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A THEOREM ON INFINITE PRODUCTS OF EIGENVALUES OF STURM-LIOUVILLE TYPE OPERATORS

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ABSTRACT. Infinite products of ratios of eigenvalues of Sturm-Liouville operators are expressed in a closed form in terms of corresponding solutions of initial-value problems.

Introduction. Gaussian path integrals are defined by

$$I = \int_{\text{paths}} D[\vec{\eta}(t)]\exp[iS[\vec{\eta}(t)]] \tag{1}$$

where

$$S[\vec{\eta}(t)] = \int_0^T (\vec{\eta}(t) \cdot \Lambda(t)\vec{\eta}(t))\, dt, \tag{2}$$

and $\vec{\eta}(t)$ are the N dimensional paths satisfying the end conditons $\vec{\eta}(0) = \vec{\eta}(T) = 0$. $\Lambda(t)$ is a differential operator of second order in (d/dt) acting on the space of paths. Such path integrals are often encountered in physical problems, e.g. the formulation of quantum mechanics [1].

The integral (1) can be evaluated in terms of the infinite product of the eigenvalues of $\Lambda(t)$ [2], [3]. In this note we present a theorem which enables one to express these infinite products in a closed form.

The theorem will be presented with a detailed proof for the simple case of a one dimensional problem ($N = 1$). The extension to $N > 1$ will be formulated but a detailed proof will not be supplied since it follows the same lines as for $N = 1$.

THEOREM 1. *Let the differential operator*

$$\Lambda(\alpha; t) = (d/dt)(p(t)d/dt) + \alpha q(t) \tag{3}$$

be defined for $0 \leqslant t \leqslant T$, $0 \leqslant \alpha \leqslant 1$, *where* $p(t)$ *and* $q(t)$ *are smooth functions of* t, $p(t) > p_0 > 0$.

Let $\lambda_k(\alpha)$ *and* $U^{(k)}(\alpha; t)$ *be the eigenvalues and eigenfunctions for the boundary-value problem*

$$\Lambda(\alpha; t)U(\alpha; t) + \lambda(\alpha)U(\alpha; t) = 0, \qquad U(\alpha; 0) = U(\alpha; T) = 0. \tag{4}$$

Let $y(\alpha; t)$ *be the solution of the initial-value problem*

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(5) $$\Lambda(\alpha; t)y(\alpha; t) = 0, \quad y(\alpha; 0) = 0, \quad (d/dt)y(\alpha; 0) = 1.$$

Then, for every α

(6) $$\left| \prod_{k=1}^{\infty} \frac{\lambda_k(\alpha)}{\lambda_k(0)} \right| = \left| \frac{y(\alpha; T)}{y(0; T)} \right|.$$

PROOF. The following points should be observed:

(a) For all k, $\lambda_k(0) > 0$ and for large k,

$$\lambda_k(0) \cong \left[\pi \Big/ \int_0^T \frac{dt}{\sqrt{p(t)}} \right]^2 \cdot k^2, \quad [4].$$

(b) The functions $\lambda_k(\alpha)$, $U^{(k)}(\alpha; t)$ and $y(\alpha; t)$ are analytic in α [5].

(c) For large k, $|\lambda_k(\alpha) - \lambda_k(0)| \leqslant \alpha \cdot M$ [4] with M independent of α or k. Hence the product on the left-hand side of (6) converges uniformly for all finite values of α.

(d) There is a finite number of nonpositive eigenvalues for a given α [6]. Hence the product on the left-hand side of (6) vanishes only at a finite number of α values, in the interval $0 \leqslant \alpha \leqslant 1$.

A necessary and sufficient condition for $y(\alpha; T)$ to vanish is that one of the $\lambda_k(\alpha)$ vanishes. Hence the two sides of equation (6) vanish simultaneously.

Consider now the functions:

(7) $$f(\alpha) = \prod_{k=1}^{\infty} \frac{\lambda_k(\alpha)}{\lambda_k(0)},$$

(8) $$\tilde{f}(\alpha) = y(\alpha; T)/y(0; T).$$

Since by construction $f(0) = \tilde{f}(0) = 1$, the theorem will be proved once it is demonstrated that $(d/d\alpha)f(\alpha)/f(\alpha) = (d/d\alpha)\tilde{f}(\alpha)/\tilde{f}(\alpha)$, for all α satisfying $f(\alpha) \neq 0$.

By virtue of (c) one can differentiate the left-hand side of equation (7) as if it were a finite product and get

(9) $$\frac{d}{d\alpha} f(\alpha)/f(\alpha) = \sum_{k=1}^{\infty} \frac{d\lambda_k(\alpha)}{d\alpha} \frac{1}{\lambda_k(\alpha)}.$$

It can easily be shown that

(10) $$\frac{d\lambda_k(\alpha)}{d\alpha} = -\int_0^T \left[U^{(k)}(\alpha; t) \right]^2 q(t)\, dt,$$

where $U^{(k)}(\alpha; t)$ are the solutions of (4) subject to the normalization

(11) $$\int_0^T \left[U^{(k)}(\alpha; t) \right]^2 dt = 1.$$

Since the integral in (10) is bounded by $\operatorname{Max}|q(t)|$ and because of (c), the series on the right-hand side of (9) converges uniformly and the summation and integration operations could be interchanged to yield

$$\frac{d}{d\alpha} f(\alpha)/f(\alpha) = \int_0^T G_{\lambda=0}(t, t, \alpha) q(t)\; dt \tag{12}$$

with

$$G_\lambda(t, t', \alpha) = \sum_{k=1}^{\infty} \frac{U^{(k)}(\alpha; t)\, U^{(k)}(\alpha; t')}{\lambda - \lambda_k(\alpha)}, \tag{13}$$

which is the spectral representation of Green's function for the boundary-value problem (4).

