this work was to show that within the range and accuracy of the numerical calculations, we could not detect spurious or missing zeros of the secular equation (1.1) which would indicate a failure of the theorem proved by Pillet for billiards with differentiable boundaries. We hope that this result will encourage our mathematical colleague to check again the necessity of the smoothness requirement.

The quasi-integrability of the exterior problem was used to explain the spectral statistics of the S matrix eigenphases, which follow the predictions of the COE ensemble of random matrices, in contrast with the established Poissonian statistics of the interior spectrum. This observation is similar to the findings of Richens and Berry [20] when they studied the spectral fluctuations of quasi integrable billiards.

Finally, we were able to check the applicability of the Kirchhoff approximation in the present case, and found it to be satisfactory.

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