Concepts in condensed matter physics - Exercise 5

Spring 2020

Due date: 24/07/2014

1. **RG analysis for the onset of Superconductivity in the presence of disorder**

Consider a thin three-dimensional superconducting material, with dimensions $W \times L \times L$ where $W \ll L$. The BCS interaction term $H_{BCS} = u_b \sum_{k,p} c_{k\downarrow}^\dagger c_{-k\uparrow}^\dagger c_{-p\uparrow} c_{p\downarrow}$ leads to a pairing instability. Assume that the bare interaction parameter u_b follows

$$u_b = \begin{cases} u_0 & \text{for} \quad \omega_D < \omega < E_F \\ u_0 - u_{ph} & \text{for} \quad \omega < \omega_D \end{cases},$$

where ω_D is the phonon Debye frequency, E_F is the Fermi energy and $u_0 > u_{ph} > 0$. The RG flow of the running coupling constant u is described by

$$\frac{d\Gamma}{dl} = \beta(\Gamma), \text{with } \Gamma = \nu u$$

where $l = \log\left(\frac{\Omega}{\omega}\right)$, ν the density of states and Ω is some ultra-violate cutoff.

a. What is the function $\beta(\Gamma)$? Using its solution find the transition temperature, T_c, of the following clean systems (ν is the density of states of the material):
   i. $\log\left(\frac{E_F}{\omega_D}\right) \approx 5$, $\Gamma_0 = \nu u_0 = 0.7$, and $\Gamma_{ph} = \nu u_{ph} = 0.3$.
   ii. $\log\left(\frac{E_F}{\omega_D}\right) \approx 5$, $\Gamma_0 = \nu u_0 = 0.8$, and $\Gamma_{ph} = \nu u_{ph} = 0.04$.

Now we would like to include the effect of disorder, which become important for frequencies below the Thouless energy given by $E_{Th} = \hbar\frac{D}{W^2}$, where D is the diffusion constant.

b. Based on the diffusion equation, $\left(\partial_t - D\partial_x^2\right)n = 0$, explain what is the physical origin of the Thouless energy.

It can be shown that, due to the diffusive motion of the electrons, for $\omega < E_{Th}$ the RG equation is modified according to

$$\beta(\Gamma) \to g + \beta(\Gamma)$$

where $g = \pi\frac{e^2}{h}R$ and R is the resistance of the sample.

c. Discussed qualitatively the effect of disorder, via g, on the flow equation. What is the critical value of g that will make the superconducting phase disappear?

d. Assume that the interaction parameter at the Thouless energy can be expressed as

$$\Gamma(E_{Th}) \equiv \Gamma_{Th} = -\frac{1}{\log\left(\frac{E_{Th}}{T_c^0}\right)},$$

where T_c^0 is the critical temperature in the clean case, and find an analytic expression for T_c as a function of u_{Th}, and g. Assume that at T_c the interaction parameter u, flows to $-\infty$. You may find the following integral helpful: $\int_{-\infty}^{-|\Gamma|} \frac{d\Gamma}{g-\Gamma^2} = \frac{1}{2\sqrt{g}} \log\left(\frac{1-\frac{\sqrt{g}}{|\Gamma|}}{1+\frac{\sqrt{g}}{|\Gamma|}}\right)$, for $g < \Gamma^2$.

e. Draw qualitatively T_c as a function of g.

2. **Mean-field approximation of the BKT transition point –** In this question you are asked to use the variational principle in order to obtain the critical temperature at which a two-dimensional superconductor undergoes a Berezinksii-Kosterlitz-Thouless (BKT) transition between a state in which the order parameter's correlations decay like a power and a state at which they decay exponentially. We will use the vairational action

$$S_V = \int d^2x \left[\frac{c}{2}(\nabla\theta)^2 + \mathrm{m}^2\,\theta^2\right]$$

where m^2 is a variational parameter to estimate the ground state of S_{SG} from question 1. Notice that we can write

$$Z_{SG} = \int D\theta\, e^{-S_{SG}} = \int D\theta\, e^{-S_V} e^{-(S_{SG}-S_V)} = Z_0\langle e^{-(S_{SG}-S_V)}\rangle$$

Where the brackets denote averaging with S_V. The variaitonal principle amounts to making the following approximation.

$$Z_{SG} \approx Z_0 e^{-\langle(S_{SG}-S_V)\rangle} \equiv Z_V.$$

a. Show that $F_{SG} \le F_V$, where F_{SG} is the free energy defined by the relation $F_{SG} = -T\log\left[Z_0\langle e^{-(S_{SG}-S_V)}\rangle\right]$ and therefore $F_V = F_0 + T\langle(S_{SG} - S_V)\rangle$.

b. Compute $\langle(S_{SG} - S_V)\rangle$ and minimize the free energy F_V with respect to m. Here you will need to introduce some ultraviolet cutoff Λ which represents a microscopic scale at which the theory breaks down.

c. Show that $m = \Lambda^{\alpha}$, find α. By taking $\Lambda \to \infty$ find the critical value of c separating the massive and massless phases.