Returning to the initial value problem (5) and differentiating it with respect to α, one gets that $z(\alpha; t) \equiv dy(\alpha; t)/d\alpha$ is a solution of the initial-value problem

$$\Lambda(\alpha; t) z(\alpha; t) = -q(t) y(\alpha; t), \quad z(\alpha; 0) = 0, \quad (d/dt) z(\alpha; 0) = 0. \tag{14}$$

The solution of (14) is expressed in terms of the solution $y(\alpha; t)$ of (5) and the solution $\tilde{y}(\alpha; t)$ of the adjoint problem

$$\Lambda(\alpha; t)\tilde{y}(\alpha; t) = 0, \quad \tilde{y}(\alpha; T) = 0, \quad (d/dt)\tilde{y}(\alpha; T) = 1. \tag{15}$$

Applying the method of variation of the parameters, one can prove that the solution of (14) is given by

$$\begin{aligned} z(\alpha; t) = \Big[& y(\alpha; t) \int_0^t y(\alpha; t')\tilde{y}(\alpha; t') q(t')\, dt' \\ & - \tilde{y}(\alpha; t) \int_0^t y(\alpha; t') y(\alpha; t') q(t')\, dt' \Big] \Big/ W, \end{aligned} \tag{16}$$

where

$$W = p(t)\left[y\, d\tilde{y}/dt - \tilde{y}\, dy/dt \right] = \text{const.} \tag{17}$$

Hence

$$\begin{aligned} z(\alpha; T) = \frac{dy(\alpha; T)}{d\alpha} &= y(\alpha; T) \int_0^T \tilde{y}(\alpha; t) y(\alpha; t) q(t)\, dt \Big/ W \\ &= y(\alpha; T) \int_0^T G_{\lambda=0}(t, t, \alpha) q(t)\, dt \end{aligned} \tag{18}$$

where $G_\lambda(t, t', \alpha)$ is again Green's function for the boundary-value problem (4), expressed in terms of the solutions of the initial-value problem (5) and its adjoint (15). A comparison of (18) with (12) completes the proof.

We now turn to the generalization of Theorem 1 to the N dimensional case. The functions $p(t)$ and $q(t)$ of (3) are now replaced by the $N \times N$ symmetric matrices $P(t)$ and $Q(t)$. $P(t)$ is required to be positive definite for all $0 \leqslant t \leqslant T$. The N dimensional vectors on which $\Lambda(\alpha; t)$ acts are denoted by $\vec{U}(\alpha; t) \equiv (U_1(\alpha; t), \ldots, U_N(\alpha; t))$.

THEOREM 2. *Let a differential operator*

(19) $$\Lambda(\alpha; t) = (d/dt)(P(t)d/dt) + \alpha Q(t)$$

be defined for $0 \leqslant t \leqslant T$, $0 \leqslant \alpha \leqslant 1$, *with* $P(t)$ *and* $Q(t)$ *as discussed above.*

Let $\lambda_k(\alpha)$ *and* $\vec{U}^{(k)}(\alpha; t)$ *be the eigenvalues and eigenfunctions for the boundary-value problem*

(20) $$\Lambda(\alpha; t)\vec{U}(\alpha; t) + \lambda(\alpha)\vec{U}(\alpha; t) = 0, \quad \vec{U}(\alpha; 0) = \vec{U}(\alpha; T) = 0.$$

Let $\vec{y}^{(j)}(\alpha; t)$ $(j = 1, \ldots, N)$ *be the* N *independent solutions of the initial-value problem*

(21) $$\Lambda(\alpha; t)\vec{y}^{(j)}(\alpha; t) = 0, \quad \vec{y}^{(j)}(\alpha; 0) = 0, \quad (d/dt)y_i^{(j)}(\alpha; 0) = \delta_{i,j}.$$

Let $D(\alpha)$ *be*

(22) $$D(\alpha) = \det\left[y_i^{(j)}(\alpha; T)\right].$$

Then, for every α

(23) $$\left|\prod_{k=1}^{\infty} \frac{\lambda_k(\alpha)}{\lambda_k(0)}\right| = \left|\frac{D(\alpha)}{D(0)}\right|.$$

OUTLINE OF THE PROOF. The proof of Theorem 2 is again based on evaluating the logarithmic derivative of the two sides of (23). Once again it is shown that the logarithmic derivative can be expanded in the form

(24) $$\frac{df(\alpha)}{d\alpha} \Big/ f(\alpha) = \sum_{j,i=1}^{N} \int_0^T dt\, G_{j,i}^{(\lambda=0)}(t, t, \alpha) Q_{ij}(t) = \frac{d\tilde{f}(\alpha)}{d\alpha} \Big/ \tilde{f}(\alpha)$$

where $f(\alpha)$ and $\tilde{f}(\alpha)$ denote the functions of α on the left- and right-hand sides of equation (23) respectively. The Green's "function" is a matrix $G_{i,j}^{(\lambda)}(t, t', \alpha)$. In proving (24) use is made of the two equivalent methods to express the Green's function, the spectral representation and the representation by means of the $2N$ independent solutions of equation (21) and its adjoint.

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