Concepts in Condensed Matter Physics: Exercise 4

Spring 2020

Due date: 13/07/2020

1. **BCS theory of superconductivity**

   In this question you will re-derive the BCS theory studied in class and also study the effect of an applied Zeeman magnetic field. Our starting point is the Hamiltonian of electrons interacting via an attractive point contact interaction ($g > 0$):

$$H = \int d^3x \left\{ \sum_{\sigma\sigma'} c^\dagger_\sigma (x) \left[\left(-\frac{\nabla^2}{2m} - \mu \right) \delta_{\sigma\sigma'} - h\sigma^z_{\sigma\sigma'} \right] c_{\sigma'} (x) - g c^\dagger_\uparrow (x)\, c^\dagger_\downarrow (x)\, c_\downarrow (x)\, c_\uparrow (x) \right\}, \tag{1}$$

   where h is a Zeeman energy, which causes a chemical potential difference between the two spin species.

   (a) Write the Hamiltonian in momentum space, and then transform it to a quadratic form by assuming the order parameter $\Delta = \frac{g}{\Omega} \sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$ is weakly fluctuating (i.e., by performing mean field). Here Ω is the system's volume.

   (b) Diagonalize the quadratic Hamiltonian using a unitary transformation and find the spectrum of excitations. How does $h \neq 0$ affect the diagonalization?

   (c) For $h = 0$. What is the ground state wavefunction? What is the ground state energy? Show that by taking $\Delta \to 0$ we recover the known non-interacting ground state energy.

   (d) For $h = 0$. Using the ground state wavefunction, write a self-consistent equation ("the BCS gap equation") for Δ. Assume the interaction is only operational in a small window of energies of width $2\omega_D$ around the Fermi energy. Solve this equation for small values of g.

   (e) We now once again consider $h \neq 0$. Find the self consistent equation relating Δ and temperature. Do this by promoting the average with respect to the ground state to a thermal average.

   (f) Find the critical temperature T_c above which superconductivity is destroyed. At zero temperature, what is the critical interaction g_c, with and without a Zeeman term? Does the Copper instability persist to finite magnetic fields?

2. **Ginzburg-Landau theory from the BCS Hamiltonian**

In this problem you will work through some of the details of going from the microscopic BCS Hamiltonian to the Ginzburg-Landau action

$$S_{GL} = \beta \int d^d x \left[\frac{r}{2} |\Delta|^2 + \frac{c}{2} \left| \left(\partial_x - 2ie\vec{A} \right) \Delta \right|^2 + u |\Delta|^4 \right]. \tag{2}$$

Specifically, you will derive the phase transition controlled by the parameter r. You will do so following these steps.

(a) Write the quantum partition function associated with the Hamiltonian Eq. (1) (with $h = 0$). We also do not concern ourselves with coupling to the electromagnetic field, so you may assume $e = 0$.

(b) Use a Hubbard-Stratonovitch transformation to decouple the interaction, as we did in the tutorial. Show explicitly what is the "fat unity" that you are multiplying the partition function by. Formally integrate out the fermions and arrive at the effective action for Δ.

(c) Now, assume the field $\Delta(x, \tau)$ is uniform, both in space and imaginary time. Expand the action up to second order in Δ (verify what happens to the first order term).

(d) Comparing your result to Eq. (2), and performing the necessary Matsubara summations and integrals, find r. (You should again assume the interaction is only operational in a small window of energies of width $2\omega_D$ around the Fermi energy, and that the density of states is roughly constant within this window.)

(e) What happens to r at the critical point? Find the exponent α which describes the behavior of r near the transition $r \propto (T - T_c)^\alpha$.

(f) Expalin in a few words how would you derive the parameters c, u in Eq. (2)?

3. **Superconductivity on the surface**

In this question you will find that above H_{c2} there is a range of fields for which superconductivity can survive on the surface. Consult "Introduction to superconductivity", by M. Tinkham, page 135.

(a) Start from the Ginzburg-Landau theory of a superconductor and neglect non-quadratic orders near the critical point. Write down the corresponding equations of motion, and using an analogy to the Schrodinger equation, find the critical field H_{c2}, above which superconductivity cannot nucleate in the interior of the sample. Write the result in terms of ϕ_0 and ξ. Can you explain the result qualitatively?

(b) Consider the same physical setting with an edge at $x = 0$ (such that for $x > 0$ there is an insulator). Show that the boundary conditions take the form $\left(\frac{\nabla}{i} - \frac{2\pi\vec{A}}{\phi_0} \right) \Delta|_n = 0$ (what is the direction $|_n$?). Show that one can automatically satisfy this boundary condition by considering an auxiliary potential, containing a mirror image of the original potential in the insulating region. Does this affect the solution from part (a) well inside the superconductor (i.e., for $|x| \gg \xi$)?

(c) Argue, using the auxiliary potential, that very close to the surface one can find a solution with lower energy, making the critical field higher near the surface.

4. **Little-Parks effect**

Consider a superconductor which has the geometry of a ring with radius R and width d.

(a) A flux ϕ penetrates the center of the ring. Write the Ginzburg-Landau theory for the ring, explain what is the condition to be in the quasi 1D limit.

(b) How does T_c depend on ϕ? and what is the corresponding coherence length $\xi(\phi)$? Discuss the limit of $R > \xi$ and $R < \xi$.

(c) So far we have implicitly ignored phase fluctuations due to vortices slipping in and out of the ring. Qualitatively, when is this a good approximation?